Free and co-free constructions for Hopf categories

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Hopf algebras

Definition

Given a braided monoidal category $(\mathcal{V}, \otimes, I, \sigma)$, a **Hopf algebra** is a category enriched in Comon($\mathcal{V}$) with one object: for the object $* \in \mathcal{G}$ we have a comonoid $(H := \mathrm{Hom}(*, *), \Delta, \epsilon)$. There are coalgebra map $m : H \otimes H \to H$. There is a unit map $j : I \to H$ and an anti comonoid map $S : H \to H$ such that:

$$m(H \otimes j) = m(j \otimes H) = H$$

$$m(H \otimes S)\Delta = j\epsilon, m(S \otimes H)\Delta = j\epsilon$$

Examples

- All groups are Hopf algebras over $\mathcal{V} = (\underline{\underline{\mathrm{Set}}}, \times, \{*\}, \sigma)$
- Group algebras kG for a group G and a field k

Hopf categories

Definition (following [3])

Given a braided monoidal category $(\mathcal{V}, \otimes, I, \sigma)$, a **Hopf category** is a category $\mathcal{H}$ enriched in Comon$(\mathcal{V})$: for the object $x, y \in \mathcal{H}$ we have a comonoid $(\mathcal{H}_{x,y}, \Delta_{x,y}, \epsilon_{x,x})$. There are coalgebra map $m_{x,y,z} : H_{x,y} \otimes H_{y,z} \to H_{x,z}$. There is a unit map $j_x : I \to H_{x,x}$ and an anti comonoid $S_{x,y} : H_{x,y} \to H_{y,x}$ such that:

$$m_{xyy}(\mathcal{H}_{xy} \otimes j_y) = m_{xxy}(j_x \otimes \mathcal{H}_{xy}) = \mathcal{H}_{xy}$$

$$m_{xyx}(\mathcal{H}_{xy} \otimes S_{xy})\Delta_{xy} = j_x \epsilon_{xy}, m_{yxy}(S_{xy} \otimes \mathcal{H}_{xy})\Delta_{xy} = j_y \epsilon_{xy}$$

Examples

- Collections of Hopf algebras: Given $(H_i)_{i \in I}$ Hopf algebras, set $0\mathrm{bj}(\mathcal{G}) = \mathrm{I}, \mathcal{H}(i,i) = H_i, \mathcal{H}(i,j) = 0$
- All groupoids are Hopf categories over $\mathcal{V} = (\underline{\underline{\mathrm{Set}}}, \times, \{*\}, \sigma)$

Why do we care about Hopf categories

Group : Hopf algebra = Groupoid : Hopf category

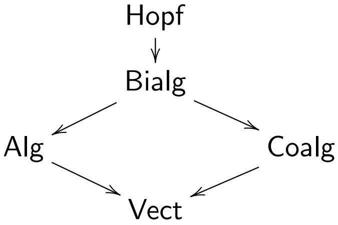
Theorem (Kellendonk-Lawson)

The partial action of a group G are in bijective correspondence with morphisms of groupoids $\mathcal{G} \to G$ that are star injective.

Is there a Hopf-version of this theorem?

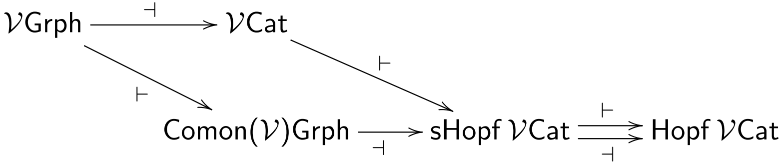
If yes, it would be a good guess that it will involve Hopf category-esque structures!

Left and right adjoint in single object case



- Takeuchi [9] in the 70s: Any coalgebra can always be embedded universally into a Hopf algebra, the so called **Hopf envelope**.
- Manin [7] in the late 80s: Extended to bialgebras.
- Sweedler [8] in 1969: Explicit construction of cofree coalgebra and mention of cofree bialgbra/ cofree Hopf algebra.
- Agore [1] in 2011: explicit proof of right adjoint to Hopf $\to$ Bialg .

A multi-object analogue



We wanted to prove he existence of these adjoints or even explicitely construct them (extending work of Vasilakopoulou [10]).

The tool box

- *Abstract nonsense*:
  locally presentable categories, does not provide insight!
- *SAFT*:

Theorem (The Special Adjoint Functor Theorem, [6])

If $\mathcal{C}$ is a complete and a well-powered category with a cogenerating family, then a functor $G : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if it is limit preserving.
Dually, if $\mathcal{C}$ is a cocomplete and co-well-powerd category with a generator, then a functor $G : \mathcal{C} \to \mathcal{D}$ has a right adjoint if and only if it is colimit preserving.

- *Explicit constructions* generalizing work of Takeuchi [9] and Manin [7]

$\mathcal{V}$-Graphs

Definition ([10])

Let X be a set and $\mathcal{V}$ an arbitrary category and A a family of objects $A_{x,y}$ in $\mathcal{V}$ indexed by $x, y \in X$. We call $A = (A_{x,y})_{x,y \in X}$ a $\mathcal{V}$-**graph**.
Let A be a graph over X and B a graph over Y. We define a **morphism of $\mathcal{V}$-graphs** $\bar{f} : A \to B$ to be a family of functions:

$$\begin{cases} f : X \to Y & \text{a morphism in } \mathsf{Set} \\ (f_{x,y} : A_{x,y} \to B_{fx,fy})_{x,y \in X} & \text{a family of morphisms in } \mathcal{V} \end{cases}$$

$\mathcal{V}$-graphs and morphisms of $\mathcal{V}$-graphs form the cateogry $\mathcal{V} - \mathsf{Grph}$.

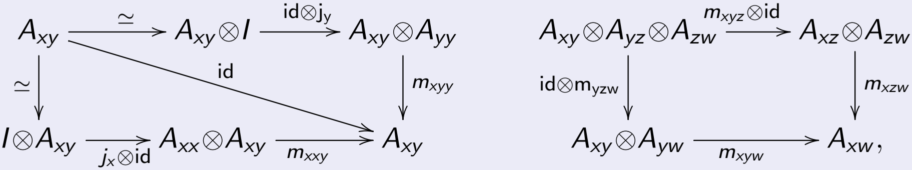
$\mathcal{V}$-categories

Definition ([10])

Given a monoidal category $\mathcal{V}$ we define a $\mathcal{V}$-**category** to be a $\mathcal{V}$-graph with a global multiplication and unit. That is a family of morphisms in $\mathcal{V}$

$$m_{xyz} : A_{xy} \otimes A_{yz} \to A_{xz} \qquad j_x : I \to A_{xx},$$

satisfying



where I denotes the monoidal unit of $\mathcal{V}$. A $\mathcal{V}$-**functor** is a $\mathcal{V}$-graph morphism that preserves multiplication and unit.
$\mathcal{V}$-categories and $\mathcal{V}$-functors form a the category $\mathcal{V} - \mathrm{Cat}$

(semi-)Hopf $\mathcal{V}$-categories

Definition ([3])

A **semi-Hopf $\mathcal{V}$-category** is a $\mathrm{Comon}(\mathcal{V})$-category.
A **Hopf $\mathcal{V}$-category** is a $\mathrm{Comon}(\mathcal{V})$-category together with an antipode.

The results

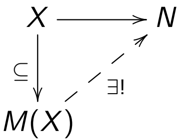
We were able to give sufficient conditions for the existence of all these adjoints!

Particularly we were able to explicitely construct a free Hopf $\mathcal{V}$-category of a semi-Hopf $\mathcal{V}$-category.

Let's see how this is done!

Recall: Free monoids

Given any set X, can we find a universal monoid $X \subseteq M(X)$?
For any monoid N and maps $X \to N$ we have a unique homomorphism:



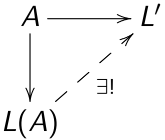
Yes! We use words in the alphabet X and concatenation of words.

$$
\begin{aligned}
M(X) &= \{\epsilon, x_1, x_2, x_3, \ldots x_1x_1, x_1x_2, x_2x_1, \ldots x_1^3, \ldots\} \\
&= \bigcup_{k=0}^{\infty} X^k,
\end{aligned}
$$

with $X^0 = \epsilon$ the empty word, $X^1 = X$ and $X^{k+1} = X^k \times X$.

Free $\mathcal{V}$-category over a $\mathcal{V}$-graph

For a cocomplete symmetric monoidal category $(\mathcal{V}, \otimes, I, \sigma)$ such that colimits commute with $\otimes$:
Given any $\mathcal{V}$-graph A can we find a universal $\mathcal{V}$-category $A \to L(A)$?
For any $\mathcal{V}$-category L' and maps $A \to L'$ we have a unique homo:



We use the same "word" approach, but must now respect target and source of every arrow!

Let A be a $\mathcal{V}$-graph. We define

$$(LA)_{xy} := \coprod_{n\in\mathbb{N}} \coprod \bigotimes_{i=0}^{n} A_{x_i,y_i}, \qquad x_0 = x, y_i = x_{i+1}, y_n = y, x \neq y$$

$$(LA)_{xx} := I \amalg \coprod_{n\in\mathbb{N}} \coprod \bigotimes_{i=0}^{n} A_{x_i,y_i}, \qquad x_0 = x, y_i = x_{i+1}, y_n = x.$$

Let $\epsilon_x : I \hookrightarrow LA_{xx}$ be the inclusion of I into the coproduct. Since we have concatenations for every two sequences, we get maps:

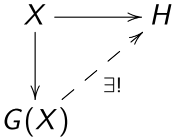
$$\bigotimes_{i=0}^{n} A_{x_i,x_{i+1}} \otimes \bigotimes_{i=n+1}^{m} A_{x_i,x_{i+1}} \to \bigotimes_{i=0}^{m} A_{x_i,x_{i+1}} \hookrightarrow (LA)_{xz}$$

that we can lift to a multiplication:

$$m_{xyz} : (LA)_{xy} \otimes (LA)_{yz} \to (LA)_{xz}.$$

Recall: Free groups

Given a set X can we find a universal group $X \to G(X)$?
For any group H and maps $X \to H$ we have a unique homomorphism:



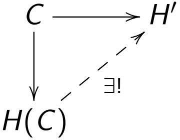
Use words in the alphabet $Y = X \cup X^{-1}$ and concatenation of words modulo

$$G(X) = M(Y)/_{\simeq},$$

where $xx^{-1} \simeq \epsilon$, $x^{-1}x \simeq \epsilon$ for any $x \in X$ and we extend this relation to words.

Takeuchi's Hopf envelope

For a (countably) cocomplete symmetric monoidal category $(\mathcal{V}, \otimes, I, \sigma)$:
Given any comonoid $C \in \mathcal{V}$ can we find a universal Hopf algebra $C \to H(C)$?
For any Hopf algebra H' and maps $C \to H'$ we have a unique homo:



We want to reconstruct the construction of a free group, but since antipodes do not generally satisfy $S^2 = \mathrm{Id}_H$. There for we need $\mathbb{N}$-indexed copies of the comonoid C.

$$V_0 = C \quad V_{i+1} = V_i^{cop}$$

$$V = \coprod_{i \in \mathbb{N}} V_i$$

We can build the tensor algebra $T(V)$ over the coalgebra V, which has a bialgebra structure (Δ, ϵ) . The anti coalgebra maps $S_i : V_i \to V_{i+1}$ given by the identity can be lifted to an anti coalgebra map $T(S) : T(V) \to T(V)^{cop,op}$, by reversing the order via the symmetric braiding σ.
We consider the maps

$$\begin{aligned} f := m(T(S)\otimes T(V))\Delta : &\quad T(V) \to T(V)\otimes T(V) \to T(V) \\ g := m(T(V)\otimes T(S))\Delta : &\quad T(V) \to T(V)\otimes T(V) \to T(V) \\ \eta\epsilon : &\quad T(V) \to I \to T(V) \end{aligned}$$

and build the consecutive coequalizers (since these maps commute)

$$T(V) \underset{f}{\overset{\eta\epsilon}{\rightrightarrows}} T(V) \xrightarrow{\pi} H(C)'$$

$$H(C)' \underset{\tilde{g}}{\overset{\tilde{\eta}\epsilon}{\rightrightarrows}} H(C)' \longrightarrow H(C)$$

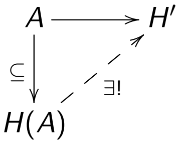
$H(C)$ is a Hopf algebra and the map $C \to V \to T(V) \to H(C)$ satisfies the diagram. Given any coalgebra map $h : C \to H'$ where $(H', m', \eta', \Delta', \epsilon', S')$ we can consider maps

$$h_i : S'^i \circ h : V_i \to H'$$

and lift them to a coalgebra map $V \to H'$. This induces a bialgebra map $h : T(V) \to H'$. Since h coequalizes both $f, \eta\epsilon$ and $g, \eta\epsilon$ we get a unique map $\bar{h} : H(C) \to H'$

Free Hopf $\mathcal{V}$-category over a Comon($\mathcal{V}$)-graph

For a cocomplete symmetric monoidal category $(\mathcal{V}, \otimes, I, \sigma)$ such that colimits commute with $\otimes$:
Given any Comon($\mathcal{V}$)-graph A can we find a universal Hopf $\mathcal{V}$-category $A \to H(A)$?
For any Hopf $\mathcal{V}$-category H' and maps $A \to H'$ we have a unique homo:



We follow the Takeuchi approach:

$$A^i_{xy} = A_{xy} \quad A^{i,op}_{xy} = A^{cop}_{xy}$$

$$A'_{xy} = \coprod_{i \in \mathbb{N}} (A^i_{xy} \amalg A^{i,op}_{yx})$$

We can build the free $\mathcal{V}$-category $L(A')$, which is a semi-Hopf $\mathcal{V}$-category via $(m_{xyz}, j_x, \Delta_{xy}, \epsilon_{xy})$. Further have anti comonoid maps $s_{xy} : A'_{xy} \to A'_{yx}$ that we lifted from the identities $A^i_{xy} \to A^{i,op}_{xy}$ and $A^{i,op}_{yx} \to A^{i+1}_{yx}$. These we can lift to a homomorphism of semi-Hopf $\mathcal{V}$-categories $S : L(A') \to L(A')^{op,cop}$ by defining

$$S_{xy}(A'_{x,x_1} \otimes A'_{x_1,x_2} \otimes \ldots \otimes A'_{x_n,y}) := s_{x_n,y}(A'_{x_n,y}) \otimes \ldots \otimes s_{x_1,x_2}(A'_{x_1,x_2}) \otimes s_{x,x_1}(A'_{x,x_1})$$

.

We consider the maps

$$
\begin{aligned}
f := m_{xyx}(L(A')_{xy} \otimes S_{xy})\Delta_{xy} : &\quad L(A')_{xy} \to L(A')_{xy} \otimes L(A')_{yx} \to L(A')_{xx} \\
g := m_{xyx}(S_{xy} \otimes L(A')_{xy})\Delta_{xy} : &\quad L(A')_{xy} \to L(A')_{yx} \otimes L(A')_{xy} \to L(A')_{yy} \\
j_x \epsilon_{xy} : &\quad L(A')_{xy} \to I \to L(A')_{xx} \\
j_y \epsilon_{xy} : &\quad L(A')_{xy} \to I \to L(A')_{yy}
\end{aligned}
$$

and build the coequalizer for every pair $x, y \in X$, to get a semi-Hopf $\mathcal{V}$-category $H(A)$.

$H(A)$ is a Hopf $\mathcal{V}$-category and the map $A \to A' \to L(A') \to H(A)$ satisfies the diagram. Given any moprhism of Comon($\mathcal{V}$)-graphs $h : A \to H'$ where $(H', m', \eta', \Delta', \epsilon', S')$ is a Hopf $\mathcal{V}$-category, we can consider maps:

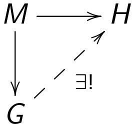
$$h^i : S'^{2i} \circ h : A^i_{xy} \to H'$$

$$h^{i,op} : S'^{2i+1} \circ h : A^{i,op}_{xy} \to H'$$

and lift them to $L(A') \to H'$. This induces is a homo of semi-Hopf $\mathcal{V}$-categories and since it equalizes both $f, j_x\epsilon_{xy}$ and $g, j_y\epsilon_{xy}$ we get a unique map $\bar{h} : H(A) \to H'$

Recall: Free group over a monoid

Given a monoid $(M, +)$ can we find a universal group $M \to G$?
For any group H and monoid maps $M \to H$ we have a unique group homo:



We can solve this by forgetting about the monoid structure on M and consider the group $G(M)$. We now factor out the relations coming from the monoid M:

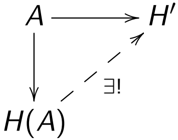
$$gh \simeq i \Leftrightarrow g + h = i \in M$$

$$g^{-1} \simeq -g$$

and reduce words by this relations.

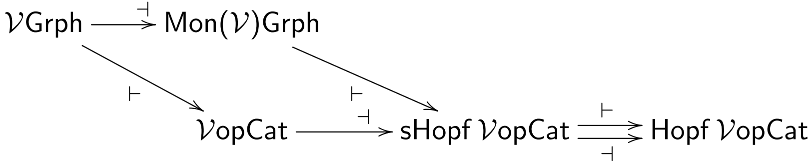
Free Hopf $\mathcal{V}$-category over a semi-Hopf ($\mathcal{V}$)-category

For a (countably) cocomplete symmetric monoidal category $(\mathcal{V}, \otimes, I, \sigma)$ such that colimits commute with $\otimes$:
Given any semi Hopf $\mathcal{V}$-category A can we find a universal Hopf $\mathcal{V}$-category $A \to H(A)$?
For any Hopf $\mathcal{V}$-category H' and homo of semi-Hopf $\mathcal{V}$-categories $A \to H'$ we have a unique homo of Hopf $\mathcal{V}$-categories.



- We forget forget the category structure on A and build a Comon($\mathcal{V}$)-graph with bigger comonoids A'_{xy}.
- We build a free category of this bigger graph resulting in a semi-Hopf $\mathcal{V}$-category, such that there is an anticomonoid map.
- We factor out a set of relations:
  a) Compatibility of S with the multiplication m in A.
  b) Compatibility of multiplication, unit in A and multiplication, unit in HA.
  c) The antipode conditions.

The caotic dual picture



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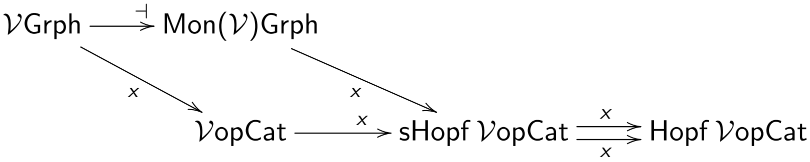

Even (co)completness fails for $\mathcal{V}$opCat. What are the right categories and duals to generalize the one-object result?

Thank you for your attention!
I am looking forward to your questions!

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