THE CLAUSIUS–MOSSOTTI FORMULA

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Abstract. In this note, we provide a short and robust proof of the Clausius–Mossotti formula for the effective conductivity in the dilute regime, together with an optimal error estimate. The proof makes no assumption on the underlying point process besides stationarity and ergodicity, and it can be applied to dilute systems in many other contexts.

1. Effective conductivity problem

We start by recalling the notion of effective conductivity in the sense of stochastic homogenization theory for an heterogeneous material made of inclusions in a given matrix of homogeneous conductivity. Let $d \geq 1$ denote the space dimension.

1.1. Stochastic setting.

We shall use a statistical description for the set of inclusions in the material. Restricting to spherical inclusions for notational simplicity, we let

$$\mathcal{B}(\mathcal{P}) := \bigcup_{x\in\mathcal{P}} B(x), \tag{1.1}$$

where $B(x) = B + x$ stands for the unit ball centered at x in $\mathbb{R}^d$, and where $\mathcal{P}$ is the set of centers of the inclusions. It remains to define statistical ensembles for the latter. We call *point set* any countable subset $\mathcal{P} \subset \mathbb{R}^d$ that is locally finite in the sense that for any bounded $E \subset \mathbb{R}^d$ the number of points of $\mathcal{P}$ in E is finite, $\mathcal{P}(E) := \sharp\{\mathcal{P} \cap E\} < \infty$. A point set $\mathcal{P}$ can be represented by the associated locally finite measure $\sum_{x\in\mathcal{P}} \delta_x$, which acts on the space of compactly supported continuous functions via $f \mapsto \mathcal{P}(f) := \sum_{x\in\mathcal{P}} f(x)$. We endow the space Ω of point sets with the smallest σ-algebra that makes all evaluation maps $\mathcal{P} \mapsto \mathcal{P}(f)$ measurable. A *random point process* is then defined as a probability measure $\mathbb{P}$ on Ω, and we denote by $\mathbb{E}[\cdot]$ the associated expectation. We further define stationarity and ergodicity with respect to translations $\mathcal{P} + z := \{x + z : x \in \mathcal{P}\}$ of point sets: The point process is said to be *stationary* (or *statistically translation-invariant*) if for any measurable set $A \subset \Omega$ we have $\mathbb{P}[A + z] = \mathbb{P}[A]$ for all $z \in \mathbb{R}^d$, where we use the notation $A + z := \{\mathcal{P} + z : \mathcal{P} \in A\}$. The process is said to be *ergodic* if any measurable set $A \subset \Omega$ that is translation-invariant, in the sense that $\mathbb{P}[A \setminus (A + z)] = \mathbb{P}[(A + z) \setminus A] = 0$ for all $z \in \mathbb{R}^d$, satisfies $\mathbb{P}[A] = 0$ or 1. In the sequel, the set of inclusions in the material is modeled by (1.1) with $\mathcal{P}$ sampled according to some stationary and ergodic random point process. Note that the inclusions are allowed to overlap in general.

1.2. Effective conductivity.

Given a stationary ergodic random point process as defined above, we consider the associated coefficient field

$$A(y) := A_1 + (A_2 - A_1)\mathbb{1}_{\mathcal{B}(\mathcal{P})}(y), \qquad y \in \mathbb{R}^d, \tag{1.2}$$

where $A_1, A_2 \in \mathbb{R}^{d\times d}$ are two strongly elliptic matrices and where $\mathcal{P}$ is sampled according to the point process. This models a homogeneous material of conductivity A_1 that is perturbed by disordered spherical inclusions of another material of conductivity A_2. On large scales, in the sense of homogenization theory, this two-phase heterogeneous material behaves like a homogeneous material with some effective conductivity $\bar{A}$ defined by

$$\bar{A}e = \mathbb{E}\left[A(\nabla\phi_e + e)\right], \tag{1.3}$$

where ϕ_e is the so-called corrector defined as the unique weak solution of the whole-space equation

$$-\nabla \cdot A(\nabla\phi_e + e) = 0, \qquad \text{in } \mathbb{R}^d, \tag{1.4}$$

in the following class: ϕ_e is almost surely in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, satisfies the anchoring condition $\fint_{B(0)} \phi_e = 0$, and its gradient $\nabla\phi_e$ is a stationary random field with vanishing expectation $\mathbb{E}\left[\nabla\phi_e\right] = 0$ and finite second moments $\mathbb{E}\left[|\nabla\phi_e|^2\right] < \infty$; see [26, 35, 25]. Note that $\bar{A}$ is not explicit in general.

2. Dilute homogenization

2.1. **The Clausius–Mossotti formula.** Homogenization was neither born in the mathematical community in the 1970s, nor in the engineering community the decade before: it emerged much earlier, in the second half of the 19th century, in the physics community, in the context of two-phase dispersed media. Motivated by the works of Poisson [37] and Faraday [14], Mossotti and Clausius were the first to investigate the question of the effective dielectric constant of a homogeneous background material perturbed by sparse spherical inclusions [31, 32, 8]. The problem was largely revisited by Maxwell [30] for the effective conductivity of two-phase media; we refer to [29] for a detailed account of the historical context. In modern language, these authors argued that the effective conductivity (1.3) associated with the two-phase model (1.2) takes on the following guise, in space dimension $d \geq 1$, in case of isotropic conductivities $A_1 = \alpha \,\mathrm{Id}$ and $A_2 = \beta \,\mathrm{Id}$,

$$\bar{A} = \alpha \,\mathrm{Id} + \varphi \frac{\alpha d(\beta - \alpha)}{\beta + \alpha(d-1)} \,\mathrm{Id} + o(\varphi), \qquad \text{as } \varphi \downarrow 0, \tag{2.1}$$

where φ stands for the volume fraction of the inclusions, that is, by the ergodic theorem,

$$\varphi := \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})}\big] = \lim_{R\uparrow\infty} \tfrac{|\mathcal{B}(\mathcal{P})\cap RB|}{|RB|}, \qquad \text{for } \mathbb{P}\text{-almost all } \mathcal{P}. \tag{2.2}$$

In case of disjoint inclusions, we have $\varphi = \lambda|B|$ where λ is the intensity of the point process (defined by $\lambda|E| = \mathbb{E}\left[\mathcal{P}(E)\right]$ for any Borel set $E \subset \mathbb{R}^d$), while in general only the inequality $\varphi \leq \lambda|B|$ holds. The dilute approximation (2.1) for the effective medium is known as the Clausius–Mossotti formula and was soon adapted to various other physical settings, in particular by Lorenz and Lorentz for the effective refractive index of two-phase media in optics [28, 27], and by Einstein for the effective viscosity of a Stokes fluid with a dilute suspension of rigid particles [12, 13]. Einstein's result was actually part of his PhD thesis, where he used it to design a celebrated experiment to measure the Avogadro number; see e.g. the inspiring historical account in [38].

We start by describing a heuristic argument for (2.1). Recalling that $\mathbb{E}\left[\nabla\phi_e\right] = 0$, the effective conductivity (1.3) can be decomposed as

$$\begin{aligned} e \cdot \bar{A}e &= \mathbb{E}\left[e \cdot A(\nabla\phi_e + e)\right] \\ &= \mathbb{E}\left[e \cdot \big(\alpha + (\beta - \alpha)\mathbb{1}_{\mathcal{B}(\mathcal{P})}\big)(\nabla\phi_e + e)\right] \\ &= \alpha|e|^2 + (\beta - \alpha)\,\mathbb{E}\left[\mathbb{1}_{\mathcal{B}(\mathcal{P})}\, e \cdot (\nabla\phi_e + e)\right], \end{aligned}$$

and thus, by the ergodic theorem, assuming for simplicity that inclusions are almost surely disjoint,

$$e\cdot\bar{A}e \;=\; \alpha|e|^2+(\beta-\alpha)\lim_{R\uparrow\infty}\tfrac{1}{|RB|}\sum_{x\in\mathcal{P}\cap RB} e\cdot\int_{B(x)}(\nabla\phi_e+e). \tag{2.3}$$

In the dilute regime $\varphi\ll 1$, inclusions are typically far from one another and therefore do not 'interact' much when solving the corrector equation (1.4). More precisely, for all $x\in\mathcal{P}$, we may heuristically approximate the corrector in the inclusion $B(x)$ by the solution of a corresponding single-inclusion problem,

$$\nabla\phi_e|_{B(x)}\simeq\nabla\psi_e(\cdot-x), \tag{2.4}$$

where ψ_e is the unique weak solution in $\dot{H}^1(\mathbb{R}^d)$ of the single-inclusion equation

$$-\nabla\cdot\big(\alpha+(\beta-\alpha)\mathbb{1}_{B(0)}\big)(\nabla\psi_e+e)=0,\qquad\text{in }\mathbb{R}^d.$$

In the present case of spherical inclusions and isotropic conductivity, this equation is explicitly solvable in form of

$$\nabla\psi_e(x)=\begin{cases}Ke, & \text{for } |x|<1;\\ \frac{K}{|x|^d}\left(e-d\frac{x\cdot e}{|x|}\frac{x}{|x|}\right), & \text{for } |x|>1;\end{cases} \tag{2.5}$$

with

$$K\;=\;\frac{\alpha-\beta}{\beta+\alpha(d-1)}.$$

Inserting this form into (2.4) and (2.3), the Clausius–Mossotti formula (2.1) heuristically follows.

2.2. **Main result.** The aim of the present note is to prove the Clausius–Mossotti formula (2.1) in the most general setting possible and to establish a sharp error bound. The above heuristic argument indicates that the error mostly comes from 'interactions' between inclusions, as it amounts to locally replacing the corrector by solutions of single-inclusion problems, cf. (2.4). To quantify this error, we need to recall the notion of *second-order intensity* λ_2 of the point process, which we introduced in [11]. For that purpose, we first define the minimal lengthscale $\ell\ge 0$ of the point process,

$$\ell\;:=\;\inf_{\substack{x,y\in\mathcal{P}\\x\neq y}}|x-y|_\infty, \tag{2.6}$$

which is (almost surely) deterministic by ergodicity. In case $\ell>0$ (that is, if the point process is hardcore), the *second-order intensity* is defined as

$$\lambda_2\;:=\;\sup_{z_1,z_2\in\mathbb{R}^d}\mathbb{E}\bigg[\sum_{\substack{x_1,x_2\in\mathcal{P}\\x_1\neq x_2}}\ell^{-d}\,\mathbb{1}_{Q_\ell(z_1)}(x_1)\,\ell^{-d}\,\mathbb{1}_{Q_\ell(z_2)}(x_2)\bigg], \tag{2.7}$$

where $Q_r(z):=rQ+z$ is the cube of sidelength r centered at z. Note that, by definition (2.6), each cube $Q_\ell(z)$ contains at most one point of $\mathcal{P}$ almost surely. In other words, λ_2 is the maximum expected number of couples of points that lie in the ℓ-neighborhood of a given element of $(\mathbb{R}^d)^2$, properly normalized by ℓ. Alternatively, recalling that the 2-*point density* is the non-negative function f_2 defined by the following relation,

$$\mathbb{E}\bigg[\sum_{\substack{x_1,x_2\in\mathcal{P}\\x_1\neq x_2}}\zeta(x_1,x_2)\bigg]\;=\;\int_{(\mathbb{R}^d)^2}\zeta f_2,\qquad\text{for all }\zeta\in C_c((\mathbb{R}^d)^2),$$

the definition (2.7) of 2-point intensity can be reformulated as

$$\lambda_2 \;=\; \sup_{z_1,z_2\in\mathbb{R}^d} \fint_{Q_\ell(z_1)\times Q_\ell(z_2)} f_2. \tag{2.8}$$

In case $\ell = 0$ (that is, if the point process is not hardcore), this definition is naturally extended to $\lambda_2 := \|f_2\|_{\mathrm{L}^\infty((\mathbb{R}^d)^2)}$. For a Poisson point process, due to the tensor structure, the 2-point intensity is simply the square of the intensity, $\lambda_2 = \lambda^2$, but for a general strongly-mixing point process it can be anything in the interval $[\lambda^2, \ell^{-d}\lambda]$ and its smallness describes some form of local independence.

We can now state the main result of this note, which is an adaptation of our recent work on Einstein's formula [11, Theorem 1] to the effective conductivity problem. It proves the validity of the Clausius–Mossotti formula (2.1) for the first time in the setting of general point processes. The error bound (2.9) below is new and is sharp in general, cf. [11, Theorem 7]. As $\varphi \le \lambda|B|$, it entails that the approximation (2.1) is only valid in general provided that

$$\lambda_2 \log\big(2+\tfrac{\lambda}{\lambda_2(1+\ell)^d}\big) \;=\; o(\varphi),$$

which we interpret as a local independence condition. In the specific case of a Poisson point process, this result follows from [10] with the improved error bound $O(\lambda_2) = O(\lambda^2)$ (without logarithmic correction). The particular case of dilute point processes obtained by Bernoulli deletion or by dilation of a given process was already treated in [9, 36]. Note that inclusions here are allowed to overlap and that no upper bound is assumed on the number of points per unit volume.

Theorem 2.1. *Consider a stationary and ergodic point process and the associated set of unit spherical inclusions, cf. (1.1). Given $\alpha, \beta > 0$, the effective conductivity (1.3) associated with the two-phase model (1.2) with isotropic conductivities $A_1 = \alpha\,\mathrm{Id}$ and $A_2 = \beta\,\mathrm{Id}$ then satisfies the following quantitative version of (2.1),*

$$\Big|\bar A - \Big(\alpha\,\mathrm{Id} + \varphi\frac{\alpha d(\beta-\alpha)}{\beta+\alpha(d-1)}\,\mathrm{Id}\Big)\Big| \;\lesssim\; \lambda_2 \log\big(2+\tfrac{\lambda}{\lambda_2(1+\ell)^d}\big). \tag{2.9}$$

$\diamondsuit$

We shall prove this result in the following slightly more general form, where conductivities A_1 and A_2 are no longer assumed to be isotropic (not even symmetric) and where A_2 may itself be heterogeneous. We consider spherical inclusions for notational convenience, but we emphasize that, as in [9, 11], this is not essential (only the above explicit form of the Clausius–Mossotti formula then needs to be changed). Note that the argument also applies to the case of strongly elliptic systems as in [9, Corollary 2.5].

Theorem 2.2. *Let $A_1 \in \mathbb{R}^{d\times d}$ be a strictly elliptic (non-necessarily symmetric) matrix, and let A_2 be a stationary and ergodic random field of uniformly elliptic and uniformly bounded (non-necessarily symmetric) matrices. Consider a stationary and ergodic point process that is independent of A_2, consider the associated set of unit spherical inclusions, cf. (1.1), and the associated coefficient field*

$$A(y) := A_1 + (A_2(y) - A_1)\mathbb{1}_{\mathcal{B}(\mathcal{P})}(y), \qquad y\in\mathbb{R}^d, \tag{2.10}$$

where $\mathcal{P}$ is sampled according to the point process. Then, the effective coefficient $\bar A$ associated via (1.3) satisfies the following expansion,

$$\big|\bar A - (A_1 + \varphi\widehat{A}_2)\big| \;\lesssim\; \lambda_2 \log\big(2+\tfrac{\lambda}{\lambda_2(1+\ell)^d}\big), \tag{2.11}$$

where the first-order effective correction $\widehat{A}_2$ is given by

$$\widehat{A}_2 e := \mathbb{E}\bigg[\fint_B (A_2 - A_1)(\nabla \psi_e + e) \bigg], \tag{2.12}$$

where ψ_e is the unique weak solution in $\dot{H}^1(\mathbb{R}^d)$ of the single-inclusion problem

$$-\nabla \cdot \big(A_1 + \mathbb{1}_B (A_2 - A_1)\big)(\nabla \psi_e + e) = 0, \qquad \text{in } \mathbb{R}^d. \tag{2.13}$$

$\Diamond$

2.3. **Previous contributions.** The asymptotic analysis of the effective conductivity in case of a periodic array of inclusions with a small volume fraction was first addressed by Berdichevskiĭ [7]; see also [25, Section 1.7]. The first justification of the Clausius–Mossotti formula in a random setting is due to Almog in dimension $d = 3$, whose results in [1, 2] precisely yield (2.1) when combined with elementary homogenization theory. The proof is based on (scalar) potential theory and crucially relies on the facts that the space dimension is $d = 3$, that A is everywhere isotropic, and that the inclusions are spherical and disjoint. Another contribution is due to Mourrat [33], who studied for all $d \geq 2$ a discrete elliptic equation (instead of a continuum one) with sparse i.i.d. perturbations of the conductivity, proving (2.1) in that setting by strongly relying on quantitative stochastic homogenization results of [21, 22]. We also highlight the inspiring work [4, 5] by Anantharaman and Le Bris, who obtained related results on sparse i.i.d. perturbations of a periodic array of inclusions; see also [3]. Those different previous results were improved in [9], where we studied the case of a general stationary and ergodic inclusion process, focusing on a dilute regime obtained by a Bernoulli deletion procedure where each inclusion is preserved independently with low probability, and where we established real analyticity with respect to the Bernoulli parameter — only assuming that the number of inclusions per unit volume be uniformly bounded. This was recently extended in [20, 10] to prove Gevrey regularity when starting from a Poisson point process (for which the uniform boundedness assumption fails). In a different vein, partly inspired by Berdichevskiĭ's approach in the periodic setting, Pertinand addressed in [36] the case when the dilution of the point process is obtained by dilating a given hardcore process, and he proved the real analyticity with respect to the inverse of the dilation parameter. Very recently, Gérard-Varet [16] proposed an alternative approach where he bypasses homogenization theory and directly quantifies in terms of the volume fraction φ the distance between the solution of a problem with sparse inclusions and that of an effective problem with conductivity given by the Clausius–Mossotti formula. Various related contributions concern the validity of Einstein's formula for the effective viscosity of a Stokes fluid with a dilute suspension of rigid particles; see [23, 24, 34, 18, 11]. Our approach in the present note is an adaptation of the recent short proof that we obtained in [11] for Einstein's formula: it allows to justify the Clausius–Mossotti formula for the first time in the setting of general point processes and to determine the optimal error estimate. Note that several points of the proof simplify in the present setting. In particular we manage to fully bypass the variational formulation of [11].

While the Clausius–Mossotti formula is universal in the sense that it only depends on the set of inclusions via its volume fraction (and on the shape of the inclusions, here assumed to be spherical), the next-order correction further depends on the two-point correlation function. The identification of this correction was first discussed in [9], and it has been the object of many recent contributions in the context of Einstein's formula [6, 17, 15, 19, 11]; we refer in particular to our recent work [11] where all higher-order corrections are systematically described in form of a cluster expansion.

3. Proof of Theorem 2.2

We denote by $\mathcal{P} = \{x_n\}_n$ the point set sampled according to the underlying random point process. In order to justify the approximation (2.4) of the corrector in terms of single-inclusion problems, we start by singling out clusters of intersecting inclusions: as we focus here on spherical inclusions with unit radius, cf. (1.1), we note that the inclusion at a point $x_n \in \mathcal{P}$ does not intersect any other inclusion if and only if

$$\rho_n := \tfrac{1}{2} \inf_{m:m\neq n} |x_n - x_m| \geq 1.$$

Let then $\mathcal{S} := \{n : \rho_n \geq 1\}$ be the set of indices corresponding to non-intersecting inclusions. Note that $\mathcal{S}$ or its complement can be empty. We shall also use the short-hand notation

$$\tilde{\mathcal{B}}(\mathcal{P}) := \textstyle\bigcup_{n\in\mathcal{S}} B(x_n)$$

for the union of non-intersecting inclusions, while we recall that the union of all inclusions is denoted by $\mathcal{B}(\mathcal{P}) = \bigcup_n B(x_n)$. Next, in order to define suitable neighborhoods of the inclusions, we consider the Voronoi tessellation $\{V_n\}_n$ associated with the point set $\mathcal{P} = \{x_n\}_n$, that is,

$$V_n := \Big\{ z \in \mathbb{R}^d : |z - x_n| < \inf_{m:m\neq n} |z - x_m| \Big\}, \tag{3.1}$$

and we then partition the whole space as

$$\mathbb{R}^d = W \cup \bigcup_{n\in\mathcal{S}} V_n, \qquad W := \mathbb{R}^d \setminus \bigcup_{n\in\mathcal{S}} V_n.$$

We shall repeatedly use the following elementary property of Voronoi tessellations, which allows to split expectations into integrals over the different Voronoi cells, see [11, proof of Lemma 2.5]: for all stationary random fields ζ with $\mathbb{E}[|\zeta|] < \infty$ (where stationarity is understood to hold jointly with the point process), we have

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|} \int_{V_n} \zeta \bigg] + \mathbb{E}\big[\mathbb{1}_W \zeta\big]. \tag{3.2}$$

With this notation at hand, we now turn to the proof of Theorem 2.2, which we split into six steps. In the spirit of the heuristic argument in Section 2.1, we start by reducing the problem to estimating in each inclusion the difference between the corrector (1.4) and the solution of the corresponding single-inclusion corrector problem. When inclusions are well-separated, which is typically the case in the dilute regime, we naturally expect this difference to be small. As an intermediate step in the estimate, it is convenient to first replace single-inclusion corrector problems by the corresponding Dirichlet problems in each Voronoi cell. The estimate relies on elliptic regularity theory in form of a mean-value property for the single-particle problem, and is quantified in terms of the distance to other particles — or equivalently, in terms of the inner radius of the Voronoi cell. Let $e \in \mathbb{R}^d$ be fixed with $|e| = 1$.

Step 1. Representation formula for the error: proof that

$$\big|\bar Ae-(A_1e+\varphi\widehat A_2e)\big|\lesssim\mathbb{E}\bigg[\sum_{n\in\mathcal S}\mathbb{1}_{B(x_n)}|\nabla(\phi_e-\psi_{e,n})|\bigg]+\mathbb{E}\big[\mathbb{1}_{\mathcal B(\mathcal P)\setminus\tilde{\mathcal B}(\mathcal P)}\big(1+|\nabla\phi_e|\big)\big],\quad(3.3)$$

where $\widehat A_2$ is given by (2.12) in the statement, and where for all n we define $\psi_{e,n}$ as the unique weak solution in $\dot H^1(\mathbb{R}^d)$ of the whole-space single-inclusion problem centered at x_n, that is,

$$-\nabla\cdot\big(A_1+\mathbb{1}_{B(x_n)}(A_2-A_1)\big)(\nabla\psi_{e,n}+e)=0,\qquad\text{in }\mathbb{R}^d.$$

Recall that, as in the statement, we also denote by ψ_e the solution of the corresponding single-inclusion problem with center x_n replaced by the origin 0, cf. (2.13).

By definition of the two-phase coefficient field A, cf. (2.10), and of the associated effective conductivity $\bar A$, cf. (1.3), we find

$$\begin{aligned}\bar Ae&=\mathbb{E}\left[A(\nabla\phi_e+e)\right]\\&=\mathbb{E}\left[A_1(\nabla\phi_e+e)\right]+\mathbb{E}\left[\mathbb{1}_{\mathcal B(\mathcal P)}(A_2-A_1)(\nabla\phi_e+e)\right]\\&=A_1e+\mathbb{E}\left[\mathbb{1}_{\mathcal B(\mathcal P)}(A_2-A_1)(\nabla\phi_e+e)\right],\end{aligned}$$

where the last identity follows from the fact that A_1 is constant and $\mathbb{E}\left[\nabla\phi_e\right]=0$. Focussing on the contribution of non-intersecting inclusions and comparing the corrector ϕ_e to single-inclusion solutions $\{\psi_{e,n}\}_n$, we can decompose

$$\begin{aligned}\bar Ae\;=\;&A_1e+\mathbb{E}\bigg[\sum_{n\in\mathcal S}\mathbb{1}_{B(x_n)}(A_2-A_1)(\nabla\psi_{e,n}+e)\bigg]\\&+\mathbb{E}\bigg[\sum_{n\in\mathcal S}\mathbb{1}_{B(x_n)}(A_2-A_1)\nabla(\phi_e-\psi_{e,n})\bigg]+\mathbb{E}\left[\mathbb{1}_{\mathcal B(\mathcal P)\setminus\tilde{\mathcal B}(\mathcal P)}(A_2-A_1)(\nabla\phi_e+e)\right].\quad(3.4)\end{aligned}$$

In order to reformulate the second right-hand side term, we appeal to (3.2), to the effect of

$$\mathbb{E}\bigg[\sum_{n\in\mathcal S}\mathbb{1}_{B(x_n)}(A_2-A_1)(\nabla\psi_{e,n}+e)\bigg]=\mathbb{E}\bigg[\sum_{n\in\mathcal S}\frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\int_{B(x_n)}(A_2-A_1)(\nabla\psi_{e,n}+e)\bigg],$$

which can be further rewritten as follows, recalling that A_2 is stationary and independent of the point process,

$$\begin{aligned}\mathbb{E}\bigg[\sum_{n\in\mathcal S}\mathbb{1}_{B(x_n)}(A_2-A_1)(\nabla\psi_{e,n}+e)\bigg]&=\mathbb{E}\bigg[\sum_{n\in\mathcal S}\frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\bigg]\mathbb{E}\bigg[\int_B(A_2-A_1)(\nabla\psi_e+e)\bigg]\\&=\mathbb{E}\big[\mathbb{1}_{\tilde{\mathcal B}(\mathcal P)}\big]\mathbb{E}\bigg[\fint_B(A_2-A_1)(\nabla\psi_e+e)\bigg].\end{aligned}$$

Recognizing the definition (2.12) of $\widehat A_2e$, noting that the definition (2.2) of the volume fraction yields

$$\mathbb{E}\big[\mathbb{1}_{\tilde{\mathcal B}(\mathcal P)}\big]=\varphi-\mathbb{E}\big[\mathbb{1}_{\mathcal B(\mathcal P)\setminus\tilde{\mathcal B}(\mathcal P)}\big],$$

and using the energy estimate $\int_{\mathbb{R}^d}|\nabla\psi_e|^2\lesssim1$ for the solution of (2.13), we deduce

$$\bigg|\mathbb{E}\bigg[\sum_{n\in\mathcal S}\mathbb{1}_{B(x_n)}(A_2-A_1)(\nabla\psi_{e,n}+e)\bigg]-\varphi\widehat A_2e\bigg|\lesssim\mathbb{E}\big[\mathbb{1}_{\mathcal B(\mathcal P)\setminus\tilde{\mathcal B}(\mathcal P)}\big].$$

Combined with (3.4), this yields the claim (3.3).

Step 2. Approximation of the corrector by local Dirichlet problems.
Instead of directly comparing the corrector ϕ_e to whole-space single-inclusion solutions $\{\psi_{e,n}\}_n$ as in (3.3), we start by comparing to solutions of single-inclusion Dirichlet problems in each Voronoi cell. More precisely, for all $n \in \mathcal{S}$, we define $\psi^\circ_{e,n}$ as the unique weak solution in $H^1_0(V_n)$ of the single-inclusion problem

$$-\nabla \cdot A(\nabla \psi^\circ_{e,n} + e) = 0, \qquad \text{in } V_n. \tag{3.5}$$

Implicitly extending $\psi^\circ_{e,n}$ by zero outside V_n, we then set

$$\psi^\circ_e := \sum_{n\in\mathcal{S}} \psi^\circ_{e,n}.$$

By definition, $\nabla\psi^\circ_e$ is stationary and we claim that it has vanishing expectation and finite second moments,

$$\mathbb{E}\left[\nabla\psi^\circ_e\right] = 0, \qquad \mathbb{E}\left[|\nabla\psi^\circ_e|^2\right] \lesssim 1. \tag{3.6}$$

Indeed, for all $R>0$, applying (3.2) to $|\nabla\psi^\circ_e|^2 \wedge R$, we find

$$\mathbb{E}\left[|\nabla\psi^\circ_e|^2 \wedge R\right] \le \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|} \int_{V_n} |\nabla\psi^\circ_e|^2\bigg],$$

and thus, using energy estimates for ψ°_e in Voronoi cells, and further applying (3.2) to the constant function 1,

$$\mathbb{E}\left[|\nabla\psi^\circ_e|^2 \wedge R\right] \lesssim \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}|V_n|\bigg] \le 1.$$

By the monotone convergence theorem, this proves the claim $\mathbb{E}\left[|\nabla\psi^\circ_e|^2\right] \lesssim 1$. Next, we can apply (3.2) to $\nabla\psi^\circ_e$, to the effect of

$$\mathbb{E}\left[\nabla\psi^\circ_e\right] = \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|} \int_{V_n} \nabla\psi^\circ_e\bigg].$$

The right-hand side vanishes due to homogeneous Dirichlet boundary conditions, and the claim (3.6) follows.

Step 3. Approximation error estimate: proof that

$$\mathbb{E}\left[|\nabla(\phi_e - \psi^\circ_e)|^2\right] + \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})}|\nabla(\phi_e-\psi^\circ_e)|\big] \lesssim \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\rho_n^{-d}\bigg] + \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}\big]. \tag{3.7}$$

We start by proving the estimate on $\mathbb{E}\left[|\nabla(\phi_e-\psi^\circ_e)|^2\right]$. Using the corrector equation (1.4) for ϕ_e in form of

$$\mathbb{E}\left[\nabla(\phi_e-\psi^\circ_e)\cdot A(\nabla\phi_e+e)\right] = 0,$$

we find

$$\begin{aligned}\mathbb{E}\left[|\nabla(\phi_e-\psi^\circ_e)|^2\right] &\lesssim \mathbb{E}\left[\nabla(\phi_e-\psi^\circ_e)\cdot A\nabla(\phi_e-\psi^\circ_e)\right]\\ &= -\mathbb{E}\left[\nabla(\phi_e-\psi^\circ_e)\cdot A(\nabla\psi^\circ_e+e)\right],\end{aligned}$$

which we can further decompose into

$$\begin{aligned}\mathbb{E}\left[|\nabla(\phi_e-\psi^\circ_e)|^2\right] \lesssim{}& -\mathbb{E}\big[\nabla(\phi_e-\psi^\circ_e)\cdot Ae\big]\\ &+\mathbb{E}\big[(\nabla\psi^\circ_e+e)\cdot A\nabla\psi^\circ_e\big] - \mathbb{E}\big[(\nabla\phi_e+e)\cdot A\nabla\psi^\circ_e\big].\end{aligned}$$

As A_1 is constant and as $\mathbb{E}\left[\nabla(\phi_e - \psi_e^\circ)\right] = 0$, the first right-hand side term is equal to

$$\mathbb{E}\big[\nabla(\phi_e - \psi_e^\circ)\cdot Ae\big] \,=\, \mathbb{E}\big[\nabla(\phi_e - \psi_e^\circ)\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)e\big],$$

hence

$$\begin{aligned}\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \,\lesssim\, &-\mathbb{E}\big[\nabla(\phi_e - \psi_e^\circ)\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)e\big]\\ &\qquad+\mathbb{E}\big[(\nabla\psi_e^\circ+e)\cdot A\nabla\psi_e^\circ\big]-\mathbb{E}\big[(\nabla\phi_e+e)\cdot A\nabla\psi_e^\circ\big].\end{aligned} \tag{3.8}$$

If A was symmetric, then the corrector equation (1.4) would ensure that the last right-hand side term vanishes. Using that A_1 is constant, we shall show that, even though this term does not vanish in the general non-symmetric case, it can be localized inside inclusions. For that purpose, we rewrite the corrector equation as

$$-\nabla\cdot A_1\nabla\phi_e \,=\, \nabla\cdot Ae + \nabla\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla\phi_e, \qquad \text{in } \mathbb{R}^d,$$

and we note that, as the coefficient A_1 is constant, it can be replaced by its transpose A_1^T in the left-hand side,

$$-\nabla\cdot A_1^T\nabla\phi_e \,=\, \nabla\cdot Ae + \nabla\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla\phi_e, \qquad \text{in } \mathbb{R}^d.$$

Testing this equation with ψ_e° then yields

$$\mathbb{E}\big[\nabla\psi_e^\circ\cdot A_1^T\nabla\phi_e\big] \,=\, -\mathbb{E}\big[\nabla\psi_e^\circ\cdot Ae\big]-\mathbb{E}\big[\nabla\psi_e^\circ\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla\phi_e\big],$$

or equivalently, adding and subtracting several terms,

$$\begin{aligned}\mathbb{E}\big[(\nabla\phi_e+e)\cdot A\nabla\psi_e^\circ\big] \,=\, &\,\mathbb{E}\big[e\cdot A\nabla\psi_e^\circ\big]-\mathbb{E}\big[\nabla\psi_e^\circ\cdot Ae\big]\\ &+\mathbb{E}\big[\nabla(\phi_e-\psi_e^\circ)\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla\psi_e^\circ\big]-\mathbb{E}\big[\nabla\psi_e^\circ\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla(\phi_e-\psi_e^\circ)\big].\end{aligned}$$

Inserting this into (3.8), we get

$$\begin{aligned}\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \,\lesssim\, &\,\mathbb{E}\big[\nabla\psi_e^\circ\cdot A(\nabla\psi_e^\circ+e)\big]-\mathbb{E}\big[\nabla(\phi_e-\psi_e^\circ)\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)e\big]\\ &+\mathbb{E}\big[\nabla\psi_e^\circ\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla(\phi_e-\psi_e^\circ)\big]\\ &-\mathbb{E}\big[\nabla(\phi_e-\psi_e^\circ)\cdot \mathbb{1}_{\mathcal{B}(\mathcal{P})}(A_2-A_1)\nabla\psi_e^\circ\big].\end{aligned} \tag{3.9}$$

We note that the first right-hand side term vanishes: indeed, appealing to (3.2), we find

$$\mathbb{E}\big[\nabla\psi_e^\circ\cdot A(\nabla\psi_e^\circ+e)\big] \,=\, \mathbb{E}\bigg[\sum_{n\in\mathcal{S}}\frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\int_{V_n}\nabla\psi_e^\circ\cdot A(\nabla\psi_e^\circ+e)\bigg],$$

where for all $n\in\mathcal{S}$ the defining equation (3.5) for $\psi_e^\circ|_{V_n}=\psi_{e,n}^\circ$ precisely gives

$$\int_{V_n}\nabla\psi_e^\circ\cdot A(\nabla\psi_e^\circ+e) \,=\, 0.$$

The estimate (3.9) then leads us to

$$\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \,\lesssim\, \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})}\big(1+|\nabla\psi_e^\circ|\big)|\nabla(\phi_e-\psi_e^\circ)|\big].$$

Appealing to (3.2), together with the Cauchy–Schwarz inequality, this entails

$$\begin{aligned}\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \,\lesssim\, &\,\mathbb{E}\bigg[\sum_{n\in\mathcal{S}}\frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\Big(1+\int_{B(x_n)}|\nabla\psi_e^\circ|^2\Big)^{\frac12}\Big(\int_{B(x_n)}|\nabla(\phi_e-\psi_e^\circ)|^2\Big)^{\frac12}\bigg]\\ &+\mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}|\nabla(\phi_e-\psi_e^\circ)|\big].\end{aligned} \tag{3.10}$$

For all $n \in \mathcal{S}$, the equation (3.5) for $\psi_e^\circ|_{V_n} = \psi_{e,n}^\circ \in H_0^1(V_n)$ yields

$$\int_{V_n} |\nabla \psi_e^\circ|^2 \lesssim \int_{V_n} \nabla \psi_{e,n}^\circ \cdot A \nabla \psi_{e,n}^\circ = -\int_{V_n} \nabla \psi_{e,n}^\circ \cdot Ae,$$

and thus, as A_1 is constant, writing $A = A_1 + (A_2 - A_1)\mathbb{1}_{B(x_n)}$ in V_n,

$$\int_{V_n} |\nabla \psi_e^\circ|^2 \lesssim -\int_{B(x_n)} \nabla \psi_{e,n}^\circ \cdot (A_2 - A_1)e,$$

which leads to the energy estimate

$$\int_{V_n} |\nabla \psi_e^\circ|^2 \lesssim 1.$$

Inserting this into (3.10), we deduce

$$\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \lesssim \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\Big(\int_{B(x_n)} |\nabla(\phi_e - \psi_e^\circ)|^2\Big)^{\frac12}\bigg] \\ + \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}|\nabla(\phi_e - \psi_e^\circ)|\big]. \quad (3.11)$$

We now appeal to the following mean-value property, which we shall prove in Step 4 below,

$$\int_{B(x_n)} |\nabla(\phi_e - \psi_e^\circ)|^2 \lesssim \rho_n^{-d} \int_{V_n} |\nabla(\phi_e - \psi_e^\circ)|^2. \quad (3.12)$$

By Young's inequality, we then get for any $R > 0$,

$$\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \lesssim R\,\mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\rho_n^{-d}\bigg] + R\,\mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}\big] \\ + \tfrac{1}{R}\mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\int_{V_n} |\nabla(\phi_e - \psi_e^\circ)|^2\bigg] + \tfrac{1}{R}\mathbb{E}\big[\mathbb{1}_W|\nabla(\phi_e - \psi_e^\circ)|^2\big].$$

As (3.2) implies that the last two right-hand side terms are equal to $\frac{1}{R}\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right]$, choosing $R \simeq 1$ large enough to absorb it in the left-hand side, we obtain

$$\mathbb{E}\left[|\nabla(\phi_e - \psi_e^\circ)|^2\right] \lesssim \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\rho_n^{-d}\bigg] + \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}\big]. \quad (3.13)$$

To conclude the proof of the claim (3.7), it remains to establish the corresponding estimate on $\mathbb{E}\left[\mathbb{1}_{\mathcal{B}(\mathcal{P})}|\nabla(\phi_e - \psi_e^\circ)|\right]$. For that purpose, we start by appealing to (3.2) in form of

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{B}(\mathcal{P})}|\nabla(\phi_e - \psi_e^\circ)|\right] \\ = \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\int_{B(x_n)} |\nabla(\phi_e - \psi_e^\circ)|\bigg] + \mathbb{E}\left[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}|\nabla(\phi_e - \psi_e^\circ)|\right].$$

By the Cauchy–Schwarz inequality, this can be bounded by same right-hand side as in (3.11), and the claim (3.7) then follows similarly as (3.13).

Step 4. Proof of the mean-value property (3.12).
We reformulate (3.12) in the following form: for all $r \geq 1$, for all $v \in H^1(rB)$ that satisfies in the weak sense,

$$-\nabla \cdot \big(A_1 + \mathbb{1}_B(A_2 - A_1)\big)\nabla v = 0, \qquad \text{in } rB,$$

we have

$$\int_B |\nabla v|^2 \lesssim r^{-d} \int_{rB} |\nabla v|^2. \tag{3.14}$$

This kind of mean-value property is well-known to hold for harmonic functions, but it is not standard in the present non-homogeneous setting. To prove this result, we appeal to a perturbative argument and decompose the solution v as $v = v_1 + v_2$, where v_1 is the unique weak solution in $v + H_0^1(rB)$ of

$$-\nabla \cdot A_1 \nabla v_1 = 0, \qquad \text{in } rB,$$

and where v_2 is the unique weak solution in $H_0^1(rB)$ of

$$-\nabla \cdot \big(A_1 + \mathbb{1}_B(A_2 - A_1)\big) \nabla v_2 = \nabla \cdot \mathbb{1}_B (A_2 - A_1) \nabla v_1, \qquad \text{in } rB.$$

As A_1 is constant, we can apply to v_1 the standard mean-value property for harmonic functions, to the effect of

$$\int_B |\nabla v_1|^2 \lesssim r^{-d} \int_{rB} |\nabla v_1|^2.$$

As the defining Dirichlet problem for v_1 yields the energy estimate

$$\int_{rB} |\nabla v_1|^2 \lesssim \int_{rB} |\nabla v|^2,$$

we deduce

$$\int_B |\nabla v_1|^2 \lesssim r^{-d} \int_{rB} |\nabla v|^2. \tag{3.15}$$

In order to estimate v_2, we start from the corresponding energy estimate

$$\int_{rB} |\nabla v_2|^2 \lesssim \int_B |\nabla v_1|^2.$$

Combined with (3.15), this yields the claim (3.14) by the triangle inequality.

Step 5. Comparison of whole-space and of Dirichlet single-inclusion problems: proof that for all $n \in \mathcal{S}$ we have

$$\int_{B(x_n)} |\nabla(\psi_{e,n} - \psi_e^\circ)|^2 \lesssim \rho_n^{-2d}. \tag{3.16}$$

For $n \in \mathcal{S}$, the difference $\delta\psi_{e,n} := \psi_{e,n} - \psi_{e,n}^\circ$ satisfies $\delta\psi_{e,n}|_{V_n} \in \psi_{e,n} + H_0^1(V_n)$ and, in the weak sense,

$$-\nabla \cdot A \nabla \delta\psi_{e,n} = 0, \qquad \text{in } V_n. \tag{3.17}$$

The mean-value property (3.14) on $B_{\rho_n}(x_n) \subset V_n$ then yields

$$\int_{B(x_n)} |\nabla \delta\psi_{e,n}|^2 \lesssim \rho_n^{-d} \int_{B_{\rho_n}(x_n)} |\nabla \delta\psi_{e,n}|^2 \le \rho_n^{-d} \int_{V_n} |\nabla \delta\psi_{e,n}|^2. \tag{3.18}$$

Given a smooth cut-off χ_n such that

$$\chi_n|_{B_{\frac12 \rho_n}(x_n)} = 0, \qquad \chi_n|_{\mathbb{R}^d \setminus B_{\rho_n}(x_n)} = 1, \qquad |\nabla \chi_n| \lesssim \rho_n^{-1},$$

the Dirichlet problem (3.17) for $\delta\psi_{e,n}$ implies the energy estimate

$$\int_{V_n} |\nabla \delta\psi_{e,n}|^2 \lesssim \int_{V_n} |\nabla(\chi_n \psi_{e,n})|^2,$$

and thus, as the single-inclusion solution $\psi_{e,n}$ enjoys the same decay properties as in (2.5), we deduce

$$\int_{V_n} |\nabla \delta\psi_{e,n}|^2 \lesssim \rho_n^{-d}.$$

Combined with (3.18), this proves the claim (3.16).

Step 6. Conclusion.
Starting from (3.3) and comparing single-inclusion solutions to their Dirichlet version ψ_e°, we have

$$\begin{aligned}\big|\bar A e - (A_1 e + \varphi \widehat A_2 e)\big| \lesssim \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})}|\nabla(\phi_e - \psi_e^\circ)|\big] + \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}\big] \\ + \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \mathbb{1}_{B(x_n)}|\nabla(\psi_{e,n} - \psi_e^\circ)|\bigg].\end{aligned}$$

Using (3.7) to estimate the first right-hand side term, appealing to (3.2) together with (3.16) in form of

$$\begin{aligned}\mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \mathbb{1}_{B(x_n)}|\nabla(\psi_{e,n} - \psi_e^\circ)|\bigg] &= \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\int_{B(x_n)} |\nabla(\psi_{e,n} - \psi_e^\circ)|\bigg] \\ &\lesssim \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in B(x_n)}}{|B|}\rho_n^{-d}\bigg],\end{aligned}$$

and recalling $\rho_n \ge 1$ for $n \in \mathcal{S}$, we deduce

$$\big|\bar A e - (A_1 e + \varphi \widehat A_2 e)\big| \lesssim \mathbb{E}\bigg[\sum_n \mathbb{1}_{0\in B(x_n)}(1+\rho_n)^{-d}\bigg] + \mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}\big]. \tag{3.19}$$

It remains to evaluate the two right-hand side terms. For that purpose, we appeal to the following observation that we first made in [11], for which a short proof is included below: for any non-increasing function $g \in \mathrm{L}^\infty(\mathbb{R}^+)$ with $g(r) \downarrow 0$ as $r \uparrow \infty$, there holds

$$\mathbb{E}\bigg[\sum_n \mathbb{1}_{0\in B(x_n)} g(\rho_n)\bigg] \lesssim \int_{\frac12\ell}^\infty |g'(r)|\,\big((\lambda_2 r^d)\wedge\lambda\big)\,dr. \tag{3.20}$$

Applying this to $g(r) = (1+r)^{-d}$, we find

$$\begin{aligned}\mathbb{E}\bigg[\sum_n \mathbb{1}_{0\in B(x_n)}(1+\rho_n)^{-d}\bigg] &\lesssim \int_{\frac12\ell}^\infty (1+r)^{-d-1}\big((\lambda_2 r^d)\wedge\lambda\big)\,dr \\ &\lesssim (\lambda_2\wedge\lambda)\log\big(2 + \tfrac{\lambda}{\lambda_2(1+\ell)^d}\big).\end{aligned}$$

Applying it to $g(r) = \mathbb{1}_{r<1}$, we further get

$$\mathbb{E}\big[\mathbb{1}_{\mathcal{B}(\mathcal{P})\setminus\tilde{\mathcal{B}}(\mathcal{P})}\big] = \mathbb{E}\bigg[\sum_n \mathbb{1}_{0\in B(x_n)}\mathbb{1}_{\rho_n<1}\bigg] \lesssim \lambda_2\wedge\lambda.$$

Combined with (3.19), this concludes the proof of (2.11).

Finally, for completeness, we include a short proof of (3.20). For that purpose, we start by rewriting the left-hand side as

$$\mathbb{E}\bigg[\sum_n \mathbb{1}_{0\in B(x_n)} g(\rho_n)\bigg] = \int_0^\infty g(r)\,d\Lambda(r), \tag{3.21}$$

where the positive measure Λ on $\mathbb{R}^+$ is defined by its distribution function

$$\Lambda([0,r]) \,:=\, \mathbb{E}\bigg[\sum_n \mathbb{1}_{0\in B(x_n)}\mathbb{1}_{\rho_n\le r}\bigg] \,=\, \mathbb{E}\bigg[\sum_n \mathbb{1}_{|x_n|<1}\,\mathbb{1}_{\exists m\ne n:\,|x_m-x_n|\le 2r}\bigg].$$

By definition of the minimal length ℓ of the point process, cf. (2.6), note that $\Lambda([0,r])=0$ for $r<\frac12\ell$. Moreover, we can bound

$$\Lambda([0,r]) \,\le\, \mathbb{E}\bigg[\sum_n \mathbb{1}_{|x_n|<1}\bigg] \,=\, \lambda|B|,$$

and alternatively, for $r\ge\frac12\ell$, by definition of the second-order intensity, cf. (2.8),

$$\begin{aligned}\Lambda([0,r]) \,\le\, \mathbb{E}\bigg[\sum_{n\ne m} \mathbb{1}_{|x_n|<1}\,\mathbb{1}_{|x_m-x_n|\le 2r}\bigg] \,&=\, \iint_{B\times B_{2r}} f_2(x,x+y)\,dxdy\\ &=\,(2r)^{-d}\iint_{B_{2r}\times B_{2r}} f_2(x,x+y)\,dxdy \,\lesssim\, \lambda_2 r^d.\end{aligned}$$

Combining these estimates yields for all $r\ge0$,

$$\Lambda([0,r]) \,\lesssim\, (\lambda_2 r^d)\wedge\lambda. \tag{3.22}$$

Under our assumptions on g, an integration by parts yields

$$\int_0^\infty g(r)\,d\Lambda(r) \,=\, -g(0)\Lambda(\{0\}) + \int_0^\infty |g'(r)|\,\Lambda([0,r])\,dr.$$

As g is nonnegative, the first right-hand side term can be dropped and the claim (3.20) follows in combination with (3.21) and (3.22). ☐

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