Partial comodules over Hopf algebras

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Partial actions

Does SO(3) act on the Atomium ?

Partial actions

A *partial action* of a group G on a set X is a collection $(D_g, \alpha_g)_{g \in G}$ where $D_g \subseteq X, \alpha_g : D_{g^{-1}} \to D_g$ are bijections such that

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- $D_e = X$ and $\alpha_e = X$;
- $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ on $D_{h^{-1}} \cap D_{(gh)^{-1}}$.

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Example

Let $\alpha : G \to \mathrm{Sym}(Y)$ be an action on a set Y and take $X \subseteq Y$. Then putting

$$D_g = X \cap g(X), \qquad \alpha_g = \alpha(g)_{|D_{g^{-1}}}$$

for every $g \in G$ defines a partial action of G on X.

Partial modules

Idea : linearisation of partial actions. Let k be a field.

$$\pi : kG \otimes kX \to kX : g \otimes x \mapsto g \cdot x = \begin{cases} \alpha_g(x) & \text{if } x \in D_{g^{-1}}, \\ 0 & \text{else.} \end{cases}$$

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Remark : in general,

$$g \cdot (h \cdot x) \neq (gh) \cdot x$$

but we have

$$g \cdot (h \cdot (h^{-1} \cdot x)) = (gh) \cdot (h^{-1} \cdot x).$$

Partial modules

Definition

A partial module over kG is a vector space M equipped with a linear map

$$\pi : kG \otimes M \to M : g \otimes m \mapsto g \cdot m$$

such that for all $g, h \in G, m \in M$

- $e \cdot m = m$;
- $g \cdot (h \cdot (h^{-1} \cdot m)) = (gh) \cdot (h^{-1} \cdot m)$;
- $g^{-1} \cdot (g \cdot (h \cdot m)) = g^{-1} \cdot ((gh) \cdot m)$.

Partial modules over Hopf algebras

Definition

A partial module over H is a vector space M equipped with a linear map

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such that for all $g, h \in H, m \in M$

- $1_H \cdot m = m$;
- $g \cdot (h_{(1)} \cdot (S(h_{(2)}) \cdot m)) = (gh_{(1)}) \cdot (S(h_{(2)}) \cdot m)$;
- $g_{(1)} \cdot (S(g_{(2)}) \cdot (h \cdot m)) = g_{(1)} \cdot ((S(g_{(2)})h) \cdot m)$.

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Does the category of partial modules ${}_H\mathrm{PMod}$ contain more information about H than ${}_H\mathrm{Mod}$?

Partial modules over Hopf algebras

Theorem (M. Alves, E. Batista, J. Vercruysse)

The category of partial modules over H is equivalent to the category of modules over the "partial Hopf algebra" H_{par} :

$$ {}_H\text{PMod} \simeq {}_{H_{par}}\text{Mod}. $$

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H_{par} is the quotient of the free algebra generated by the symbols $[h]$ for $h \in H$ by the relations

$$
\begin{aligned}
[1_H] &= 1_{H_{par}} \\
[g][h_{(1)}][S(h_{(2)})] &= [gh_{(1)}][S(h_{(2)})] \\
[g_{(1)}][S(g_{(2)})][h] &= [g_{(1)}][S(g_{(2)})h]
\end{aligned}
$$

Partial modules over Hopf algebras

Suppose the antipode S is invertible.
Let A be the subalgebra generated by

$$\{[h_{(1)}][S(h_{(2)})] \mid h \in H\}.$$

Then H_{par} is a *Hopf algebroid* over A and there is a strong monoidal and closed forgetful functor

$${}_{H_{par}}\mathrm{Mod} \to {}_A\mathrm{Mod}_A.$$

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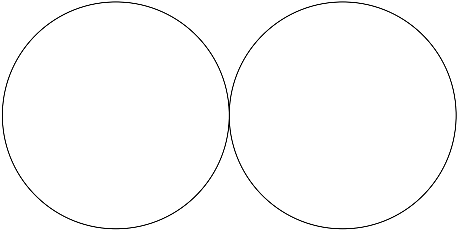
Remark

The forgetful functor $U : {}_H\mathrm{PMod} \to \mathrm{Vect}_k$ has a left adjoint

$$\mathrm{Vect}_k \to {}_{H_{par}}\mathrm{Mod} \simeq {}_H\mathrm{PMod} : V \mapsto H_{par} \otimes V$$

Partial comodules



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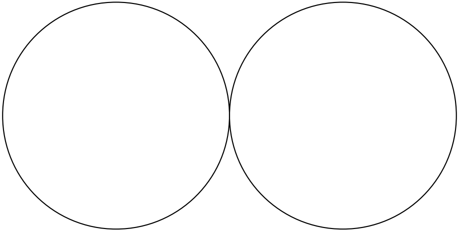

$$G = \mathrm{SO}(2) \times C_2$$

A "representation"

$$\pi : G \to \mathrm{End}(V)$$

induces a linear map

$$\rho : V \to V \otimes \mathcal{O}(G).$$

Partial comodules

Definition

A partial comodule over H is a vector space M equipped with a linear map

$$\rho : M \to M \otimes H : m \mapsto m^{(0)} \otimes m^{(1)}$$

such that for all $m \in M$

- $m^{(0)}\epsilon(m^{(1)}) = m$;
- $m^{(0)(0)} \otimes m^{(0)(1)}{}_{(1)} \otimes m^{(0)(1)}{}_{(2)}S(m^{(1)}) =$ $m^{(0)(0)(0)} \otimes m^{(0)(0)(1)} \otimes m^{(0)(1)}S(m^{(1)})$;
- $m^{(0)(0)} \otimes m^{(0)(1)}S(m^{(1)}{}_{(1)}) \otimes m^{(1)}{}_{(2)} =$ $m^{(0)(0)(0)} \otimes m^{(0)(0)(1)}S(m^{(0)(1)}) \otimes m^{(1)}$.

Is $\mathrm{PMod}^H \simeq \mathrm{Mod}^C$ for some coalgebra C ?

Example

Let H_4 be the Sweedler Hopf algebra, $H_4 = \langle 1, g, x, y \rangle$. Let $M = k[z]$ equipped with

$$\rho : k[z] \to k[z] \otimes H_4 : z^n \mapsto z^n \otimes \frac{1+g}{2} + z^{n+1} \otimes y.$$

Then $(k[z], \rho)$ is a partial comodule over H_4.

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However, M contains and infinite descending chain of partial subcomodules

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Every element of a comodule over a coalgebra is contained in a finite-dimensional subcomodule, hence there exists no coalgebra C such that

$$\mathrm{PMod}^{H_4} \simeq \mathrm{Mod}^C.$$

The right adjoint

Using SAFT :

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Theorem

The category of partial comodules over H is comonadic over Vect_k, hence it is equivalent to the Eilenberg-Moore category of the comonad $\mathbb{C} = UR$.

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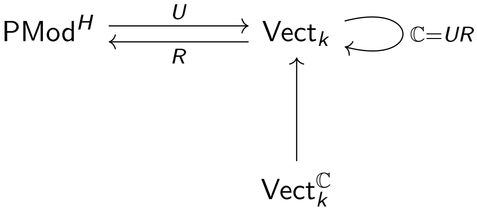

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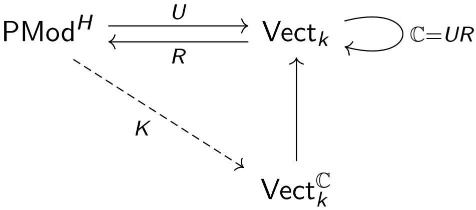

The right adjoint

Sketch of the explicit construction :

- For a vector space V, consider

$$V \hat{\otimes} \hat{C}(H) = \prod_n V \otimes H^{\otimes n},$$

  with the linear map

$$\sigma : \prod_n V \otimes H^{\otimes n} \to \prod_n (V \otimes H^{\otimes n} \otimes H)$$

- Let $R^0(V) \subseteq \prod_n V \otimes H^{\otimes n}$ be maximal such that

$$\sigma(R^0(V)) \subseteq R^0(V) \otimes H.$$

- Let $R(V) \subseteq R^0(V)$ be maximal such that $(R(V), \sigma)$ is a partial comodule.

The right adjoint

Unit : for a partial comodule (M, ρ)

$$\eta_M : M \to RU(M) \subseteq \prod_n M \otimes H^{\otimes n} : m \mapsto (\rho^n(m))_n.$$

Counit : for a vector space V, the restriction of

$$\pi_0 : \prod_n V \otimes H^{\otimes n} \to V$$

to $UR(V)$.

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Is kG regular, i. e. is $(kG^*)_{par}$ finite dimensional ?

Topological considerations 1

Considering topological partial comodules $(M, \rho : M \to M \hat{\otimes} H)$, the forgetful functor

$$\hat{U} : \mathsf{TPMod}^H \to \mathsf{CTVS}_k$$

has a right adjoint

$$\hat{R}(V) \subseteq V \hat{\otimes} \hat{C}(H)$$

and $\hat{R} \cong - \hat{\otimes} \hat{H}^{par}$.

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Theorem

- TPMod^H is equivalent to $\mathrm{TMod}^{\hat{H}^{par}}$, the topological comodules over $\hat{H}^{par}$.
- PMod^H is equivalent to the category of discrete topological comodules over $\hat{H}^{par}$.

Topological considerations 2

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$$\mathrm{PMod}^H \underset{R}{\overset{U}{\rightleftarrows}} \mathrm{Vect}_k$$

Put on $R(V)$ the topology generated by

$$\ker[(\epsilon_V \otimes H^{\otimes n})\rho^n_{R(V)}] \qquad \text{for } n \in \mathbb{N}$$

and let $\overline{R(V)}$ be its completion.

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and let $\overline{R(V)}$ be its completion.

$$\tilde{\mathbb{C}} : \mathrm{Vect}_k \to \mathrm{CTVS}_k : V \mapsto \overline{R(V)}$$

is a *relative comonad* in the sense of

T. Altenkirch, J. Chapman, T. Uustalu, *Monads need not be endofunctors*

Relative comonads

Let $\mathcal{C}, \mathcal{D}$ be categories and $J : \mathcal{C} \to \mathcal{D}$ a functor.

Definition

A relative comonad on J is a triple

- a functor $S : \mathcal{C} \to \mathcal{D}$,
- a natural transformation $\epsilon : S \Rightarrow J$ (the counit),
- a natural transformation

$$(-)_* : \mathrm{Hom}_{\mathcal{D}}(S(-), J(-)) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(S(-), S(-))$$

  (the Kleisli-extension)

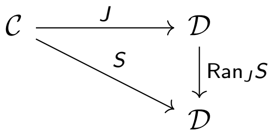
satisfying certain axioms.

Relative comonads

With right Kan extension along J :

$$\mathsf{Ran}_J S : \mathcal{D} \to \mathcal{D}$$

such that



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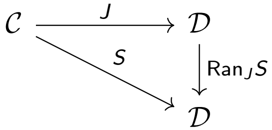

in a universal way.

$$S \circ_J S = \mathrm{Ran}_J S \circ S.$$

Relative comonads

$D : \mathsf{Vect}_k \to \mathsf{CTVS}_k$ is well-behaved, hence the relative comonad $\tilde{\mathbb{C}}$ induces a comonad

$$\overline{\mathbb{C}} = \mathrm{Ran}_D \tilde{\mathbb{C}} : \mathsf{CTVS_k} \to \mathsf{CTVS}_k.$$

If $(V_\alpha)_\alpha$ is a basis of open subspaces of V, then

$$\overline{\mathbb{C}}(V) = \varprojlim \tilde{\mathbb{C}}(V/V_\alpha).$$

Theorem

If H is finite dimensional, then $\overline{\mathbb{C}} = -\hat{\otimes}\hat{H}^{par}$.

Thank you for your attention ! Shoot your questions !