Measure theory : exercises
Bachelor 3
Academic year 2017-2018

Chapter 2 : Integration

Measurable functions

1. Let $(\Omega, \mathcal{F})$ be a measurable space. Assume that $f_1, f_2 : \Omega \to [0, \infty]$ are $\mathcal{F}$-measurable functions. Show that this also holds for $f_1 + f_2$, for $f_1 f_2$, and for $\log(1 + |f_1| e^{f_2})$.
2. Let $(\Omega, \mathcal{F})$ be a measurable space and let $A \subset \Omega$ be any subset.
   (a) Let $\mathcal{F}_A := \{A \cap B : B \in \mathcal{F}\}$ and show that this defines a σ-algebra on A.
   (b) Show that if $f : \Omega \to [-\infty, \infty]$ is $\mathcal{F}$-measurable then the restriction $f|_A : A \to [-\infty, \infty]$ is $\mathcal{F}_A$-measurable.
   (c) Let $g : A \to [-\infty, \infty]$ a $\mathcal{F}_A$-measurable function. Show that there exists an $\mathcal{F}$-measurable extension $\hat{g} : \Omega \to [-\infty, \infty]$ of g on Ω.

Lebesgue's integral

3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For all $A \in \mathcal{A}$ and all measurable functions $f : \Omega \to [0, \infty]$, we recall
$$\int_A f \, d\mu := \sup \left\{ \int_A s \, d\mu \; : \; s \text{ simple and } \mathcal{F}\text{-measurable with } 0 \leq s \leq f \right\}.$$
Show that the following properties hold for all measurable functions f and g and all $A, B \in \mathcal{F}$:
   (a) $A \subset B$ and $f \geq 0 \Rightarrow \int_A f \, d\mu \leq \int_B f \, d\mu$;
   (b) $f \geq 0$ and $c \in [0, \infty] \Rightarrow \int_A cf \, d\mu = c \int_A f \, d\mu$;
   (c) $f(x) = 0$, $\forall x \in A \Rightarrow \int_A f \, d\mu = 0$, even if $\mu(A) = \infty$;
   (d) $\mu(A) = 0$, $f \geq 0 \Rightarrow \int_A f \, d\mu = 0$, even if $f(x) = \infty$, $\forall x \in A$;
   (e) $f \geq 0 \Rightarrow \int_A f \, d\mu = \int_\Omega \chi_A f \, d\mu$; deduce that for $A, B \in \mathcal{A}$ and $A \cap B = \varnothing$ there holds $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$.
4. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ denotes the counting measure. Show that any function $f : \mathbb{N} \to [0, \infty]$ is semi-integrable and that
$$\int_{\mathbb{N}} f d\mu = \sum_{n \geq 1} f(n).$$
5. Consider the functions
$$f_1(x) = \begin{cases} +\infty & \text{if } x = 0, \\ \ln |x| & \text{if } 0 < |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{x^2 - 1} & \text{if } |x| \neq 1, \\ 20 & \text{if } |x| = 1, \end{cases} \qquad f_3(x) \equiv 1.$$
Determine whether these functions are integrable on $(\mathbb{R}, \mathcal{R})$ with respect to the measure m, in each of the following two cases, and if possible compute the value of the integrals.
   (a) $m = \lambda$ is the Lebesgue measure ;
   (b) m is defined by
$$m(B) = \sum_{n \in B \cap \mathbb{Z}} \frac{1}{1 + (n+1)^2}$$
   for all $B \in \mathcal{R}$.

6. Let $f, g : [0,1] \to \mathbb{R}^+$ be Lebesgue-integrable functions. Show that $f+g$ is also Lebesgue-integrable, but that fg is in general not.
7. Let $f \in L^1(\mu)$. Show that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\int_A |f|\, d\mu < \epsilon$ if $\mu(A) < \delta$. (This is the so-called continuity property for the Lebesgue integral.)
8. Let $f \in L^1(\mathbb{R}, \mathcal{R}, \lambda)$ and assume that $\int_K f(x)dx = 0$ holds for all compact subset $K \subset \mathbb{R}$. Show that $f = 0$ almost everywhere.
9. Let $f \in L^1(\mathbb{R}, \mathcal{R}, \lambda)$.
   (a) Show that for all $\varepsilon > 0$ there exists a simple function $g : \mathbb{R} \to \mathbb{R}$ of the form $g = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$ with $A_1, \ldots, A_k$ bounded intervals, such that $\int_{\mathbb{R}} |f(x) - g(x)| < \varepsilon$.
   (b) Conclude that for all $\varepsilon > 0$ there exists a bounded smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |f(x) - h(x)| dx < \varepsilon$.

Convergence theorems

10. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f_n : \Omega \to [0, \infty]$, $n \geq 1$, be measurable functions such that
$$f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0.$$
For all $x \in \Omega$, define $f(x)$ as the limit of $f_n(x)$ as $n \to \infty$. Show that if $f_1 \in L^1(\mu)$ then
$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$
Give a counterexample showing that the conclusion becomes false without the integrability assumption on f_1.
11. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $(f_n)_n$ be a sequence of complex measurable functions on Ω such that $f_n \to f$ uniformly on Ω. Show that if $f_n \in L^1(\Omega)$ for all n, then
$$f \in L^1(\Omega) \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$
Also show that the result is in general false if $\mu(\Omega) = \infty$.
12. Determine the limits of
$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx \quad \text{et} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx$$
as $n \to \infty$.
13. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, and let $f_n, f : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable functions with $f_n(x) \to f(x)$ for all x and with $\int f_n d\mu \to c$ for some $c > 0$. Show with examples that $\int f d\mu$ can take any value in $[0, c]$.
14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. The goal of this exercise is to establish the Egorov and the Lusin theorems. [1]
   (a) Let $f_n, f : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable functions and assume that there exists $A \in \mathcal{F}$ with $\mu(A) < \infty$ such that $f_n(\omega) \to f(\omega)$ for all $\omega \in A$. Prove the Egorov theorem : for all $\varepsilon > 0$, there exists $A_\varepsilon \subset A$ with $A_\varepsilon \in \mathcal{F}$ and $\mu(A \setminus A_\varepsilon) < \varepsilon$ such that $f_n \to f$ uniformly on A_ε. In other words, almost everywhere convergence is equivalent to the so-called almost uniform convergence on sets with finite measure.
   (b) Show that the Egorov theorem may not hold if $\mu(A) = \infty$.

1. This exercise establishes Littlewood's three principles of real analysis : "Every (measurable) set is nearly a finite sum of intervals; every function is nearly continuous; every convergent sequence of functions is nearly uniformly convergent." The rigorous versions of these principles are given respectively by the regularity of the Lebesgue measure, the Lusin theorem, and the Egorov theorem.

(c) Deduce from item (a) the Lusin theorem : given $a, b \in \mathbb{R}$, if $f : [a, b] \to \mathbb{C}$ is Lebesgue-measurable, then for all $\varepsilon > 0$ there exists a compact set $K \subset [a, b]$ such that $\lambda([a, b] \setminus K) < \varepsilon$ and such that the restriction $f|_K$ is continuous.

(d) As an application of item (a) together with exercise 11, deduce a new proof of the bounded convergence theorem.

Miscellaneous

15. Let f be defined on $[0, 1)$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

Show that f is Lebesgue-measurable and that

$$\lambda(f^{-1}(E)) = \lambda(E)$$

for all measurable $E \subset [0, 1)$ (which means that f is a homomorphism of the Lebesgue space $[0, 1)$). Show that f is ergodic, that is, $\lambda(f^{-1}(E)\Delta E) = 0$ implies $\lambda(E) = 0$ or 1.

16. The goal of this exercise is to define spherical coordinates on $\mathbb{R}^n$.

(a) Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Show that the function $\varphi : x \mapsto (|x|, \frac{x}{|x|})$ is a bijection between $\mathbb{R}^n \setminus \{0\}$ and $(0, \infty) \times S^{n-1}$.

(b) Let λ denote the Lebesgue measure on $\mathbb{R}^n$. Show that we may define as follows a measure σ on S^{n-1} : for any Borel set $A \subset S^{n-1}$, define $\tilde{A} = \varphi^{-1}((0, 1) \times A)$ and $\sigma(A) = n\lambda(\tilde{A})$.

(c) For a Borel function $f \geq 0$, establish the formula

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty r^{n-1} \left(\int_{S^{n-1}} f(ru)\, d\sigma(u) \right) dr.$$

We may proceed as follows :

i. Show that for all $N \in \mathbb{N}$,

$$\mathcal{S}_N = \{\varphi^{-1}((a, b] \times A)\, ;\, 0 \leq a \leq b \leq N \text{ and } A \text{ Borel}\}$$

is a semi-algebra on $\varphi^{-1}((0, N] \times S^{n-1})$.

ii. Check the formula for $f = \mathbb{1}_E$, with $E \in \mathcal{S}_N$.

iii. Deduce that the formula holds if $f = \mathbb{1}_B$, for any Borel set B.

iv. Generalize to a simple Borel function $f \geq 0$, then to any Borel function $f \geq 0$.

(d) Show that $f(x) = (1 + |x|^2)^{-m/2} \in L^1(\mathbb{R}^n)$ if and only if $g(x) = (1 + |x|)^{-m} \in L^1(\mathbb{R}^n)$ if and only if $m > n$.