Measure theory — Exercise part

MATH-F-3001

15 January 2018

Please justify all your statements carefully. Answers can be written in English or French.

Question 1. Let X be a set and let $\mu_1^*, \mu_2^* : 2^X \to [0, \infty]$ be two outer measures.

(a) Show that $\mu_1^* + \mu_2^*$ is also an outer measure.

(b) If $\mu_1^*(X) + \mu_2^*(X) < \infty$, also show that $\mathcal{M}(\mu_1^*) \cap \mathcal{M}(\mu_2^*) = \mathcal{M}(\mu_1^* + \mu_2^*)$.

(c) Can you find a counterexample for (b) if $\mu_1^*(X) = \infty$?

Question 2. Let $E \subset \mathbb{R}$ be Lebesgue-measurable and let $\delta > 0$. If $\lambda(E \cap (a, b)) \geq \delta(b - a)$ holds for all intervals $(a, b) \subset \mathbb{R}$, deduce that $\lambda(\mathbb{R} \setminus E) = 0$.
Hint : First deduce that $\lambda(E \cap A) \geq \delta\lambda(A)$ holds for all $A \subset \mathbb{R}$ Lebesgue-measurable.

Question 3. Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) := \begin{cases} 1, & \text{if } x \geq 0 \text{ and } x \leq y < x + 1; \\ -1, & \text{if } x \geq 0 \text{ and } x + 1 \leq y < x + 2\,; \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is Borel-measurable and that $\int_{\mathbb{R}}(\int_{\mathbb{R}} f(x, y)dy)dx \neq \int_{\mathbb{R}}(\int_{\mathbb{R}} f(x, y)dx)dy$. Does this contradict Fubini's theorem ?

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Prove that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x - h) - f(x)|\, dx = 0.$$

Hint : Use (without proof) the density of the set of continuous functions in $L^1(\mathbb{R})$.

Question 5. Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \leq p < \infty$, and $\epsilon > 0$.

(a) Let $f_n, f \in L^p(X, \mu)$ be such that $\|f_n\|_{L^p(X,\mu)} = 1 = \|f\|_{L^p(X,\mu)}$ for all $n \in \mathbb{N}$ and such that $f_n \to f$ μ-almost everywhere. Consider the probability measure ν defined by $\nu(A) := \int_A |f|^p d\mu$. Show that there exists $E \in \mathcal{A}$ such that $f_n/f \to 1$ uniformly on E and $\nu(X \setminus E) < \epsilon$.
Hint : Use (without proof) Egorov's theorem.

(b) For f_n, f, E as in (a), show that $\limsup_n \int_{X \setminus E} |f_n|^p d\mu < \epsilon$.

(c) For f_n, f, E as in (a), deduce from (a) and (b) that $f_n \to f$ in $L^p(X, \mu)$.

(d) If $g_n, g \in L^p(X, \mu)$ are such that $g_n \to g$ in μ-measure and $\|g_n\|_{L^p(X,\mu)} \to \|g\|_{L^p(X,\mu)}$, then $g_n \to g$ in $L^p(X, \mu)$.