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Quantum Hall-like effect for cold atoms in non-Abelian gauge potentials

N. Goldman and P. Gaspard

Center for Nonlinear Phenomena and Complex Systems - Université Libre de Bruxelles (U.L.B.), Code Postal 231, Campus Plaine, B-1050 Brussels, Belgium

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Abstract. - We study the transport of cold fermionic atoms trapped in optical lattices in the presence of artificial Abelian or non-Abelian gauge potentials. Such external potentials can be created in optical lattices in which atom tunneling is laser assisted and described by commutative or non-commutative tunneling operators. We show that the Hall-like transverse conductivity of such systems is quantized by relating the transverse conductivity to topological invariants known as Chern numbers. We show that this quantization is robust in non-Abelian potentials. The different integer values of this conductivity are explicitly computed for a specific non-Abelian system which leads to a fractal phase diagram.

During the last decades, remarkable quantum phenomena have been discovered such as the Aharonov-Bohm effect [1], the geometric phases [2,3], and the integer quantum Hall effect (IQHE) [4,5]. These important phenomena manifest themselves when gauge potentials are present in non-trivial topological spaces, in which cases the wave functions describing the system may acquire topological or geometrical phases. They are elegantly described in terms of differential geometry and topology in which the topological phases are related to the concept of holonomy [6]. In the IQHE, Hall's transverse conductivity of two-dimensional electronic systems submitted to high magnetic fields turns out to be quantized according to $\sigma_H = -C\,(e^2/h)$ where C is a topologically invariant integer called the first Chern number [6–8]. Since the proportionality factor is a combination of the electric charge e and Planck's constant h, such quantization phenomena play a key role in metrology [4].

Very recently, great experimental advances have been performed on the control of cold atomic systems [9–12]. In such systems, artificial gauge potentials can be created in optical lattices in order to reproduce the dynamics of periodically constrained fermionic atoms submitted to the analogue of a magnetic field. Such systems would thus offer the possibility to observe the corresponding fractal energy spectrum known as the Hofstadter butterfly [13,14]. More recently the realization of non-Abelian gauge potentials has also been envisaged, allowing the observation of a non-Abelian Aharonov-Bohm effect [15], magnetic monopoles [16], and particular metal-insulator transitions [17].

The purpose of this Letter is to report the possibility of an integer quantum Hall-like effect in the transport of cold fermionic atoms trapped in optical lattices submitted to artificial gauge potentials. We consider this effect in a family of non-Abelian gauge potentials, which contains Abelian potentials as particular cases. In spite of the new topological context provided by the non-Abelian gauge structure, we show that an integer quantum Hall-like effect is possible and that the transverse conductivity is indeed quantized in terms of Chern numbers in systems of fermionic cold atoms in optical lattices with artificial Abelian or non-Abelian gauge potentials. Since the current induced in our system is electrically neutral, this quantized conductivity has the distinctive units of the inverse of Planck's constant, which could be important for metrological purposes. In this Letter, we first define the system Hamiltonian characterized by its non-Abelian gauge structure and evaluate the transverse conductivity with Kubo's formula which gives the linear response of the system to an external static perturbation. We then express the Hall-like conductivity in terms of Chern numbers in order to prove the quantization of this physical quantity. We compute these topological invariants for a specific system which leads to a fractal phase diagram. We then discuss possible experimental setups for the validation of our theoretical considerations.

We are interested in systems described by the following Hamiltonian

$$\mathcal{H} = \sum_{m,n} t_a \, \{c^\dagger_{m,n}|U_x|c_{m+1,n}\} + t_b \, \{c^\dagger_{m,n}|U_y|c_{m,n+1}\} + \text{h.c.} \tag{1}$$

which determines the dynamics of fermions in a two-dimensional lattice within a tight-binding approximation as studied experimentally in cold atom systems trapped in optical lattices [9–12]. For $U_{x,y}$ belonging to the Abelian group of unitary complex numbers $U(1)$, this Hamiltonian reproduces the evolution of an electronic system constrained by a periodic potential and submitted to a magnetic field [14]. In the latter case, the single-particle wave function gets a phase factor U_x (U_y) when tunneling is performed along the x (y) direction and their product around a unit cell gives a total phase proportional to the penetrating magnetic flux. In this Letter, we study the case in which the system has a $U(2)$ gauge structure, which implies that the operators U_x and U_y are 2×2 unitary matrices and the single-particle wave function has two components: when tunneling occurs along the x (y) direction, the matrix U_x (U_y) acts as a tunneling operator on the two-component wave function and the system is non-Abelian for $[U_x, U_y] \neq 0$ [15,17]. The operators $U_{x,y}$ are related to the gauge potential $\boldsymbol{A}$ present in the system via the usual relation $U_{x,y} = e^{iA_{x,y}}$. In order to indicate multicomponent wave functions and operators, we use the following notations introduced in Ref. [3]: A *row* of n orthogonal kets is denoted by $\{|\psi\rangle| = (|\psi\rangle_1, ..., |\psi\rangle_n)$ and a *column* of n kets by $||\psi\rangle\}$.The coefficients t_a and t_b describe transfers in the $x = ml_x$ and $y = nl_y$ directions, where (l_x, l_y) are the unit cell's lengths, and $|c^\dagger_{m,n}\}$ (resp. $|c_{m,n}\}$) are the 2-component fermionic creation (resp. annihilation) field operators on site (m, n). After a change of basis, the latter's components are transformed into $c_{m,n,j} = \sum_\lambda \psi_{\lambda j}(m,n)\, c_\lambda$, where c_λ is the operator of annihilation of a fermion in the j^{th} component $\psi_{\lambda j}(m,n)$ of the single-particle wave function $\{\psi_\lambda(\boldsymbol{r})| = (\psi_{\lambda 1}(\boldsymbol{r}), \psi_{\lambda 2}(\boldsymbol{r}))$ with $\boldsymbol{r} = (m,n)$ and $j = 1, 2$.

The current density of the system $\boldsymbol{j}_{m,n}$ associated with the Hamiltonian operator (1) is written in terms of the fermionic operators and its components are given by

$$\begin{aligned} j_{m,n,x} =& \frac{it_a}{\hbar} \sum_{\lambda\lambda' jj'} \psi^*_{\lambda j}(m-1,n) (U_x)_{jj'} \psi_{\lambda' j'}(m,n) c^\dagger_\lambda c_{\lambda'} \\ &+ \text{h.c.} \end{aligned}$$

and a similar expression for $j_{m,n,y}$.

It has been recently suggested that non-Abelian gauge potentials can be created in optical lattices [15, 17]. In such setups, atoms with doubly degenerate Zeeman sublevels hop from a site to another with the assistance of additional lasers and their tunnelings are indeed described by unitary non-commutative operators [15]. The corresponding gauge potential is given by

$$\boldsymbol{A} = \left[\begin{pmatrix} -\frac{\pi}{2} & \frac{\pi}{2} e^{i\phi} \\ \frac{\pi}{2} e^{-i\phi} & -\frac{\pi}{2} \end{pmatrix}, \begin{pmatrix} 2\pi m \alpha_1 & 0 \\ 0 & 2\pi m \alpha_2 \end{pmatrix}, 0 \right] \tag{2}$$

and induces an effective "magnetic" field characterized by three parameters α_1, α_2 and ϕ, which can be controlled with external lasers [15]. We notice that this gauge potential is Abelian under the condition that $\alpha_1 - \alpha_2$ is an integer and non-Abelian otherwise.

We evaluate the linear response of the system to an external static force applied along the y direction $\mathcal{H}_{\text{ext}} = -\Delta_y \sum_{m,n} n \, \{c^\dagger_{m,n}|c_{m,n}\}$. A generalized Lorentz force produced by the effective "magnetic" field induces a transverse current which satisfies $\langle j_x \rangle = \sigma_{xy} \, \Delta_y$. Kubo's formula expresses the transverse conductivity in terms of the total current $\boldsymbol{J} \equiv \sum_{m,n} \boldsymbol{j}_{m,n}$

$$\sigma_{xy} = \frac{i\hbar}{V} \sum_b \frac{\langle b|J_x|0\rangle\langle 0|J_y|b\rangle}{(E_b - E_0)^2} - \frac{\langle 0|J_x|b\rangle\langle b|J_y|0\rangle}{(E_b - E_0)^2} \tag{3}$$

where $|b\rangle$ denotes the eigenstates of the Hamiltonian (1) of eigenvalue E_b and V is the system volume or area.

The system presents symmetries under translations defined by the operators $|\big(T^q_x\, \psi\big)(m,n)\} = |\psi(m+q,n)\}$ and $|\big(T_y\, \psi\big)(m,n)\} = |\psi(m,n+1)\}$. The operator T_y commutes with the single-particle Hamiltonian H, since the potential (2) only depends on the x coordinate. On the other hand, the commutation relations $[T^q_x, H] = [T^q_x, T_y] = 0$ are satisfied under the conditions $\alpha_j = p_j/q$ with $j = 1, 2$ and p_1, p_2, q some integer numbers. The translational symmetries allow us to write $|\psi(m,n)\} = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|u(m)\}$ with $|u(m)\}$ q-periodic and the first Brillouin zone is a 2-torus $\mathbb{T}^2$ defined by $k_x \in [0, \frac{2\pi}{q}]$ and $k_y \in [-\pi, \pi]$. The latter change can be viewed as a gauge transformation which modifies the Hamiltonian by substituting $A_\mu \to A'_\mu = A_\mu + k_\mu$ so that the single-particle Hamiltonian takes the following form

$$\begin{aligned} &H(\boldsymbol{k})|u(m)\} = \\ &t_a e^{i(A_x + k_x)}|u(m+1)\} + t_a e^{-i(A_x + k_x)}|u(m-1)\} \\ &+ t_b e^{i(A_y + k_y)}|u(m)\} + t_b e^{-i(A_y + k_y)}|u(m)\} \end{aligned} \tag{4}$$

This equation can be differentiated with respect to k_μ and a straightforward calculation yields

$$\begin{aligned} \sum_b \frac{\langle b|J_x|0\rangle\langle 0|J_y|b\rangle}{(E_b - E_0)^2} = \frac{1}{\hbar^2} \sum_{\epsilon_\lambda < \epsilon_{\text{F}}} \sum_{\epsilon_{\lambda'} > \epsilon_{\text{F}}} \{\langle u_{\lambda'}|\partial_{k_x} H|u_\lambda\rangle\} \\ \times \{\langle u_\lambda|\partial_{k_y} H|u_{\lambda'}\rangle\}/(\epsilon_\lambda - \epsilon_{\lambda'})^2 \end{aligned} \tag{5}$$

where $H|u_\lambda\} = \epsilon_\lambda |u_\lambda\}$ are the single-particle eigenstates and the scalar product is defined by $\{\langle u_\lambda | u_{\lambda'}\rangle\} = \sum_{m=1}^q \sum_{j=1,2} u^*_{\lambda j}(m)\, u_{\lambda' j}(m)$. The Fermi energy ϵ_{F} is supposed to be situated inside a gap of the spectrum. We find that $\sigma_{xy} = -\sigma_{yx}$ and the identity (5) allows us to write the conductivity as

$$\sigma_{xy} = \frac{1}{(2\pi)^2 i\hbar} \sum_{\epsilon_\lambda < \epsilon_{\mathrm{F}}} \int_{\mathbb{T}^2} \sum_j \Big[\langle \partial_{k_x} u_{\lambda j} | \partial_{k_y} u_{\lambda j} \rangle - \langle \partial_{k_y} u_{\lambda j} | \partial_{k_x} u_{\lambda j} \rangle \Big] d\boldsymbol{k} \tag{6}$$

This result is important because it relates the transverse conductivity to integer topological invariants called Chern numbers. The latter are defined on a *fibre bundle* which is a topological space that locally resemble the direct product of the parameter space $\mathbb{T}^2$ with the non-Abelian gauge group $U(2)$ and that we note $P(\mathbb{T}^2, U(2))$. This product space may be globally non-trivial and the structure *twists* when the Chern number is non-zero. In the Abelian framework, the interpretation of the IQHE in terms of topological arguments [6,7] showed that Berry's curvature $\mathcal{F} = \big[\langle \partial_{k_x} \psi(\boldsymbol{k}) | \partial_{k_y} \psi(\boldsymbol{k}) \rangle - \langle \partial_{k_y} \psi(\boldsymbol{k}) | \partial_{k_x} \psi(\boldsymbol{k}) \rangle\big] dk_x dk_y$, with $|\psi\rangle$ an eigenstate depending on the wave vector $\boldsymbol{k}$, defines a curvature on the fibre bundle $P(\mathbb{T}^2, U(1))$ and that its integral over the parameter space $\mathbb{T}^2$ corresponds to the first Chern number $C = \frac{i}{2\pi} \int_{\mathbb{T}^2} \mathcal{F}$, which is an integer. These concepts can be generalized in the actual non-Abelian framework [18, 19] for which Berry's curvature $\mathcal{F}$ is a 2×2 matrix and the Chern number is then defined by

$$C = \frac{i}{2\pi} \int_{\mathbb{T}^2} \mathrm{tr}\big(|\langle \partial_{k_x} u| \} \{ |\partial_{k_y} u \rangle | - |\langle \partial_{k_y} u| \} \{ |\partial_{k_x} u \rangle | \big) dk_x dk_y \tag{7}$$

where $\mathrm{tr}\mathcal{F}$ denotes the trace of the matrix $\mathcal{F}$. One eventually finds that the Hall-like conductivity (6) is given by a sum of Chern numbers

$$\sigma_{xy} = -\frac{1}{h} \sum_{\epsilon_\lambda < \epsilon_{\mathrm{F}}} C(\epsilon_\lambda) \tag{8}$$

where the integer numbers $C(\epsilon_\lambda)$ are associated with each energy band. It follows that the transverse Hall-like conductivity of the system evolves by steps corresponding to integer multiples of the inverse of Planck's constant and that it remains constant under small perturbations. Besides, a measure of the transverse conductivity in this non-Abelian system provides a direct evaluation of these topological invariants.

The Chern numbers were explicitly obtained in the usual Abelian framework [20], and we here show that they can be computed explicitly in non-Abelian systems as well. Using Stokes' theorem one gets the following result

$$\sigma_{xy} = \sum_{\epsilon_\lambda < \epsilon_{\mathrm{F}}} \frac{1}{2\pi h} \oint_{\partial \mathbb{T}^2} \left(\frac{\partial \theta_\lambda}{\partial k_x} dk_x + \frac{\partial \theta_\lambda}{\partial k_y} dk_y \right) \tag{9}$$

where θ_λ is the phase accumulated by u_λ around the torus and evaluated on its border $\partial \mathbb{T}^2$. These phases can be evaluated in a way similar as in the Abelian case [20]. Let us write the Schrödinger equation associated with the non-Abelian Hamiltonian (4) in the form

$$t_a \begin{pmatrix} e^{i\phi + ik_x} u_2(m+1) + e^{i\phi - ik_x} u_2(m-1) \\ e^{-i\phi + ik_x} u_1(m+1) + e^{-i\phi - ik_x} u_1(m-1) \end{pmatrix} + t_b \begin{pmatrix} 2\cos(2\pi\alpha_1 m + k_y) u_1(m) \\ 2\cos(2\pi\alpha_2 m + k_y) u_2(m) \end{pmatrix} = \epsilon \begin{pmatrix} u_1(m) \\ u_2(m) \end{pmatrix} \tag{10}$$

When $t_a \to 0$, this equation gives a $2q$-degenerate single band and the dispersion law for $u_j(m)$ is

$$\epsilon_j(m, k_y) = 2t_b \cos(2\pi p_j m/q + k_y) =: 2t_b \cos(k_y^m) \tag{11}$$

First, we consider the case where $\alpha_1 = p_1/q$ and $\alpha_2 = (p_1 + nq)/q$ with n integer. This case allows the explicit computation of Chern numbers for Fermi energies situated inside the various gaps of the spectrum in the approximation of weak coupling with t_a small and $t_b = 1$. Since the Chern numbers are topological invariants, their weak-coupling values extend to other couplings as long as the band gaps deform continuously. As t_a increases from 0 to 1, different regimes are observed. For $t_a > 0$ and small, $q-1$ gaps open at the *crossing points* $k_y = \pm\frac{l\pi}{q}$ with l integer. Thereafter, the dispersion branches separate and no crossing remains. When $t_a = 1$, the spectra $\epsilon = \epsilon(\alpha_1 = \frac{p_1}{q}; \alpha_2 = \frac{p_1+nq}{q})$ precisely depict a Hofstadter butterfly [13]. Finally when $t_a > 1$, q new gaps open so that the number of gaps in the spectrum becomes $2q-1$.

In order to obtain the Chern numbers, one can study how the eigenfunctions change at the different gaps as k_y varies in the first Brillouin zone [20]. Near these gaps, the four functions $u_{1,2}(m)$ and $u_{1,2}(m')$ with their dispersion branch crossing each other in the limit $t_a = 0$ are strongly coupled together. Anticrossing appear for $t_a \neq 0$ which can be calculated by perturbation theory giving an effective Schrödinger equation for the vector $\boldsymbol{v}(m, m', r)$ formed with these four functions. Considering the r^{th} gap and setting $t_r = m - m'$, this equation can be written as $M(q, t_r, k_x)\, \boldsymbol{v}(m, m', r) = \epsilon\, \boldsymbol{v}(m, m', r)$. The latter system has been solved in order to find the four eigenvectors $\boldsymbol{v}_i$. When k_y passes a *crossing point*, the eigenvectors undergo some transformation $\boldsymbol{v}_i \to \tilde{\boldsymbol{v}}_i = R(i, t_r, k_x)\boldsymbol{v}_i$. Following the eigenvector of the r^{th} band through the successive anticrossings in the first Brillouin zone $k_y \in [-\pi, \pi]$, one finds that the eigenvector goes under the total transformation $|u_\lambda\} \to e^{iq(t_r - t_{r-1})k_x} |u_\lambda\}$. We can now evaluate the r^{th} band's contribution to the transverse transport coefficient σ_{xy} by considering the formula (9) which gives $\sigma_{xy}(r^{\mathrm{th}}\,\mathrm{band}) = (t_r - t_{r-1})/h$.

The integers t_r have still to be evaluated. From the definition (11), one has $\epsilon_1(m, k_y) = \epsilon_2(m, k_y)$ in the present situation where $\alpha_2 = (p_1 + nq)/q$. Two conditions are necessary for *crossing points* that determine the r^{th} gap when $t_a \neq 0$: $k_y^{m'} = -k_y^m + 2\pi s_r$ with s_r integer, and $k_y^{m'} = k_y^m - 2\pi p_1 t_r/q$ with $t_r = m - m'$, which lead to the Diophantine equation $r = p_1 t_r + q s_r$, with r the position of the gap and $\alpha_1 = p_1/q, \alpha_2 = (p_1 + nq)/q$. For $t_a > 1$, all

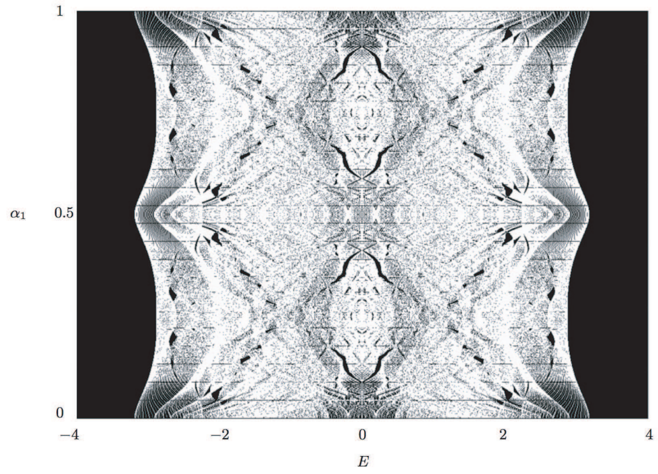

Fig. 1: Energy spectrum $\epsilon = \epsilon(\alpha_1)$ for $\alpha_2 = 31/701$. Gaps are shown in black. A Hofstadter butterfly crosses this structure at $\alpha_1 = \alpha_2 = 31/701$, where its larger gaps are put forward through thin black lines.

the $2q-1$ gaps open, so that $\sigma_{xy} = \sum_{j=1}^{r}(t_j - t_{j-1})/h = t_r/h$, if the Fermi energy lies in the r^{th} gap. For $t_a \leq 1$, only $q-1$ gaps open and the bands have a doubled multiplicity. Accordingly, one finds $\sigma_{xy} = 2\,t_r/h$ if the Fermi energy lies in the r^{th} gap.

When resolving the non-Abelian Harper equation (10) ($t_a = t_b = 1$) one finds the energy spectrum $\epsilon = \epsilon(\alpha_1, \alpha_2)$ which forms, in the $3D$ space (ϵ, α_1, α_2), a structure called the *Hofstadter moth* [15]. This $3D$ structure is characterized by a complex distribution of small gaps as shown in Fig. 1 which illustrates a typical cut through the "moth". If the spectrum $\epsilon = \epsilon(\alpha_1)$ is depicted in a particular plane $\alpha_2 = (p_1 + nq)/q$ with n integer, one recovers the Hofstadter butterfly [13,21–23]. In this spectrum, each gap is characterized by the two integers s_r and t_r given by the aforementioned Diophantine equation [23]. Consequently, it is possible to draw a phase diagram similar to Osadchy and Avron's [24] for this new non-Abelian case. This new phase diagram represents the integer values of the transverse transport coefficient σ_{xy} inside the infinitely many gaps of the butterfly, which are given by $\sigma_{xy} = 2\,t_r/h$. In order to check the robustness of this result, we depict in Fig. 2 the energy spectra for different values of $\alpha_1 - \alpha_2$ and anisotropy ratio t_b/t_a. The gauge potential is Abelian in Figs. 2(a) and (b), but non-Abelian in Figs. 2(c) and (d). We observe in this latter case that the spectrum thickens in particular around $\alpha_1 = 0.5$. Under the effect of the perturbations, some gaps disappear as in the Hofstadter model [25], but the principal gaps where the Hall-like conductivity is quantized remain.

An important issue is whether this result could be experimentally probed. A possible system described by the Hamiltonian (1) which includes the gauge potential (2) has already been proposed by Osterloh *et al.* [15] who put forward the possibility to submit cold fermionic atoms to such artificial non-Abelian gauge potentials. Each vertex of a $3D$ optical lattice contains an atom with degenerate Zeeman sublevels in the hyperfine ground state manifolds $\{|g\rangle_j, |e\rangle_j\}$ with $j = 1, 2$, so that the states we are dealing with are represented by the rows $\{|e\rangle| = (|e\rangle_1, |e\rangle_2)$ or $\{|g\rangle| = (|g\rangle_1, |g\rangle_2)$. We note that fermionic atoms with such properties, i.e. ^{40}K in states $|F = 9/2, m_F = 9/2, 7/2, ...\rangle$ and $|F = 7/2, m_F = -7/2, -5/2, ...\rangle$ as suggested in Ref. [15], are already studied experimentally in optical lattices [10–12]. Lasers are used in order to create the lattice, as well as the non-Abelian gauge potential through state-dependent control of hoppings which take place within every plane z = constant. In this framework, we consider a possible setup that exhibits a current in the y direction consisting of an accelerating optical lattice [14,26], with the (possibly gravitational) acceleration $\boldsymbol{a} = (0, a_y)$. Consequently, a perturbative term $\mathcal{H}_{\text{ext}}$ with the intersite energy difference $\Delta_y = M a_y l_y$ where M is the mass of the particles is added to the Hamiltonian [14,27]. In this case, the transverse transport coefficient σ_{xy} would give the relation between the external forcing and the transverse atomic current through the lattice. We notice that the current is electrically neutral so that this conductivity has the distinctive units of the inverse of Planck's constant and relates the energy difference Δ_y to a number of atoms per unit time in the current. Recently, it has been suggested that such atomic currents could be experimentally produced and measured in optical lattices. In particular, Ponomarev *et al.* [27] have investigated theoretically the possibility of such currents in a system of fermions coupled to bosons. The latter acts as a cold bath for the fermions, which induces a relaxation of the Bloch oscillations. If a transverse current is present in such systems under the conditions described in this work, we have shown that the transverse Hall-like conductivity should be quantized. In order to observe experimentally the effects emphasized in this work, it is crucial that the energy gaps represented in Fig. 2 remain sufficiently clear under thermal effects. The energy resolution required is approximatively 100 Hz which corresponds to temperatures of the order of 10 nK. These temperatures are realized in experiments involving ^{40}K in state $|F = 9/2, m_F = 9/2\rangle$ as mentionned in Refs. [10–12]. We finally point out that recent studies put forward the possibility to create and measure currents in atomic periodic systems, such as optical lattices, by connecting two reservoirs in order to create a chemical potential gradient through the system which generates a current [28]. Such devices, in which the trapping potential present in the optical lattice can be taylored, are envisaged in the new context of *atomtronics* [28] and could be considered for the detection of the quantized Hall-like conductivity in optical lattices submitted to Abelian or non-Abelian gauge potentials.

In this Letter, we have shown the existence of topological invariants in the transverse transport properties of cold atomic systems. It appears that fermionic atoms submitted to artificial Abelian or non-Abelian gauge potentials

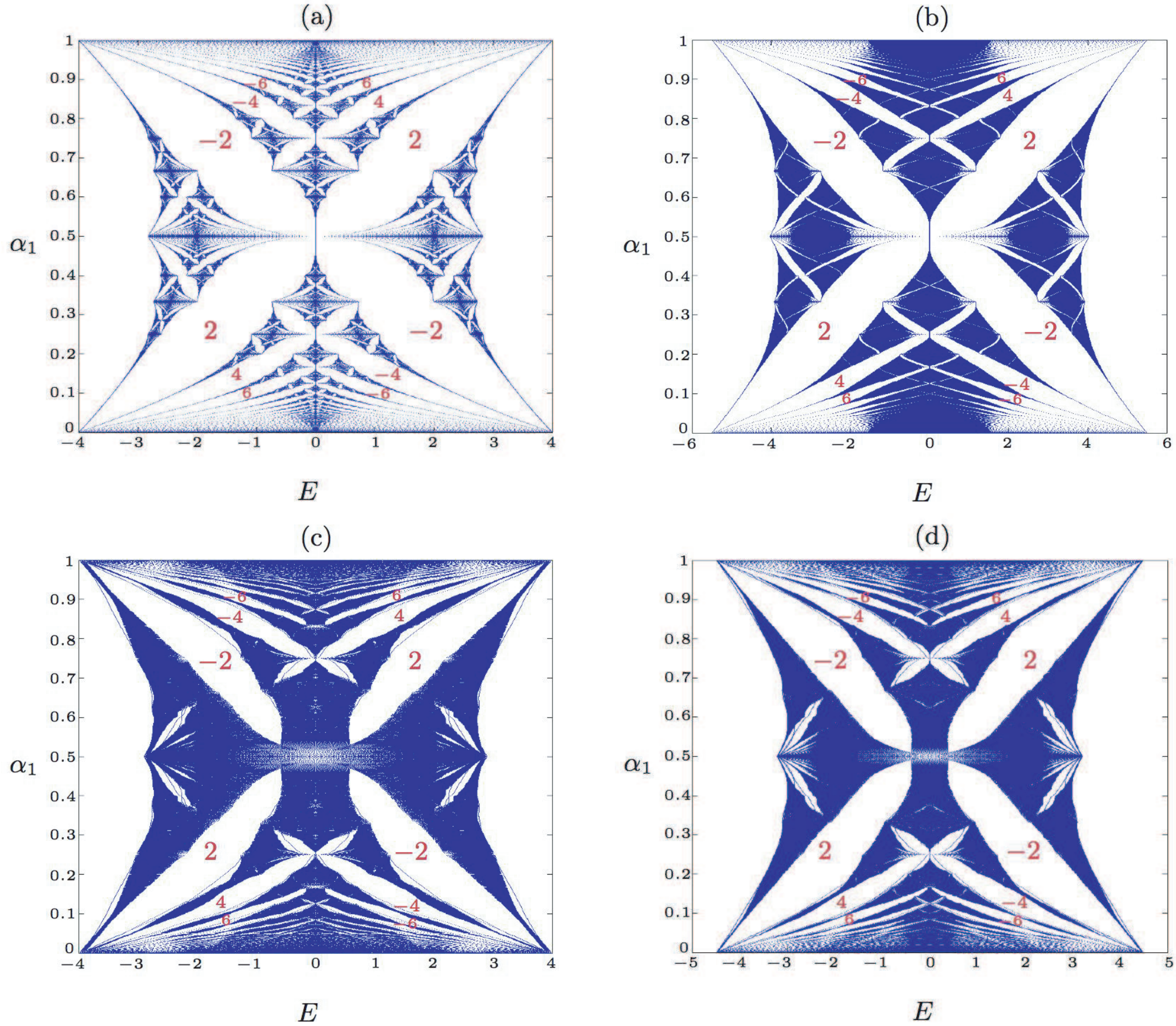

Fig. 2: Spectra for $\alpha_1 = p_1/701$ in the Abelian cases: (a) $\alpha_2 = p_1/701 + n$ with $\Lambda = 2t_b/t_a = 2$; (b) $\alpha_2 = p_1/701 + n$ with $\Lambda = 3.5$; and non-Abelian cases: (c) $\alpha_2 = (p_1 + 0.1)/701 + n$ with $\Lambda = 2$; (d) $\alpha_2 = (p_1 + 0.05)/701 + n$ with $\Lambda = 2.5$. The integers in the energy gaps give the corresponding values of $h\sigma_{yx}$.

and perturbed by an external forcing should exhibit an integer quantum Hall-like effect. We have here shown that the quantization of the transverse conductivity is robust if the gauge potential is set non-Abelian. The different values of the Hall-like conductivity are computed for a specific $U(2)$ system which depict a fractal phase diagram that should partly resist perturbations encountered in a realistic setup. Cold atom systems trapped in optical lattices present the versatility propitious to the study of the fractional quantum Hall effect (FQHE) [29]. These systems also constitute good candidates for the exploration of the quantization and fractal structures we have here emphasized. We can speculate that the quantization of the transverse conductivity of neutral fermionic cold atom currents could open a new window in metrology.

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