Measure theory : exercises

Bachelor 3

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Chapter 1 : Measure spaces

Classes of subsets

1. Let Ω be a set, and let $\mathcal{F} \subset 2^\Omega$ be some set of subsets of Ω.
   (a) Assume that $\Omega \in \mathcal{F}$ and that $A \setminus B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$. Show that $\mathcal{F}$ is an algebra.
   (b) Assume that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed by complementation and by finite disjoint union. Show that $\mathcal{F}$ does not need to be an algebra.
2. Let Ω be a finite set of cardinality $\#\Omega = 2p$, $p \in \mathbb{N}$. Consider the collection
$$\mathcal{K} = \{A \subset \Omega : \#A = 2r \text{ for some } r \in \{0, \cdots, p\}\}.$$
   (a) Show that $\mathcal{K}$ is a λ-system.
   (b) For which values of p is $\mathcal{K}$ an algebra and/or a σ-algebra ?
3. Let Ω be a set, and let $\mathcal{F} \subset 2^\Omega$ be some set of subsets of Ω. Show that for any $B \in \sigma(\mathcal{F})$ there exists a countable subset $\mathcal{F}_B \subset \mathcal{F}$ such that $B \in \sigma(\mathcal{F}_B)$.
4. Let $A_1, \ldots, A_n$ be subsets of some set Ω. For all $\alpha \in \{0,1\}^n$, let
$$F(\alpha) = \bigcap_{i=1}^n A_i^{(\alpha_i)},$$
where $A_i^{(0)} := A_i$ and $A_i^{(1)} := A_i^c$. Then the collection $\{F(\alpha) : \alpha \in \{0,1\}^n\}$ is a partition of Ω and there holds
$$f(\{A_1, \ldots, A_n\}) = \Big\{ \bigcup_{\alpha \in J} F(\alpha) : J \subset \{0,1\}^n \Big\}.$$
Conclude that $\#f(\{A_1, \ldots, A_n\}) \le 2^{2^n}$. Write explicitly $f(\{A\})$, $f(\{A_1, A_2\})$, $f(\{A_1, A_2, A_3\})$. Similarly, given a countable collection $(A_n)_n$ of subsets of Ω, conclude that $f(\{A_n\}_{n=1}^\infty)$ is (at most) countable. Is it true that $\sigma((A_n)_n)$ is also always (at most) countable ?
5. Let $(A_n)_n$ be a sequence of subsets of some set Ω. Considering the construction of exercise 4, show that $\sigma((A_n)_n)$ is either finite or has cardinality at least that of the continuum. [1] In particular, if $\sigma((A_n)_n)$ is infinite, then it is uncountable.
6. Let $(A_n)_n$ be a countable partition of some set Ω. Show that
$$\sigma((A_n)_n) = \Big\{ \bigcup_{n \in J} A_n : J \subset \mathbb{N} \Big\}.$$
7. Let Ω be an countable set. Show that every σ-algebra $\mathcal{F}$ on Ω is generated by a partition of Ω as in exercise 6. Show that this fails in general if Ω is uncountable.

Set functions

8. Let $(\Omega, \mathcal{F})$ be a measurable space. The lim sup and the lim inf of a sequence $(C_n)_n \subset \mathcal{F}$ are defined as follows,
$$\liminf_{n\to\infty} C_n = \bigcup_{n\ge1} \bigcap_{k\ge n} C_k \qquad \text{and} \qquad \limsup_{n\to\infty} C_n = \bigcap_{n\ge1} \bigcup_{k\ge n} C_k.$$
Note that both clearly belong to $\mathcal{F}$. Given two sequences $(A_n)_n, (B_n)_n \subset \mathcal{F}$, show from these definitions that

1. It is actually either finite or has cardinality exactly that of the continuum. This can be shown by transfinite induction on the Borel hierarchy. In particular, the cardinality of the Borel σ-algebra on $\mathbb{R}^k$ is that of the continuum.

(a) $\limsup_n A_n \supset \liminf_n A_n$;

(b) $(\limsup_n A_n)^c = \liminf_n A_n^c$;

(c) $\limsup_n (A_n \cap B_n) \subset (\limsup_n A_n) \cap (\limsup_n B_n)$;

(d) $\limsup_n (A_n \cup B_n) = (\limsup_n A_n) \cup (\limsup_n B_n)$;

(e) $(\liminf_n A_n) \cup (\liminf_n B_n) \subset \liminf_n (A_n \cup B_n)$;

(f) $\liminf_n (A_n \cap B_n) = (\liminf_n A_n) \cap (\liminf_n B_n)$.

Give examples showing that the inclusions in items (c) and (e) can be strict. In addition,

(g) if $A_n \to A$ (that is, $\liminf_n A_n = A = \limsup_n A_n$) and $B_n \to B$, then show that

$$A_n \cup B_n \longrightarrow A \cup B \qquad \text{and} \qquad A_n \cap B_n \longrightarrow A \cap B;$$

(h) if μ is a measure on $\mathcal{F}$, show that

$$\mu(\liminf_n A_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

What can be said about $\mu(\limsup_n A_n)$?

9. Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a finite measure μ. Given a sequence $(A_n)_n \subset \mathcal{F}$, establish the following alternative (the so-called *Borel-Cantelli lemma*).
   (a) If $\sum_n \mu(A_n) < \infty$, then $\mu(\limsup_n A_n) = 0$.
   (b) If $\sum_n \mu(A_n) = \infty$ and if the sets A_n are independent, then $\mu(\limsup_n A_n) = \mu(\Omega)$.

10. Consider the σ-algebra 2^Ω on Ω, and for all $A \in 2^\Omega$ define

$$\mu(A) \quad := \quad \begin{cases} \#A, & \text{if } \#A < \infty; \\ \infty, & \text{if } A \text{ is infinite.} \end{cases}$$

Show that μ is a measure. It is called the *counting measure*. When is μ a finite measure? When is it σ-finite?

11. Let Ω be a set, consider the algebra $\mathcal{F} = \{A \subset \Omega : A \text{ or } A^c \text{ is finite}\}$, and for all $A \in \mathcal{F}$ define

$$\mu(A) \quad := \quad \begin{cases} 0, & \text{if } A \text{ is finite;} \\ \infty, & \text{if } A \text{ is infinite.} \end{cases}$$

Show that μ is an additive set function on $\mathcal{F}$ but is in general not a measure on $\mathcal{F}$. When is μ a measure on $\mathcal{F}$?

12. Let Ω be an uncountable set and let $\mathcal{F}$ be the collection of all subsets $A \subset \Omega$ such that either A or A^c is (at most) countable. Define

$$\mu(A) := \begin{cases} 0, & \text{if } A \text{ is (at most) countable;} \\ 1, & \text{if } A^c \text{ is (at most) countable.} \end{cases}$$

Show that $\mathcal{F}$ is a σ-algebra and that μ is a measure on $\mathcal{F}$.

13. A σ-algebra $\mathcal{F}$ is said to be *countably generated* if there exists a countable family $(A_n)_n \subset \mathcal{F}$ such that $\mathcal{F} = \sigma((A_n)_n)$. Show that a sub-$\sigma$-algebra of a countably generated σ-algebra does not need to be countably generated.
Hint: Show that the Borel σ-algebra $\mathcal{F}$ on $\mathbb{R}$ is countably generated, and consider the σ-algebra $\mathcal{F}'$ defined in exercice 12 (with $\Omega = \mathbb{R}$).

Extension theorem

14. Let μ be a finite measure on a ring $\mathcal{F}$. Given two sequences $(A_n)_n, (B_n)_n \subset \mathcal{F}$ with $\bigcup_n A_n, \bigcup_n B_n \in \mathcal{F}$ and with $B_n \subset A_n$ for all n, show that
$$\mu\Big(\bigcup_n A_n\Big) - \mu\Big(\bigcup_n B_n\Big) \leq \sum_n \left(\mu(A_n) - \mu(B_n)\right).$$
15. Let $\mathcal{F}$ be a semiring. Denote by $\mathcal{F}^+$ the collection of all finite disjoint unions of elements of $\mathcal{F}$. Show that $\mathcal{F}^+$ is a ring, and conclude that $\mathcal{F}^+$ is the smallest ring that contains $\mathcal{F}$. For that purpose, we may argue as follows :
    (a) Show that $\mathcal{F}^+$ is a π-system.
    (b) Write $A \cup B$ as a disjoint union $B \cup (A \setminus B)$. Note that $\mathcal{F}^+$ is closed by finite disjoint unions, so that it is enough to show that $A, B \in \mathcal{S}^+ \Rightarrow A \setminus B \in \mathcal{S}^+$.
16. For $i = 1, 2$, let Ω_i be a nonempty set and let $\mathcal{F}_i$ be a semiring on Ω_i. Define $\mathcal{F} := \{A_1 \times A_2 : A_i \in \mathcal{F}_i, i = 1, 2\}$. Show that $\mathcal{F}$ is a semiring on the product $\Omega_1 \times \Omega_2$.
17. Let $(\Omega, \mathcal{F})$ be a measurable space, and let μ_1 and μ_2 be two measures on this space. Let $\mathcal{S} \subset \mathcal{F}$ be a semiring that generates $\mathcal{F}$, that is, $\sigma(\mathcal{S}) = \mathcal{F}$. Assume that μ_1 and μ_2 are σ-finite on $\mathcal{S}$ and satisfy $\mu_1 \leq \mu_2$ on $\mathcal{S}$. Then show that $\mu_1 \leq \mu_2$ holds on $\mathcal{F}$. Give a counterexample without the σ-finiteness assumption.
18. Let $(\Omega, \mathcal{F})$ be a measurable space and let $(\mu_n^*)_n$ be a sequence of outer measures on $(\Omega, \mathcal{F})$. Show that $\sum_n \mu_n^*$ and $\sup_n \mu_n^*$ are also outer measures.
19. Let μ be a finite measure on $\mathcal{R}^k$, that is, the Borel σ-algebra on $\mathbb{R}^k$. Define
$$\mu_1(A) = \sup\{\mu(F) \,:\, F \subseteq A, F \,\text{closed}\}, \qquad \mu_2(A) = \inf\{\mu(G) \,:\, A \subseteq G, G \,\text{open}\}.$$
Show that $\mu = \mu_1 = \mu_2$ on $\mathcal{R}^k$. This proves that any finite Borel measure on $\mathbb{R}^k$ is *regular*.
20. Prove the following theorem due to Steinhaus : if $E \subset \mathbb{R}$ is Lebesgue measurable and has positive measure, then the difference set $E - E := \{x - y : x, y \in E\}$ contains a neighborhood of the origin.
Hint : Use the regularity of the Lebesgue measure (cf. exercice 19) to find an compact set K and an open set U such that $K \subset E \subset U$ and $\lambda(U) < 2\lambda(K)$. Consider $\varepsilon := d(K, U^c) > 0$ and use the translation invariance of the Lebesgue measure to show that $(-\varepsilon, \varepsilon) \subset K - K$.

Measurable functions

21. On the Borel measurable space $(\mathbb{R}, \mathcal{R})$, consider the functions f and g defined by
$$f(x) = \begin{cases} |\sin x| & \text{if } -\pi < x \leq \pi, \\ 1 & \text{if } 10 < x < 20, \\ 0 & \text{elsewhere}; \end{cases}$$
and
$$g(x) = \begin{cases} 1 - x^2 & \text{if } -1 \leq x \leq 1 \text{ and } x \notin \mathbb{Q}, \\ 0 & \text{elsewhere.} \end{cases}$$
Show that f and g are measurable.
22. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = f(x)$ for all $x \in \mathbb{R} \setminus D$, for some countable set $D \subset \mathbb{R}$. Show that g is also Borel measurable.

Pathological sets

23. Construct a subset of $[0,1)$ that is not Lebesgue measurable.[2] We proceed as follows (Vitali's counterexample)
   (a) Consider the relation $\sim$ on $[0,1)$ defined by
   $$x \sim y \text{ if and only if } x - y \in \mathbb{Q},$$
   and show that $\sim$ is an equivalence relation on $[0,1)$.
   (b) Consider the quotient space $\mathcal{A} = [0,1)/\!\sim$, and use the axiom of choice to select a function $\psi : \mathcal{A} \to [0,1)$ such that $\psi(\alpha) \in \alpha$ holds for all $\alpha \in \mathcal{A}$. Set $A := \psi(\mathcal{A})$, and show that
   $$\{A + q\,;\, q \in \mathbb{Q},\, |q| < 1\}$$
   is a family of disjoint subsets whose union contain $[0,1)$.
   (c) Show that the measurability of A would lead to a contradiction.

24. Construct a subset $E \subset \mathbb{R}$ such that any Lebesgue measurable set that is included in E or in E^c has measure 0. Can such a set E be Lebesgue measurable? Show that this implies that any set $B \subset \mathbb{R}^d$ with positive Lebesgue outer measure contains a set that is not Lebesgue measurable.
   For the construction of E, we proceed as follows.
   (a) Define
   $$G_0 := \{r + 2n\sqrt{2} : r \in \mathbb{Q}, n \in \mathbb{Z}\}, \qquad G_1 := \{r + (2n+1)\sqrt{2} : r \in \mathbb{Q}, n \in \mathbb{Z}\},$$
   and $G := G_0 \cup G_1 = \{r + n\sqrt{2} : r \in \mathbb{Q}, n \in \mathbb{Z}\}$. Check that G and G_0 are subgroups of $(\mathbb{R}, +)$, that $G_0 \cap G_1 = \varnothing$, and that $G_1 = \sqrt{2} + G_0$.
   (b) Consider the relation $\sim$ on $\mathbb{R}$ defined by
   $$x \sim y \text{ if and only if } x - y \in G,$$
   and show that $\sim$ is an equivalence relation on $\mathbb{R}$.
   (c) Consider the quotient space $\mathcal{A} = \mathbb{R}/\sim$, and use the axiom of choice to select a function $\psi : \mathcal{A} \to \mathbb{R}$ such that $\psi(\alpha) \in \alpha$ holds for all $\alpha \in \mathcal{A}$. Set $A := \psi(\mathcal{A})$ and $E := A + G_0$. Show that any Lebesgue measurable subset of E has measure 0.
   Hint : Suppose $B \subset E$ is Lebesgue measurable with $\lambda(B) > 0$. Use the Steinhaus theorem (cf. exercise 20) to find $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset E - E$. Due to the density of G_1 in $\mathbb{R}$, show that this would lead to a contradiction.
   (d) Show that $E^c = A + G_1 = \sqrt{2} + E$, and deduce that any Lebesgue measurable subset $B \subset E^c$ is also of measure 0.

25. The triadic Cantor set C is the subset of $[0,1]$ that remains after having removed successively the interval $(\frac{1}{3}, \frac{2}{3})$, then the 2 intervals $(\frac{1}{3^2}, \frac{2}{3^2})$ and $(\frac{7}{3^2}, \frac{8}{3^2})$, then the 4 intervals
   $$\left(\frac{1}{3^3}, \frac{2}{3^3}\right), \quad \left(\frac{7}{3^3}, \frac{8}{3^3}\right), \quad \left(\frac{19}{3^3}, \frac{20}{3^3}\right), \quad \left(\frac{25}{3^3}, \frac{26}{3^3}\right),$$
   and so on.
   (a) Give a more formal definition of C.
   (b) Show that C is compact.

2. What the construction shows is that the axiom of choice implies the existence of a non Lebesgue measurable set. Moreover, it was essentially proven by Solovay that the failure of the axiom of choice is consistent with the measurability of all subsets of $\mathbb{R}$, meaning that it is not possible to construct a non Lebesgue measurable set without using some part of the axiom of choice.

(c) Show that C has zero Lebesgue measure.

(d) Let $x \in [0,1)$. Show that there exists a sequence $(\alpha_n)_n \subset \{0,1,2\}$, such that all α_n's are not all equal to 2 after some index, and such that

$$x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Set $x = 0 \cdot \alpha_1 \alpha_2 \dots$ (this is the triadic representation of x).

(e) Show that $C \supset \{x \in [0,1)\,;\, x = 0 \cdot \alpha_1\alpha_2 \dots \text{ with } \alpha_n \in \{0,2\}\}$. Deduce that C is uncountable.

26. Define as follows a function $f : [0,1) \to C$ (where C denotes the Cantor set introduced in exercise 25). If

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n},$$

with $a_n \in \{0,1\}$ for all n, where all a_n's are not equal to 1 after some index, we define

$$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}.$$

(a) Show that f is not continuous, but is (strictly) increasing.

(b) Let $A \subset [0,1)$ be non Lebesgue measurable. Show that $f(A)$ is Lebesgue measurable but not Borel measurable.

27. The Smith-Volterra-Cantor (SVC) set is the subset E of $[0,1]$ obtained as follows : first remove the middle $1/4 = 1/2^2$, so the remaining set is $[0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$, then remove the middle $1/2^4$ of each remaining subinterval, so the remaining set is

$$\left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right],$$

then remove the middle $1/2^6$ of each remaining subinterval, and so on.

(a) Give a more formal definition of E.

(b) Show that E is compact.

(c) Show that E has Lebesgue measure $1/2 > 0$. (In passing, this implies that E has Hausdorff dimension 1.)

(d) Although the set E has positive measure, show that it is nowhere dense (that is, the interior of the closure of E is empty), which means that E contains no interval !

(e) Show that the bounded function $\mathbb{1}_E$ is Lebesgue integrable, but is not Riemann integrable, and is in addition not equivalent (that is, equal Lebesgue-a.e.) to any Riemann integrable function.

(f) Consider the generalization of the definition of E consisting in removing at step n the middle r_n fraction of each remaining subinterval (while E corresponds to the choice $r_n = 2^{-2n}$). Under what condition on the sequence $(r_n)_n$ has the constructed set a positive Lebesgue measure ?