# Exercises Measure Theory

Bachelor 3

Academic Year 2017-2018

## Chapter 5: Signed Measures

1. Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mu, \nu$ be two **finite** measures on $\mathcal{F}$. Prove that $\nu \ll \mu$ if and only if $\forall \epsilon > 0, \exists \delta > 0, \forall A \in \mathcal{F} : \mu(A) < \delta \implies \nu(A) < \epsilon$. Is this also true if $\mu$ and $\nu$ are $\sigma$-finite measures?
   *Hint:* Let $\nu$ be the counting measure for all subsets of $\mathbb{N}$ and let $\mu$ be the discrete measure on this $\sigma$-field satisfying $\mu(\{n\}) = (n+1)^{-2}$ for $n \in \mathbb{N}$.

2. Show that the Radon-Nikodym theorem is NOT true in general if $\mu$ is not $\sigma$-finite.
   *Hint:* Look at the Lebesgue measure $\lambda$ and the counting measure $\mu$ on $\big([0,1], \mathcal{R}_{[0,1]}\big)$.

3. Let $\mu$ be a $\sigma$-finite measure. Show that there exists a **finite** measure $\nu$ such that $\mu \equiv \nu$.

4. Show that there is a Lebesgue decomposition in the $\sigma$-finite as well as in the finite case. Prove that it is unique.

5. Let $(\Omega, \mathcal{F})$ be a measurbale space and let $\lambda$, $\mu$, and $\nu$ be $\sigma$–finite measures on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu \ll \lambda$. Denote the $\mu_2$-density of a measure $\mu_1$ always by $\frac{d\mu_1}{d\mu_2}$ provided that it exists.

   (a) If $f : \Omega \to \mathbb{R}$ is $\mu$–integrable, show that
   $$\int f d\mu = \int f \frac{d\mu}{d\lambda} d\lambda$$

   (b) Show that
   $$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \qquad \lambda\text{–}a.e.$$

   (c) If we also have $\mu \ll \nu$, show that
   $$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1} \qquad \mu, \nu\text{–}a.e.$$