# **Final exam** — May 8th

Please provide complete and well-written solutions to the following exercises. All answers must be properly explained and justified.

**Exercise 1.** (/20) Consider the numbers $1, 2, 3, \ldots, 12$ written around a ring as they usually are on a clock. Consider a Markov chain that from any number jumps with equal probability to the two adjacent numbers.

(i) What is the expected number of steps that $X_n$ will take to return to its starting point?

(ii) What is the probability that $X_n$ will visit all the other states before returning to its starting point?

**Exercise 2.** (/10) Consider the Markov chain with state space $\{1, 2, \ldots\}$ and transition probability

$$\begin{array}{rcll} P_{m,m+1} & = & \frac{m}{2m+2} & \text{for } m \geq 1, \\ P_{m,m-1} & = & \frac{1}{2} & \text{for } m \geq 2, \\ P_{m,m} & = & \frac{1}{2m+2} & \text{for } m \geq 2, \end{array}$$

and $P_{1,1} = \frac{3}{4}$. Show that this chain admits no stationary distribution.

**Exercise 3.** (/15) A doctor is working at night in an emergency room. Emergencies come in at times of a Poisson process with rate 0.5 per hour. The doctor can only get to sleep when it has been 36 minutes (.6 hours) since the last emergency. For example, if there is an emergency at 1:00 and a second one at 1:17 then she will not be able to get to sleep until at least 1:53, and it will be even later if there is another emergency before that.

(i) What is the long-run fraction of time the doctor spends sleeping? For that purpose, formulate a renewal reward process in which the reward in the $i$th interval is the amount of time the doctor gets to sleep in that interval.

(ii) The doctor alternates between sleeping for an amount of time $s_i$ and being awake for an amount of time $w_i$. Use (i) to compute $\mathbb{E}w_i$.

**Exercise 4.** (/30) Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{P}[X_1 = 1] = p$ and $\mathbb{P}[X_1 = -1] = q = 1 - p$ with $0 \leq p < \frac{1}{2}$. Consider the asymmetric random walk $S_n = S_0 + X_1 + \ldots + X_n$, and let $V_0 := \min\{n \geq 0 : S_n = 0\}$.

(i) Check that $S_n - (p - q)n$ and $(S_n - (p - q)n)^2 - n(1 - (p - q)^2)$ are martingales.

(ii) Use (i) to show that for all $x \geq 0$,

$$\mathbb{E}_x V_0 = \frac{1}{q - p}\, x \qquad \text{and} \qquad \operatorname{Var}_x V_0 = \frac{1 - (q - p)^2}{(q - p)^3}\, x.$$

(iii) Show that for all $\theta \leq 0$ there holds

$$e^{\theta x} = \mathbb{E}_x \phi(\theta)^{-V_0},$$

in terms of the moment generating function $\phi(\theta) := \mathbb{E}e^{\theta X_1}$.

(iv) Let $0 < s < 1$. Solve the equation $\phi(\theta) = \frac{1}{s}$ and use (iii) to conclude

$$\mathbb{E}_x s^{V_0} = \left( \frac{1 - \sqrt{1 - 4pqs^2}}{2ps} \right)^x.$$

Show how this allows to recover the results in (ii).

**Exercise 5.** (/25) Consider two machines that are maintained by a single repairman. Machine $i$ functions for an exponentially distributed amount of time with rate $\lambda_i$ before it fails. The repair times for each unit are exponential with rate $\mu_i$. They are repaired in the order in which they fail. Suppose that $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$.

(i) Formulate a Markov chain model for this situation with state space $\{0, 1, 2, 12, 21\}$.

(ii) Find the stationary distribution. What is the long-run frequency that machine 1 is out of service?

(iii) Compute the average time before both machines are out of service.