The aim of the course is an introduction to Lagrangian and Hamiltonian mechanics. The fundamental equations as well as standard applications of Newtonian mechanics are derived from variational principles.

From a mathematical perspective, the course and exercises (i) constitute an application of differential and integral calculus (primitives, inverse and implicit function theorem, graphs and functions); (ii) provide examples and explicit solutions of first and second order systems of differential equations; (iii) give an implicit introduction to differential manifolds and to realizations of Lie groups and algebras (change of coordinates, parametrization of a rotation, Galilean group, canonical transformations, one-parameter subgroups, infinitesimal transformations).

From a physical viewpoint, a good grasp of the material of the course is indispensable for a thorough understanding of the structure of quantum mechanics, both in its operatorial and path integral formulations. Furthermore, the discussion of Noether’s theorem and Galilean invariance is done in a way that allows for a direct generalization to field theories with Poincaré or conformal invariance.
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Chapter 1

Extremisation, constraints and Lagrange multipliers

The purpose of this chapter is to re-derive in the finite-dimensional case standard results on constrained extremisation so as to motivate and clarify analogous results needed in the functional case. It is based on chapters II.1, II.2, II.5 of [8]. The discussion on constraints and regularity conditions is taken from chapter 1 of [11].

1.1 Unconstrained extremisation

The problem consists of finding the extrema (=stationary points) of a function $F(x^1, \ldots, x^n)$ of $n$ variables $x^a$, $a = 1, \ldots, n$ in the interior of a domain. We write the variation of such a function under variations $\delta x^a$ of the variables $x^a$ as

$$\delta F = \delta x^a \frac{\partial F}{\partial x^a}. \tag{1.1}$$

Here and throughout, we use the summation convention over repeated indices, $\delta x^a \frac{\partial F}{\partial x^a} = \sum_{a=1}^{n} \delta x^a \frac{\partial F}{\partial x^a}$.

Since for a collection of functions $f_a(x^b)$,

$$f_a \delta x^a = 0, \forall \delta x^a \implies f_a = 0, \tag{1.2}$$

it follows that the extrema $\bar{x}^a$ are determined by the vanishing of the gradient of $F$,

$$\delta F = 0 \iff \frac{\partial F}{\partial x^a} \bigg|_{x^a = \bar{x}^a} = 0. \tag{1.3}$$

Recall further that the nature (maximum, minimum, saddle point) of each extremum is determined by the eigenvalues of the matrix

$$\frac{\partial^2 F}{\partial x^a \partial x^b} \bigg|_{\bar{x}}. \tag{1.4}$$

1.2 Constraints and regularity conditions

Very often, one is interested in a problem of extremisation, where the variables $x^a$ are subjected to constraints, that is to say where $r$ independent relations

$$G_m(x) = 0, \quad m = 1, \ldots, r, \tag{1.5}$$

5
are supposed to hold. Standard regularity on the functions $G_m$ are that the rank of the matrix $\frac{\partial G_m}{\partial x^a}$ is $r$ (locally, in a neighborhood of $G_m = 0$). This means that one may choose locally a new system of coordinates such that $G_m$ are $r$ of the new coordinates, $x^\alpha \leftrightarrow q^\alpha, G_m$, with $\alpha = 1, \ldots n - r$.

**Lemma 1.** A function $f$ that vanishes when the constraints hold can be written as a combination of the functions defining the constraints:

$$f|_{G_m=0} = 0 \iff f = g^m G_m$$

(1.6)

for some coefficients $g^m(x)$.

That the condition is sufficient ($\iff$) is obvious. That it is necessary ($\implies$) is shown in the new coordinate system, where $f(q^\alpha, G_m = 0) = 0$ and thus

$$f(q^\alpha, G_m) = \int_0^1 d\tau \frac{d}{d\tau} f(q^\alpha, \tau G_m) = G_m \int_0^1 d\tau \frac{\partial f}{\partial G_m}(q^\alpha, \tau G_m),$$

(1.7)

so that $g^m = \int_0^1 d\tau \frac{\partial f}{\partial G_m}(q^\alpha, \tau G_m)$. \qed

In the following, we will use the notation $f \approx 0$ for a function that vanishes when the constraints hold, so that the lemma becomes $f \approx 0 \iff f = g^m G_m$.

**Lemma 2.** Suppose that the constraints hold and that one restricts oneself to variations $\delta x^a$ tangent to the constraint surface,

$$G_m = 0, \quad \frac{\partial G_m}{\partial x^a} \delta x^a \approx 0.$$  

(1.8)

It follows that

$$f_a \delta x^a \approx 0, \quad \forall \delta x^a \text{ satisfying } (1.8) \iff f_a \approx \mu^m \frac{\partial G_m}{\partial x^a}.$$  

(1.9)

for some $\mu^m$.

NB: In this lemma, the use of $\approx$ means that these equalities in (1.8) and (1.9) are understood as holding when $G_m = 0$.

That the condition is sufficient ($\iff$) is again direct when contracting the expression for $f_a$ in the RHS of (1.9) with $\delta x^a$ and using the second of (1.8).

That the condition is necessary ($\implies$) follows by first subtracting from the LHS of (1.9) the combination $-\lambda^m \frac{\partial G_m}{\partial x^a} \delta x^a \approx 0$, with arbitrary $\lambda^m$, which gives

$$(f_a - \lambda^m \frac{\partial G_m}{\partial x^a}) \delta x^a \approx 0.$$  

(1.10)

Without loss of generality (by renaming if necessary some of the variables), we can assume that

$$\left|\frac{\partial G_m}{\partial x^a}\right| \neq 0, \text{ for } a = n - r + 1, \ldots, n.$$  

(1.11)

This implies that one may locally solve the constraints in terms of the $r$ last coordinates,

$$G_m = 0 \iff x^\Delta = X^\Delta(x^\alpha), \quad \Delta = n - r + 1, \ldots, n, \quad \alpha = 1, \ldots, n - r.$$  

(1.12)

In this case, one refers to the $x^\Delta$ as dependent and to the $x^\alpha$ as independent variables. This also implies that, locally, there exists an invertible matrix $M^{m\Delta}$ such that

$$M^{m\Delta} \frac{\partial G_m}{\partial x^\Delta} = \delta^m_{m'}, \quad \frac{\partial G_m}{\partial x^\Delta} M^m = \delta^\Delta.$$  

(1.13)
In terms of the various types of variables, (1.10) may be decomposed as
\[
(f_\Delta - \lambda^m \frac{\partial G_m}{\partial x^\Delta}) \delta x^\Delta + (f_\alpha - \lambda^m \frac{\partial G_m}{\partial x^{\alpha}}) \delta x^{\alpha} \approx 0. \tag{1.14}
\]
Since the \(\lambda^m\) are arbitrary, we are free to fix
\[
\lambda^m = M^m \Gamma f_\Gamma := \mu^m \implies f_\Delta = \mu^m \frac{\partial G_m}{\partial x^\Delta}, \tag{1.15}
\]
so that the first term (1.14) vanishes. We then remain with
\[
(f_\alpha - \mu^m \frac{\partial G_m}{\partial x^{\alpha}}) \delta x^{\alpha} \approx 0. \tag{1.16}
\]
Since the variables \(x^{\alpha}\) are unconstrained and the variations \(\delta x^{\alpha}\) independent, this implies as in (1.2) that
\[
f_\alpha \approx \mu^m \frac{\partial G_m}{\partial x^{\alpha}}. \tag{1.17}
\]

### 1.3 Constrained extremisation

Extremisation of a function \(F(x^{\alpha})\) under \(r\) constraints \(G_m = 0\) can then be done in two equivalent ways:

**Method 1:** The first is to solve the constraints as in (1.12) and to extremise the function
\[
F^G(x^{\alpha}) = F(x^{\alpha}, X^\Delta(x^{\beta})). \tag{1.18}
\]
In terms of the unconstrained variables \(x^{\alpha}\), we are back to an unconstrained extremisation problem, whose extrema are determined by
\[
\frac{\partial F^G}{\partial x^{\alpha}} = 0. \tag{1.19}
\]
In order to relate to the previous discussion and the second method to be explained below, note that, by using Lemma 1, \(G_m = 0 \iff x^\Delta - X^\Delta(x^{\alpha}) = 0\) implies
\[
x^\Delta - X^\Delta(x^{\alpha}) = g^\Delta m G_m, \tag{1.20}
\]
for \(g^\Delta m(x^{\alpha})\) invertible. Differentiation with respect to \(x^\Gamma\) yields
\[
\delta_l^\Delta = \frac{\partial g^\Delta m}{\partial x^l} G_m + g^\Delta m \frac{\partial G_m}{\partial x^l}. \tag{1.21}
\]
Hence, if the constraints are satisfied, i.e., if \(G_m = 0\), \(g^\Delta m\) is the inverse matrix to \(\frac{\partial G_m}{\partial x^\Delta}\), \(g^\Delta m \approx M^\Delta m\). On the one hand,
\[
\frac{\partial F^G}{\partial x^{\alpha}} = \frac{\partial F}{\partial x^{\alpha}} \bigg|_{x^\Delta = X^\Delta} + \frac{\partial F}{\partial x^\Gamma} \bigg|_{x^\Delta = X^\Delta} \frac{\partial X^\Gamma}{\partial x^{\alpha}}. \tag{1.22}
\]
On the other hand,
\[
\frac{\partial X^\Gamma}{\partial x^{\alpha}} = \frac{\partial (X^\Gamma - x^{\alpha})}{\partial x^{\alpha}} = -\frac{\partial g^\Gamma m}{\partial x^{\alpha}} G_m - g^\Gamma m \frac{\partial G_m}{\partial x^{\alpha}}, \tag{1.23}
\]
so that, if \(G_m = 0\), or equivalently, if one substitutes \(x^\Delta\) by \(X^\Delta\), (1.19) may be written as
\[
\frac{\partial F}{\partial x^{\alpha}} \approx \mu^m \frac{\partial G_m}{\partial x^{\alpha}}, \quad \mu^m \approx \frac{\partial F}{\partial x^\Delta} g^\Delta m. \tag{1.24}
\]
Furthermore, if \( G_m = 0 \), one may also write
\[
\frac{\partial F}{\partial x}\delta_x = \frac{\partial F}{\partial x}\delta_x \approx \frac{\partial F}{\partial x} \gamma_m \frac{\partial G_m}{\partial x} \approx \mu^m \frac{\partial G_m}{\partial x}. \tag{1.25}
\]
by using (1.21) and the definition of \( \mu^m \) in (1.24).

**Method 2:** Instead of restricting oneself to the space of independent variables \( x^a \), one extends the space of variables \( x^a \) by additional variables \( \lambda^m \), called Lagrange multipliers, and one considers the unconstrained extremisation of the function,
\[
F^\lambda(x^a, \lambda^m) = F(x^a) - \lambda^m G_m, \quad \delta F^\lambda = 0. \tag{1.26}
\]
By unconstrained extremisation, we understand that the variables \( x^a, \lambda^m \) are considered as unconstrained at the outset, with independent variations \( \delta x^a, \delta \lambda^m \). That this gives the same extrema can be seen as follows:
\[
0 = \delta F^\lambda = (\frac{\partial F}{\partial x^a} - \lambda^m \frac{\partial G_m}{\partial x^a}) \delta x^a - \delta \lambda^m G_m, \tag{1.27}
\]
implies
\[
\frac{\partial F}{\partial x^a} = \lambda^m \frac{\partial G_m}{\partial x^a}, \quad G_m = 0. \tag{1.28}
\]
A variation of the second equation then implies that \( \frac{\partial G_m}{\partial x^a} \delta x^a = 0 \), while contraction of the first equation with \( \delta x^a \) gives us (1.10) of the proof of lemma 2. We can then continue as in the proof of this lemma to conclude that \( \frac{\partial F}{\partial x^a} \approx \mu^m \frac{\partial G_m}{\partial x^a} \) for some \( \mu^m \), and thus that both methods define the same extrema.
Chapter 2
Constrained systems and d’Alembert principle

2.1 Holonomic constraints

Consider \( \mathcal{N} \) particles moving in space \( \mathbb{R}^3 \), with coordinates denoted by \( x_i^\alpha, i = 1, \ldots, \mathcal{N}, \alpha = 1, 2, 3. \) Alternatively, we can label them by \( x^a \), with \( a = 1, \ldots, 3\mathcal{N}. \) We assume that they are submitted to \( r \) constraints of the form

\[
G_m(x^a, t) = 0. \tag{2.1}
\]

Such constraints that only involve the positions but not the velocities of the particles are called \textit{holonomic}. As before, we assume regularity conditions: the rank of matrix \( \frac{\partial G_m}{\partial x^a} \) is maximal for all values of \( t. \) When the constraints do not depend explicitly on time, they are called \textit{scleronomic}, if they do, they are called \textit{rheonomic}.

During a motion, the particles are assumed to remain on the surface defined by the constraints. If this motion \( dx^a \) takes place during an infinitesimally small time interval \( dt \), this means that

\[
dG_m = dx^a \frac{\partial G_m}{\partial x^a} + dt \frac{\partial G_m}{\partial t} = 0. \tag{2.2}
\]

In other words, the constraints are constraints on the trajectories of the particles and gives rise by differentiation with respect to time to constraints on the velocities,

\[
\dot{x}^a \frac{\partial G_m}{\partial x^a} + \frac{\partial G_m}{\partial t} = 0, \tag{2.3}
\]

where, for a trajectory \( x^a(t) \), the notation \( \dot{x}^a = \frac{dx^a}{dt} \) and \( \ddot{x}^a = \frac{d^2 x^a}{dt^2} \) is used.

Consider two trajectories \( x^a(t) \) and \( \tilde{x}^a(t) \). What we will be interested in below is (instantaneous) \textit{virtual displacements} i.e., the difference of these trajectories \( \Delta x^a(t) = \tilde{x}^a(t) - x^a(t) \) at fixed times \( t. \) It follows that

\[
\Delta \frac{dx^a(t)}{dt} = \Delta \lim_{\epsilon \to 0} \frac{x^a(t + \epsilon) + x^a(t)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\tilde{x}^a(t + \epsilon) - x^a(t + \epsilon) - (\tilde{x}^a(t) - x^a(t))}{\epsilon} = \frac{d\tilde{x}^a(t)}{dt} - \frac{dx^a(t)}{dt} = \frac{d}{dt} \Delta x^a(t). \tag{2.4}
\]

More specifically we will be interested in infinitesimal displacements \( \delta x^a \) of this type that are compatible with the constraints,

\[
\delta x^a \frac{\partial G_m}{\partial x^a} \approx 0. \tag{2.5}
\]
Lemma 3. The elementary work $\delta W$ of the reaction forces due to the constraints vanishes for all infinitesimal virtual displacements that are compatible with the constraints if and only if the reaction forces are perpendicular to the constraints,

$$\delta W = R_\alpha \delta x^\alpha \approx 0 \iff R_\alpha \approx \mu^m \partial G_m \partial x^\alpha,$$

for some $\mu^m(x^\alpha, t)$.

Indeed, except for the interpretation in terms of work and forces as well as the geometric meaning of a linear combination of the gradient of constraints, this follows directly from the previous lemma. The notation $\delta W$ means that this elementary work is not necessarily the variation of a function.

In what follows, we limit ourselves (almost) exclusively to such constraints, i.e., to constraints whose reaction forces do not do any work during a virtual displacement. In particular this means that there can be no friction due to these constraints. Such constraints are called ideal.

2.2 d’Alembert’s theorem

The dynamics of the $N$ particles is governed by Newton’s equations,

$$m_{(i)} \ddot{x}_i^\alpha = F_i^\alpha + R_i^\alpha,$$

where the parenthesis around the index $i$ means that there is no summation, and where $F_i^\alpha$ are the components of the applied forces acting on the $i$th particles, while $R_i^\alpha$ are the reaction forces due to the constraints, assumed to be ideal. The latter keep the particles confined to the constraint surface.

Theorem 1 (d’Alembert). Newton’s equation for a system of point particles subjected to ideal constraints are equivalent to the condition that the elementary work of the sum of the applied forces and of the forces of inertia ($-m_{(i)} \ddot{x}_i^\alpha$) vanishes for all infinitesimal virtual displacements compatible with the constraints,

$$\left(F_i^\alpha - m_{(i)} \ddot{x}_i^\alpha\right) \delta x_i^\alpha \approx 0.$$

The theorem states that the Newton’s equations (2.7), together with the fact that the reaction forces associated to ideal constraints are of the form $R_i^\alpha \approx \mu^m \partial G_m \partial x_i^\alpha$ on account of equation (2.6) of Lemma 3 are equivalent to (2.8) when the constraints (2.1) hold, and when limiting oneself to compatible infinitesimal virtual displacements, (2.5). That the condition is sufficient ($\Longrightarrow$) follows by contracting (2.7) with $\delta x_i^\alpha$ and using (2.6) together with (2.5) to conclude that (2.8) holds. That the condition is necessary ($\iff$) follows because (2.1), (2.5) together with (2.8) imply, as in lemma 3, that $F_i^\alpha - m_{(i)} \ddot{x}_i^\alpha \approx -\mu^m \partial G_m \partial x_i^\alpha$, for some $\mu^m$, which are Newton’s equation (2.7), together with (2.6).

2.3 Non-holonomic constraints

Non-holonomic constraints are constraints on the velocities of point particles as in (2.3), but that do not originate from holonomic constraints, i.e., from constraints on positions, through differentiation with respect to time.
As an example, consider a vertical disk or coin rolling without sliding on a horizontal plane, \( z = 0 \) in standard Cartesian coordinates. Let \((x_C, y_C, z_C)\) the coordinates of the center of the disk. Only \(x_C, y_C\) are relevant since \(z_C = a\), with \(a\) the radius of the disk. Let \(\phi\) be the angle of rotation of the disk starting from a fixed initial direction and \(\theta\) the angle made by \(\vec{I}_x\) and the axes of rotation of the disk. Since the velocity \(\vec{v}\) of the center of the disk is normal to this axes, and lies in a plane of constant \(z\), it follows that the angle between \(\vec{1}_x\) and \(\vec{v}\) is \(\theta - \frac{\pi}{2}\), so that

\[
x_C = v \cos(\theta - \frac{\pi}{2}) = v \sin \theta, \quad y_C = v \sin(\theta - \frac{\pi}{2}) = -v \cos \theta.
\]  

(2.9)

Futhermore, the velocity of a point on the edge of the disk is \(a \dot{\phi}\). This is the module of the velocity of the point of contact of the disk with the plane, and since there is no sliding this is also the velocity of the center of the disk, \(v = a \dot{\phi}\). By injecting into the previous relations, we thus get the following relations between velocities for this motion

\[
\begin{align*}
\dot{x}_C - a \sin \theta \dot{\phi} &= 0, \\
\dot{y}_C + a \cos \theta \dot{\phi} &= 0,
\end{align*}
\]  

(2.10)

and the question is whether these constraints on velocities come from constraints on positions by differentiation with respect to time.

More generally, consider \(r\) constraints that are at most linear in velocities \(\dot{x}^a\),

\[
K^m_a \dot{x}^a + K^m_0 = 0,
\]  

(2.11)

where \(K^m_a(x^b, t), K^m_0(x^b, t)\) are functions that depend on the coordinates \(x^a, a = 1, \ldots, n\) (which may or may not simply be the positions \(x_1^a\) in Cartesian space of the \(N\) particles), and possibly an explicit dependence on time \(t\). The question then is whether the relation on the velocities (2.11), or any equivalent relation on the velocities

\[
\Lambda^m_{m'}(K^m_a \dot{x}^a + K^m_0) = 0,
\]  

(2.12)

with \(\Lambda^m_{m'}(x^b, t)\) an invertible matrix, can be written as

\[
\Lambda^m_{m'}(K^m_a \dot{x}^a + K^m_0) = \partial G^m_{\dot{x}^a} \dot{x}^a + \frac{\partial G^m}{\partial t}.
\]  

(2.13)
for \( r \) functions \( G^m(x^a, t) \). Since the velocities are arbitrary, this relation can be decomposed as

\[
\begin{align*}
\frac{\partial G^m}{\partial x^a} &= \Lambda^m_{m'} K^m'_{A'} , \\
\frac{\partial G^m}{\partial t} &= \Lambda^m_{m'} K^m'_{0} ,
\end{align*}
\]  

(2.14)

In order to answer this question, it is useful to introduce \( \Lambda^m_{m'} \) consequence of the following theorem.

Necessary conditions for \( n+1 \) equations can be written in a unified way as

\[
\frac{\partial G^m}{\partial x^A} = \Lambda^m_{m'} K^m'_{A'} .
\]  

(2.15)

Necessary conditions\(^1\) for the existence of such functions \( G^m \) is that mixed second derivatives of the right hand sides vanish,

\[
\frac{\partial^2 G^m}{\partial x^A \partial x^B} = \frac{\partial^2 G^m}{\partial x^B \partial x^A} \implies \frac{\partial}{\partial x^A} (\Lambda^m_{m'} K^m'_{B}) = \frac{\partial}{\partial x^B} (\Lambda^m_{m'} K^m'_{A}) .
\]  

(2.16)

This is however not the condition that we need because it still contains the unknown functions \( \Lambda^m_{m'} \). What we need is a condition on the functions \( K^m_{A} \) alone. A standard technical assumption on these functions, that we assume to hold is that they are linearly independent in the following sense: if there exist some functions \( \lambda_m(x^B) \) such that \( \lambda_m K^m_{A} = 0 \) then necessarily \( \lambda_m = 0 \). The condition we need is then a consequence of the following theorem.

**Theorem 2** (Frobenius). For linearly independent functions \( K^m_{A} \), the set of conditions

\[
\left( \frac{\partial}{\partial x^A} K^m_{A_2} \right) K^1_{A_3} \ldots K^r_{A_{r+2}} = 0 ,
\]  

(2.17)

is necessary and sufficient for the existence of functions \( \Lambda^m_{m'}(x^B) \) (invertible) and \( G^m(x^B) \) such that

\[
\frac{\partial G^m}{\partial x^A} = \Lambda^m_{m'} K^m'_{A} .
\]  

(2.18)

In equation (2.17), the square bracket denotes complete antisymmetrization of the indices \( A_1, \ldots A_{r+2} \), divided by \( (r+2)! \), the number of permutations of \( r+2 \) objects. More explicitly,

\[
\left( \frac{\partial}{\partial x^A_1} K^m_{A_2} \right) K^1_{A_3} \ldots K^r_{A_{r+2}} = \frac{1}{(r+2)!} \sum_{\sigma \in P_{r+2}} (1)^{\sigma} (\frac{\partial}{\partial x^{A_{\sigma(1)}}} K^m_{A_{\sigma(2)}}) K^1_{A_{\sigma(3)}} \ldots K^r_{A_{\sigma(r+2)}} ,
\]  

(2.19)

where \( P_{r+2} \) denotes a permutation of \( r+2 \) elements and \( |\sigma| = 0 \) if the permutation is even, while \( |\sigma| = 1 \) if the permutation is odd.

Let us prove that these conditions are indeed necessary (\( \Leftarrow \)). We start from

\[
0 = \left( \frac{\partial}{\partial x^{A_1}} \frac{\partial G^m}{\partial x^{A_2}} \right) K^1_{A_3} \ldots K^r_{A_{r+2}} ,
\]  

(2.20)

which holds because the second order derivatives \( \frac{\partial^2 G^m}{\partial x^A \partial x^B} \) are symmetric in \( A_1, A_2 \), so that, when completely antisymmetrizing in the right hand side of equation (2.20), one finds always 0. Since we assume that (2.18) is true, we find by substitution

\[
0 = \left( \frac{\partial}{\partial x^{A_1}} (\Lambda^m_{m'} K^m'_{A_2}) \right) K^1_{A_3} \ldots K^r_{A_{r+2}} = \left( \frac{\partial}{\partial x^{A_1}} \Lambda^m_{m'} K^m'_{A_2} K^1_{A_3} \ldots K^r_{A_{r+2}} + \Lambda^m_{m'} (\frac{\partial}{\partial x^{A_1}} K^m'_{A_2}) K^1_{A_3} \ldots K^r_{A_{r+2}} \right)
\]  

(2.21)

\(^1\)It can in fact be shown, as done later in the course, that these conditions are also locally sufficient.
The first term on the last line vanishes because the antisymmetry in $A_2, \ldots A_{r+2}$ implies the antisymmetry in $m', 1, \ldots, r$, but $m'$ takes a value between 1 and $r$ so that one of the values arises twice and a completely antisymmetric expression then vanishes. Equation (2.17) then follows because $\Lambda^m_{m'}$ is invertible.

The proof that the condition is sufficient ($\Rightarrow$) can be found in [3] or [10].

In the particular case of the disk, we have

\[ x^A = (t, x_C, y_C, \phi, \theta), \]  

(2.22)

and we have two constraints, $r = 2$. More explicitly,

\[
\begin{align*}
K^1_0 &= 0, & K^1_1 &= 1, & K^1_2 &= 0, & K^1_3 &= -a \sin \theta = -a \sin x^4, & K^1_4 &= 0, \\
K^2_0 &= 0, & K^2_1 &= 0, & K^2_2 &= 1, & K^2_3 &= a \cos \theta = a \cos x^4, & K^2_4 &= 0.
\end{align*}
\]

(2.23)

Let us show that the constraints are non-holonomic, i.e., that there is at least one of the conditions in (2.17) that is not satisfied. If we take $m = 1$, they become

\[
\left( \frac{\partial}{\partial x^A} K^1_{A_1} K^1_{A_2} K^2_{A_3} K^2_{A_4} \right) = 0.
\]

(2.24)

Choosing $A_1 = 4, A_2 = 3, A_3 = 1, A_4 = 2$, one can check that only the first term in the complete antisymmetrization is non-vanishing, so that the result of the left hand side is given by $\frac{1}{4!} (-a \cos x^4)$ which does not vanish.

### 2.4 d’Alembert’s principle

In the proof of d’Alembert theorem, nothing crucially depended on the fact that the constraints were holonomic. It can thus be extended to non-holonomic ones (by the replacement $\frac{\partial G^m}{\partial x^a} \rightarrow K^m_a$ in the proof). Suppose then that there are $r$ non holonomic constraints

\[ K^m_a \dot{x}^a + K^m_0 = 0, \]

(2.25)

and consider infinitesimal virtual displacements $\delta x^a$ that are compatible,

\[ K^m_a \delta x^a = 0. \]

(2.26)

d’Alembert’s principle then states that Newton’s equation

\[ m(a) \ddot{x}^a = F^a + R^a, \]

(2.27)

together with the condition that the forces of reaction are of the form

\[ R_a \approx \lambda_m K^m_a, \]

(2.28)

are equivalent to the condition that the sum of the applied forces and of the forces of inertia produce no work during any compatible infinitesimal virtual displacement $\delta x^a$,

\[ (F^a - m(a) \ddot{x}^a) \delta x^a \approx 0. \]

(2.29)

Systems covered by d’Alembert’s theorem and principle are

- systems of point particles that move on polished surfaces that are fixed or moving (holonomic constraints),
• motions of solids with one point having a prescribed trajectory
• sliding of solids on polished surfaces
• rolling without sliding of solids on rough surfaces
• fixed links between solids

What is not covered by the theorem and principle are constraints that involve friction or constraints that are expressed by inequalities.
Consider trajectories $x^a = x^a(t)$ that pass through a fixed point at some initial time $t_i$ and another fixed point at a final time $t_f$, $x^a(t_i) = x^a_i$, $x^a(t_f) = x^a_f$. A natural trajectory is a trajectory that satisfies Newton’s equations together with the constraints. Virtual trajectories
Chapter 4

Lagrangian mechanics
Chapter 5

Applications of the Lagrangian formalism
Chapter 6

Covariance of the Lagrangian formalism

6.1 Change of dependent variables

Consider a non-singular change of variables of the generalized Lagrange coordinates,

\[ q^\alpha = q^\alpha(q^{\beta'}, t), \quad \det \left( \frac{\partial q^\alpha}{\partial q^{\beta'}} \right) \neq 0, \quad (6.1) \]

with inverse transformation denoted by \( q^{\beta'} = q^{\beta'}(q^\alpha, t) \). The Lagrangian in the new coordinates is

\[ L'(q', \dot{q}', t) = L(q(q', t), \frac{d}{dt}q(q', t), t) := L | \quad (6.2) \]

where the bar means that one has done the change of variables inside the function, with \( \frac{d}{dt}q^\alpha(q^{\beta'}, t) = \frac{\partial q^\alpha}{\partial q^{\beta'}} \dot{q}^\alpha + \frac{\partial q^\alpha}{\partial t} \).

By direct computation one finds

\[ \frac{\delta L'}{\delta q^{\beta'}} = \frac{\partial L'}{\partial q^{\beta'}} - \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}^{\beta'}} = \frac{\partial L}{\partial q^\alpha} \frac{\partial q^\alpha}{\partial q^{\beta'}} + \frac{\partial L}{\partial q^\alpha} \left( \frac{\partial^2 q^\alpha}{\partial q^{\beta'} \partial \dot{q}^\gamma} \dot{q}^\gamma + \frac{\partial^2 q^\alpha}{\partial q^{\beta'} \partial t} \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} \frac{\partial q^\alpha}{\partial q^{\beta'}} \]

\[ = \frac{\partial q^\alpha}{\partial q^{\beta'}} \left[ \frac{\partial L}{\partial q^\alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} \right]. \quad (6.3) \]

Indeed, when \( \frac{d}{dt} \) in the last term of the first line hits \( \frac{\partial q^\alpha}{\partial q^{\beta'}} \), the previous term in the first line is cancelled. As a consequence, out of the second and third term of the first line, one only remains with \( \frac{d}{dt} \) that hits \( \frac{\partial L}{\partial \dot{q}^\alpha} \).

Furthermore,

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\alpha} \right) = \frac{\partial^2 L}{\partial q^\beta \partial q^\alpha} \frac{d}{dt} q^\beta(q', t) + \frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\alpha} \frac{d^2}{dt^2} q^\beta(q', t) + \frac{\partial^2 L}{\partial t \partial \dot{q}^\alpha} \frac{d}{dt} q^\alpha | = \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} \right), \quad (6.4) \]

where \( f(q, \dot{q}, \ddot{q}, t) | = f(q(q', t), \frac{d}{dt}q(q', t), \frac{d^2}{dt^2}q(q', t), t) \). Hence,

\[ \left[ \frac{\delta L'}{\delta q^{\beta'}} \right] = \left[ \frac{\partial q^\alpha}{\partial q^{\beta'}} \left( \frac{\delta L}{\delta q^\alpha} \right) \right]. \quad (6.5) \]

This equation captures the covariance of Euler-Lagrange equations under changes of Lagrange coordinates. The meaning is as follows. Suppose that \( \bar{q}^\alpha(t) \) is a natural trajectory, i.e., a solution to the Euler-Lagrange equations of motion,

\[ \frac{\delta L}{\delta q^\alpha} (\bar{q}^\alpha(t), \frac{d}{dt} \bar{q}^\alpha(t), t) = 0. \quad (6.6) \]
If one changes variables, \(q'^\alpha = q'^\beta(q^\alpha, t)\), with inverse given in (6.1), it follows that \(\bar{q}'(t) = q'^\beta(q^\alpha(t), t)\) is a solution for the variational principle defined by \(L'(q', \dot{q'}, t)\). Indeed, (6.5) is equivalent to

\[
\frac{\partial q'^\beta}{\partial q^\alpha} \frac{\delta L'}{\delta q'^\beta} \left( \dot{q}'(q, t), \frac{d}{dt} q'(q, t), \frac{d^2}{dt^2} q'(q, t), t \right) = \frac{\delta L}{\delta q^\alpha}(q, \dot{q}, \ddot{q}, t). \tag{6.7}
\]

When evaluating at \(\bar{q}'(t)\), the RHS vanishes. The result that \(\frac{\delta L'}{\delta q'^\beta}\) evaluated at \(\bar{q}'(t)\) vanishes then follows from invertibility of \(\frac{\partial q'^\beta}{\partial q^\alpha}\).

### 6.2 Change of the independent variable

The action is a functional, it produces a number if one evaluates it on a trajectory,

\[
S[q] = \int_{t_i}^{t_f} dt \, L\left( q(t), \frac{d}{dt} q(t) \right) \in \mathbb{R}. \tag{6.8}
\]

Under an arbitrary reparametrisation of time, \(t = t(t')\), with inverse denoted by \(t' = t'(t)\), one finds the same number provided that

\[
S[q] = \int_{t'_{i}}^{t'_{f}} dt' \, L\left( q(t'(t')), \frac{dt}{dt'} q(t'(t')), (t'(t')) \right). \tag{6.9}
\]

This suggests that the Euler-Lagrange equations are covariant under \(t = t(t')\) and \(q'^\alpha(t') = q^\alpha(t(t'))\) provided that

\[
L'(q', \frac{dq'}{dt'}, t') = \frac{dt}{dt'} L\left( q(t'(t')), \frac{dt'}{dt} \frac{d}{dt} q(t'(t')), (t'(t')) \right), \tag{6.10}
\]

with

\[
\frac{dq'^\alpha}{dt'} = \frac{dq^\alpha}{dt} \bigg|_{t=t'(t')} \frac{dt}{dt'} = \dot{q}^\alpha \bigg|_{t=t'(t')} \frac{dt}{dt'}. \tag{6.11}
\]

Consider first a simple example. If \(L = \frac{1}{2} m q^2\), \(\delta L = 0 \iff \ddot{q} = 0 \iff q(t) = vt + c\) with \(v, c \in \mathbb{R}\). Let \(t, t' > 0\) and take \(t = \sqrt{t'} \iff t' = t^2\).

It follows that \(\frac{dt'}{dt} = 2t = 2\sqrt{t'}\), and the solution becomes \(q'(t') = v\sqrt{t'} + c\). The new Lagrangian is

\[
L' = \frac{dt}{dt'} \frac{1}{2} m \left( \frac{dt'}{dt} \frac{dq'}{dt'} \right)^2 = m\sqrt{t'}(\frac{dq'}{dt'})^2. \tag{6.12}
\]

The new equations of motion are

\[
\frac{\delta L'}{\delta q'} = \frac{\partial L'}{\partial q'} - \frac{d}{dt'} \frac{\partial L'}{\partial \dot{q}'} = -\frac{d}{dt'} \left( 2m\sqrt{t'} \frac{dq'}{dt'} \right) = -m \frac{1}{\sqrt{t'}} \frac{dq'}{dt'} - 2m\sqrt{t'} \frac{d^2 q'}{dt'^2} = 0, \tag{6.13}
\]

which is equivalent to

\[
\frac{d^2 q'}{dt'^2} = -\frac{1}{2t'} \frac{dq'}{dt'}. \tag{6.14}
\]

This equation holds since

\[
\frac{dq'}{dt'} = \frac{1}{2} v \sqrt{t'}, \quad \frac{d^2 q'}{dt'^2} = -\frac{1}{4} v \frac{1}{( \sqrt{t'})^3}. \tag{6.15}
\]
The proof in the general case proceeds as follows.

\[
\frac{\delta L'}{\delta q'^{\alpha}} = \frac{\partial L'}{\partial q'^{\alpha}} - \frac{d}{dt'} \frac{\partial L'}{\partial \dot{q}'^{\alpha}} = \frac{dt}{dt'} \frac{\partial L}{\partial q^{\alpha}} \left| \partial \frac{\partial L}{\partial \dot{q}^{\alpha}} \right| - \frac{d}{dt'} \left( \frac{df}{dt'} \frac{\partial L}{\partial \dot{q}^{\alpha}} \right) = \frac{dt}{dt'} \left[ \frac{\partial L}{\partial \dot{q}^{\alpha}} \right] - \frac{d}{dt'} \left( \frac{\partial L}{\partial \dot{q}^{\alpha}} \right) \right] = \frac{dt}{dt'} \left[ \frac{\delta L}{\delta q^{\alpha}} \right],
\]

where the last equality holds provided one can show that

\[
\frac{d}{dt'} \left( \frac{\partial L}{\partial \dot{q}^{\alpha}} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^{\alpha}} \right).
\]

As in the preceding section, \(\frac{\delta L'}{\delta q'^{\alpha}} = \frac{d}{dt'} \left( \frac{\delta L}{\delta q^{\alpha}} \right)\) means that, if \(q^{\alpha}(t)\) is a natural trajectory for \(L(q, \dot{q}, t)\), then so is \(q'^{\alpha}(t') = q^{\alpha}(t(t'))\) for \(L'(q', \dot{q}', t')\)

In order to show (6.17), we have

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^{\alpha}} \right) = \frac{\partial^2 L}{\partial \dot{q}^{\alpha} \partial \dot{q}^{\beta}} |_{-} + \frac{\partial^2 L}{\partial \dot{q}^{\alpha} \partial q^{\beta}} |_{-} + \frac{\partial^2 L}{\partial \dot{q}^{\alpha} \partial q^{\beta}} |_{-} + \frac{\partial^2 L}{\partial \dot{q}^{\alpha} \partial \dot{q}^{\beta}} |_{-}.
\]

When taking into account (6.11), the first terms on the RHS do agree, since so do the last terms, we only have to show that middle terms do as well, which is the case if

\[
\dot{q}^{\beta} = \frac{d^2 t'}{dt^2} \frac{d q^{\beta}}{dt'} + \left( \frac{dt'}{dt} \right)^2 \frac{d^2 q^{\beta}}{dt'^2}.
\]

Differentiating (6.11) with respect to \(t'\) gives

\[
\frac{d^2 q^{\beta}}{dt'^2} = \frac{d}{dt'} \left( \frac{dt}{dt'} \dot{q}^{\beta} \right) = \frac{d^2 t}{dt'^2} \dot{q}^{\beta} + \left( \frac{dt'}{dt} \right)^2 \frac{d^2 \dot{q}^{\beta}}{dt'^2},
\]

which implies that

\[
\dot{q}^{\beta} = \left( \frac{dt'}{dt} \right)^2 \frac{d^2 \dot{q}^{\beta}}{dt'^2} - \left( \frac{dt'}{dt} \right)^2 \frac{d^2 \dot{q}^{\beta}}{dt'^2}.
\]

This gives the result when using (6.11) and

\[
0 = \frac{d}{dt} \left( \frac{dt'}{dt} \right) = \frac{d^2 t'}{dt^2} \dot{q}^{\beta} + \left( \frac{dt'}{dt} \right)^2 \frac{d^2 t'}{dt'^2} \Rightarrow \frac{d^2 t'}{dt'^2} = -\left( \frac{dt'}{dt} \right)^3 \frac{d^2 t'}{dt'^2}.
\]

### 6.3 Combined change of variables

When combining the results of the two previous sections, Euler-Lagrange equations are covariant under

\[
\left\{ \begin{array}{l}
q^{\alpha} = q^{\alpha}(q^{\beta}, t') \\
t = t(t')
\end{array} \right.
\]

with inverse

\[
\left\{ \begin{array}{l}
q'^{\alpha} = q'^{\alpha}(q^{\beta}, t) \\
t' = t'(t)
\end{array} \right.
\]
provided that

$$L'(q^{\alpha'}(t'), \frac{dq^{\alpha'}}{dt'}(t'), t') = \frac{dt}{dt'} L\left(q^{\alpha}(q^{\alpha'}(t'), t'), \frac{dt'}{dt} \frac{d}{dt} q^{\alpha}(q^{\alpha'}(t'), t'), t(t')\right),$$  \hspace{1cm} (6.26)$$

with $$\frac{d}{dt} q^{\alpha'}(q^{\alpha'}(t'), t') = \frac{\partial q^{\alpha'}}{\partial q^{\gamma'}} \frac{dq^{\gamma'}}{dt'} + \frac{\partial q^{\alpha'}}{\partial t'}.$$ Indeed, showing that

$$\frac{\delta L'}{\delta q^{\alpha'}} = \frac{dt}{dt'} \left( \frac{\delta L}{\delta q^{\alpha}} \right) \frac{\partial q^{\alpha}}{\partial q^{\alpha'}}.$$  \hspace{1cm} (6.27)$$
can now be done in a straightforward way by combining the intermediate steps of the proofs of the previous two sections.
Chapter 7

Noether’s theorem and Galilean invariance

7.1 Noether’s first theorem

A first integral or conservation law is a function \( K(q, \dot{q}, t) \) that is constant on all natural trajectories,
\[
dK \bigg| \dot{q} = 0, \quad \text{with} \quad \delta L \bigg| \dot{q} = 0.
\] (7.1)

An infinitesimal symmetry is a virtual variation that leaves the Lagrangian invariant up to a total derivative,
\[
\delta_Q q^\alpha = Q^\alpha(q, \dot{q}, t) \text{ such that } \delta_Q L = \frac{dF(q, \dot{q}, t)}{dt}
\] (7.2)

Noether’s first theorem states that:

Every infinitesimal symmetry gives rise to a conservation law.

Indeed,
\[
\delta_Q L = \delta_Q q^\alpha \frac{\partial L}{\partial q^\alpha} + \delta_Q \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} = Q^\alpha \frac{\partial L}{\partial q^\alpha} + \frac{dQ^\alpha}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} = \frac{dF}{dt}
\] (7.3)

where the second equality follows because virtual variations commute with the total derivative, while the last equality follows by the assumption that the variation is a symmetry. Using
\[
\frac{dQ^\alpha}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} = \frac{d}{dt} \left( Q^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} \right) - Q^\alpha \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha}
\] (7.4)

the latter combines with the first term to produce the Euler-Lagrange derivatives, while the former combines with the total derivative on the RHS,
\[
Q^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} = \frac{d}{dt} \left( F - \frac{\partial L}{\partial \dot{q}^\alpha} Q^\alpha \right).
\] (7.5)

Since the LHS vanishes when evaluated at any natural trajectory, it follows that associated the conservation law is given by
\[
K = F - \frac{\partial L}{\partial \dot{q}^\alpha} Q^\alpha.
\] (7.6)

Remark: Neither the theorem nor the expression depends on the choice of representative for the action: if \( L' = L + \frac{df(q,t)}{dt} \), it follows that \( \delta_Q \) is also an infinitesimal symmetry of \( L' \),
\[
\delta L' = \delta L + \frac{d}{dt} (\delta_Q f) = \frac{d}{dt} (F + \delta_Q f),
\] (7.7)

with associated conservation law
\[
K' = F + \frac{\partial f}{\partial q^\alpha} Q^\alpha - (\frac{\partial L}{\partial q^\alpha} + \frac{\partial f}{\partial q^\alpha}) Q^\alpha = K.
\] (7.8)

\(^{1}\)This means that the variation leaves the action functional invariant if one considers only trajectories that vanish at the end-points and restricts \( f \) to satisfy \( f(0, t) = 0 \).
7.2 Applications

7.2.1 Explicit time independence and conservation of energy

Consider a Lagrangian that does not depend explicitly on time, \( L = L(q, \dot{q}) \). It follows that
\[
\frac{dL}{dt} = \frac{\partial L}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \ddot{q}^\alpha = \frac{\delta L}{\delta q^\alpha} \dot{q}^\alpha + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha \right),
\]
(7.9)
In other words,
\[
K = \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha - L,
\]
(7.10)
is the conservation law associated to the infinitesimal symmetry
\[
\delta Q q^\alpha = -\dot{q}^\alpha.
\]
(7.11)
In particular, if
\[
L = \frac{1}{2} g_{\alpha\beta}(q) + A_\alpha(q) \dot{q}^\alpha - V(q),
\]
(7.12)
the associated conservation law
\[
K = g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + A_\alpha \dot{q}^\alpha - \frac{1}{2} g_{\alpha\beta} - A_\alpha \dot{q}^\alpha + V = E
\]
(7.13)
corresponds to the mechanical energy.

7.2.2 Cyclic variables and conservation of canonical momentum

Suppose that a given variable \( q^\alpha \) is “cyclic” which means that \( L \) does not depend on \( q^\alpha \) but may depend on the associated velocity \( \dot{q}^\alpha \). To fix ideas, suppose that the first variable \( q^1 \) is cyclic,
\[
\frac{\partial L}{\partial q^1} = 0.
\]
It follows that
\[
\delta Q q^1 = -\dot{q}^1
\]
(7.14)
Since \( F = 0 \), the associated conservation law is
\[
K = -\frac{\partial L}{\partial \dot{q}^1} (-\dot{q}^1) = \frac{\partial L}{\partial \dot{q}^1},
\]
(7.15)
which is the canonical momentum associated to \( q^1 \).

7.2.3 One-parameter groups of transformations

Suppose that the Lagrangian is invariant under a set of invertible transformations \( q^\alpha' = q^\alpha'(q^\beta, t; s) \) that depend continuously on a parameters \( s \), and reduce to the identity at \( s = 0 \), \( q^\alpha'(q^\beta, t; 0) = q^\alpha \). By invariance we mean that the transformed Lagrangian in terms of the new variables is equal to the old Lagrangian in terms of the same variables,
\[
L'(q', \dot{q}', t) = L(q(q', t; s), \frac{d}{dt} q(q', t; s), t) = L(q', \dot{q}', t),
\]
(7.16)
where the first equality merely expresses how the new Lagrangian in terms of the new coordinates is obtained in terms of the old one, while the second equality expresses the invariance of the Lagrangian. When using the inverse transformation, the latter is equivalent to
\[
L(q(q, t; s), \frac{d}{dt} q(q, t; s), t) = L(q, \dot{q}, t).
\]
(7.17)
In this case, the transformation
\[
\delta R^\alpha = R^\alpha(q, t) := \frac{\partial q^\alpha'}{\partial s} \big|_{s=0},
\]
(7.18)
defines an infinitesimal symmetry.

Indeed, \(q^\alpha' = q^\alpha + s R^\alpha(q, t) + O(s^2)\), while \(\frac{dq^\alpha'}{dt} = \dot{q}^\alpha + s \frac{dR^\alpha}{dt} + O(s^2)\). If \(\delta R^\alpha = \frac{\partial}{\partial s} \frac{dq^\alpha'}{dt} \big|_{s=0}\), \(\delta R^\alpha = \frac{dR^\alpha}{dt}\), as it should for a virtual variation. In these terms, the invariance condition (7.17) becomes
\[
L(q + s R + O(s^2), \dot{q} + s \frac{dR}{dt} + O(s^2), t) = L(q, \dot{q}, t), \quad \forall s.
\]
(7.19)
In particular, for \(s = 0\), this is an identity, while the terms linear in \(s\) on the LHS have to vanish. The latter can be obtained by taking \(\frac{\partial}{\partial s}\) at \(s = 0\), and yield
\[
R^\alpha \frac{\partial L}{\partial q^\alpha} + \frac{dR^\alpha}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} = 0,
\]
(7.20)
which can also be written as \(\delta_R L = 0\). According to Noether’s theorem, the associated conservation law is given by
\[
K = -\frac{\partial L}{\partial \dot{q}^\alpha} \delta_R q^\alpha.
\]
(7.21)
Note that a cyclic variable can be interpreted as a particular case. Indeed, the transformation that leaves the Lagrangian invariant is given by \(q^\alpha' = q^\alpha - s \delta^\alpha_1\).

Another particular case is a system of particles in three-dimensional Euclidean space, with a Lagrangian of the type
\[
L = T - V, \quad T = \frac{1}{2} \sum_{i=1}^{N} m_{(i)} \dot{x}_{\alpha(i)} \dot{x}_{\alpha(i)}; \quad V = V(r_{ij}).
\]
(7.22)
In this expression, the summation in the kinetic term over the different particles has been written out explicitly, while the summation over the 3 components of the position of a single particle is implicit through the summation convention. Note also that we have defined \(x_{\alpha(i)} = \delta_{\alpha\beta} x_{\beta(i)}\), and that the potential \(V(r_{ij})\) with \(i < j\) is assumed to depend only on the relative distances between the different particles,
\[
r_{ij} = \sqrt{(x_{\alpha(i)} - x_{\alpha(j)})(x_{\alpha(i)} - x_{\alpha(j)})}.
\]
(7.23)
If one now considers translations in Euclidean space,
\[
x_{i}^\alpha' = x_i^\alpha - a^\alpha,
\]
(7.24)
for constant \(a^\alpha\) (which correspond to three different parameters \(s\)), invariance of the Lagrangian in the form (7.17) is manifest since \(\frac{dx_{i}^\alpha'}{dt} = \frac{dx_{i}^\alpha}{dt}\) and \(r_{ij}' = r_{ij}\). The associated infinitesimal transformations are
\[
R_{i}^{\alpha} = \frac{\partial x_{i}^\alpha}{\partial a^\gamma} \big|_{a=0} = \delta_{i}^{\alpha},
\]
(7.25)
with associated conservation laws given by the components of total linear momenta,
\[
P_{\gamma} = \sum_{i} \frac{\partial L}{\partial x_{\gamma(i)}} = \sum_{i} m_{(i)} \dot{x}_{\gamma(i)}.
\]
(7.26)
For rotations, we have
\[
x_{i}^\alpha = A_{i}^{\alpha} \delta x_{i}^{\beta}, \quad A_{\alpha}^{\beta} \delta_{\alpha\gamma} A_{\gamma}^{\delta} = \delta_{\beta\delta} \iff A^T A = I,
\]
(7.27)
with $A$ a constant $3 \times 3$ orthogonal matrix. Again, the invariance of the Lagrangian is manifest since orthogonality implies that
\[
\frac{dx^\alpha_i}{dt} \frac{dx^\gamma_j}{dt} = \frac{dx^\alpha_i}{dt} \frac{dx^\gamma_j}{dt} = \frac{dx^\alpha_i}{dt} \frac{dx^\gamma_j}{dt}
\]
while distances are also left invariant. In order to find the parameters, we note that if $A^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta + O(\omega^2)$, with $\omega^\alpha_\beta$ a “small” matrix, the orthogonality condition
\[
(\delta^\alpha_\beta + \omega^\alpha_\beta + O(\omega^2))\delta_{\alpha\gamma}(\delta^\gamma_\delta + \omega^\gamma_\delta + O(\omega^2)) = \delta_{\beta\delta},
\]
implies to first order in $\omega$
\[
\omega_{\beta\delta} + \omega_{\delta\beta} = 0 \iff \omega + \omega^T = 0.
\]
This implies that $\omega$ can be written in terms of three independent rotation parameters $\theta^\gamma$ as
\[
\omega^\alpha_\beta = \theta^\gamma e^\alpha_\gamma \beta,
\]
and that the associated infinitesimal field transformations are
\[
R^\alpha_\gamma = \frac{\partial x^\gamma_i}{\partial \theta^\alpha_j} |_{\theta=0} = e^\alpha_\gamma \beta x^\beta_i,
\]
whith associated conservation laws given by the components of the total angular momentum,
\[
J_\gamma = \epsilon_{\gamma\alpha\beta} \sum_i m_i \dot{x}^\alpha_i \dot{x}^\beta_i.
\]
We will show in the appendix that a rotation matrix that is connected to the identity can be written as a rotation around an axis
\[
\vec{\theta} = \theta \vec{e}, \theta = \sqrt{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2},
\]
with $\vec{e}_1 = \vec{I}_x, \vec{e}_2 = \vec{I}_y, \vec{e}_3 = \vec{I}_z$, as
\[
A^\alpha_\beta = \delta^\alpha_\beta \cos \theta + \theta^\alpha \theta_\beta \left( \frac{1 - \cos \theta}{\theta^2} \right) - e^\alpha_\gamma \theta_\beta \sin \theta \frac{\theta^\gamma}{\theta}.
\]
As before, one then finds that
\[
\frac{\partial A^\alpha_\beta}{\partial \theta^\gamma} |_{\theta=0} = -e^\alpha_\gamma \beta.
\]

### 7.2.4 One-parameter groups of transformations involving time

Consider now a one-parameters group of invertible transformations that affects time as well,
\[
\left\{ \begin{array}{l}
q^{\alpha'} = q^{\alpha'}(q^\beta, t; s), \\
t' = t'(t; s)
\end{array} \right. \implies q^{\alpha'}(t') = q^{\alpha'}(q^\beta(t(t'; s)), t(t'; s); s),
\]
with $q^{\alpha'} = q^{\alpha'}(q^\beta, t; 0) = q^\alpha, t'(t; 0) = t$. According to the equation (6.26) of the previous section, the transformed Lagrangian is
\[
L'(q'(t'), \frac{dq'(t')}{dt'}, t') = \frac{dt}{dt'} L(q(q'(t'), t'), \frac{dt'}{dt} \frac{d}{dt'} q(q'(t'), t'), t(t(t'))) = \frac{dt}{dt'} L(q(q'(t'), t'), \frac{dt'}{dt} q(q'(t'), t') - \frac{d}{dt'} G'(q'(t'), t'),
\]
where we have temporarily suppressed the $s$ dependence. Let us assume that the transformed Lagrangian is equal to the starting point Lagrangian (in terms of the new variables), up to a total derivative,
\[
L'(q'(t'), \frac{dq'(t')}{dt'}, t') = L(q'(t'), \frac{dq'(t')}{dt'}, t') = \frac{d}{dt'} G'(q'(t'), t'),
\]
with \( G'|_{s=0} = 0 \). When using the inverse transformation and multiplying by \( \frac{dt'}{dt} \), the generalized invariance condition (7.37) can also be written as

\[
\frac{dt'}{dt} L(q'(q, t), \frac{dt}{dt} q'(q(t), t), t'(t)) - \frac{d}{dt} G(q(t), t) = L(q(t), \frac{dt}{dt} q(t), t),
\]

(7.39)

with \( G(q(t), t) = G'(q'(q(t), t), t'(t)) \). In terms of the action, this generalized invariance condition means that as a functional of the new trajectories and the transformed end-points, the

\[
S'[q'] = \int_{t'=-t'(t_i)}^{t'=t'(t_f)} dtL(q'(q, t), \frac{dt}{dt} q'(t), t),
\]

(7.40)

equals the action as a functional of the old trajectories, up to a boundary term. Indeed, by first changing the name of the dummy integration variable, and then doing the change of variables from primed to unprimed ones, we get

\[
S[q'] = \int_{t'=-t'(t_i)}^{t'=t'(t_f)} dt'L(q'(q, t), \frac{dt}{dt} q(t), t) = \int_{t=-t(t_i)}^{t=t(t_f)} dt\frac{dt'}{dt} L(q'(q, t), \frac{dt}{dt} q'(t), t).
\]

(7.41)

When using the generalized invariance condition (7.39), this gives

\[
S[q'] = \int_{t=-t(t_i)}^{t=t(t_f)} dtL(q(t), \frac{dt}{dt} q(t), t) + \left[ G(q(t), t) \right]_{t_i}^{t_f}.
\]

(7.42)

Note that the boundary term does not contribute when computing the variation of trajectories with fixed end-points.

Expanding the transformed trajectories to first order in \( s \) gives

\[
\begin{align*}
q^{\alpha'}(t') &= q^{\alpha}(t) + sR^{\alpha}(q^{\beta}, t) + O(s^2), \\
t' &= t + s\xi(t) + O(s^2),
\end{align*}
\]

(7.43)

the associated equal-time virtual variation is

\[
\left[ \frac{\delta S[q]}{\delta q^\alpha(t)} \right] = \lim_{s \to 0} \frac{1}{s} \left( q^{\alpha'}(t) - q^{\alpha}(t) \right) = \lim_{s \to 0} \frac{1}{s} \left( q^{\alpha'}(t' - s\xi(t) + O(s^2)) - q^{\alpha}(t) \right) = \lim_{s \to 0} \frac{1}{s} \left( q^{\alpha'}(t') + sR^{\alpha}(q^\beta(t), t) - s\frac{dq^{\alpha'}}{dt}(t')\xi(t) + O(s^2) - q^{\alpha}(t) \right) = \left[ R^{\alpha} - q^{\alpha}\xi \right]
\]

(7.44)

Such a variation leaves the Lagrangian invariant up to a total derivative. Indeed, the generalized invariance condition (7.37) can be written as

\[
(1 + s\dot{\xi})(L(q + sR, (1 - s\dot{\xi}) \frac{dt}{dt} (q + sR), t + s\xi) - s\frac{d}{dt} G_1 + O(s^2) = L(q, \dot{q}, t).
\]

(7.45)

To order 0 in \( s \), this is an identity, while the terms of order 1 yield

\[
\dot{\xi} L(q, \dot{q}, t) + \frac{\partial L}{\partial q^\alpha} R^{\alpha} + \frac{\partial L}{\partial \dot{q}^\alpha} \left( \frac{d}{dt} R^{\alpha} - \dot{\xi} q^\alpha \right) + \frac{\partial L}{\partial \dot{q}^\alpha}(q, \dot{q}, t)\dot{\xi} - \frac{d}{dt} G_1 = 0,
\]

(7.46)

The first term can be integrated by parts

\[
\dot{\xi} L = \frac{d}{dt}(\xi L) - \xi \left( \frac{\partial L}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \ddot{q}^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \dot{\xi} \right),
\]

(7.47)

and, when combining like terms, we get

\[
\frac{\partial L}{\partial q^\alpha} \left( R^{\alpha} - \dot{q}^\alpha \xi \right) + \frac{\partial L}{\partial \dot{q}^\alpha} \frac{d}{dt} \left( R^{\alpha} - \dot{q}^\alpha \xi \right) = \frac{d}{dt} (G_1 - \xi L) \Leftrightarrow \delta S L = \frac{d}{dt} (G_1 - \xi L).
\]

(7.48)
The associated conservation law is

$$K = G - \xi L - \frac{\partial L}{\partial \dot{q}^\alpha} \delta_{\dot{q}}^\alpha. \quad (7.49)$$

In particular, a Lagrangian does not depend explicit on time, $\frac{\partial L}{\partial t} = 0$ iff it possesses the symmetry $q^{\prime \alpha} = q^\alpha$, $t^\prime = t + s$. In this case $G = 0$, $\xi = 1$, $R^\alpha = 0$, $\delta_{\dot{q}}^\alpha = -\dot{q}^\alpha$. The associated conservation law is

$$E = \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha - L, \quad (7.50)$$
discussed previously, which now appears from the viewpoint of the symmetry of the system under constant time translations.

### 7.2.5 Galilean boosts

Galilean transformations on a system of particles are described by

$$x^{\prime \alpha}_i = A^\alpha_\beta x^\beta_i + v^\alpha t - a^\alpha, \ t^\prime = t + b, \ A^T A = I. \quad (7.51)$$

They consists of a combination of spatial translations, rotations, time-translations, described by the constant parameter $b$, and changes of observers with a constant velocity in a given direction, sometimes called “Galilean boosts” and described by the 3 parameters $v^\alpha$.

We have already analyzed invariance of the particle Lagrangian (7.22) under spatial translations (parameters $a^\alpha$) and rotations (parameters $\omega^\alpha$ of the rotation matrix $A$) and the associated conservation laws $P_\alpha, J_\alpha$. For the time-translation, explicit time-independence implies that the associated conservation law is $E = \sum_i \frac{\partial L}{\partial \dot{q}^{(i)}_i} \dot{q}^{(i)}_i - L = T + V$. So only the Galilean boosts remain to be analyzed explicitly. More generally, for the whole set of Galilean transformations, we have $\frac{dt^\prime}{dt} = 1, \frac{dx^{\prime \alpha}_i(t^\prime)}{dt^\prime} = A^\alpha_\beta \frac{dx^\beta(t)}{dt} + v^\alpha$. For the kinetic energy of the particle Lagrangian (7.22), this gives

$$T \left( \frac{dx^{\prime \alpha}_i(t^\prime)}{dt^\prime} \right) = \frac{1}{2} \sum_i m(i)(A^\alpha_\beta \frac{dx^\beta_i}{dt} + v^\alpha)(A^\alpha_\gamma \frac{dx^\gamma_i}{dt} + v_\gamma)$$

$$= \frac{1}{2} \sum_i m(i) \frac{dx^{\alpha}_i}{dt} \frac{dx^{\alpha}_i}{dt} + \sum_i m(i) v^\alpha A^\alpha_\beta \frac{dx^\beta_i}{dt} + \frac{1}{2} \sum_i m(i) v^\alpha v_\gamma$$

$$= T \left( \frac{dx^{\alpha}_i}{dt} \right) + \frac{d}{dt} \left( \sum_i m(i) v^\alpha \left( A^\alpha_\beta x^\beta_{(i)} + \frac{1}{2} t v^\alpha \right) \right). \quad (7.52)$$

Since relative distances are invariant even if one translates the individual coordinates linearly in time (by the same amount), the potential is also invariant. Since

$$G = \sum_i m(i) v^\alpha (A^\alpha_\beta x^\beta_{(i)} + \frac{1}{2} t v^\alpha), \quad (7.53)$$

it follows that for Galilean boosts parametrized by $v^\gamma$, $R^{\alpha}_{\gamma (i)} = \delta^\alpha_\gamma t, \ \xi = 0$, $\frac{\partial G}{\partial v^\gamma} |_{a,b,\theta,v=0} = \sum_i m(i) x^{\gamma}_{(i)}$, and the associated conservation laws are

$$K_\gamma = \sum_i m(i) (x^{\gamma}_{(i)} - \dot{x}^{\gamma}_{(i)} t). \quad (7.54)$$
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Bibliography


