Please provide complete and well-written solutions to the following exercises.
Due on May 22nd before noon.

Homework 6

Exercise 1. Thomas catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmons and 60% are trouts. A salmon weighs an average of 8 pounds with a standard deviation of 2 pounds, and a trout weighs an average of 4 pounds with a standard deviation of 1 pound.

(i) Find the mean and standard deviation of the total weight of fish he catches in two hours.

(ii) What is the probability that he will catch exactly 1 salmon and 2 trouts if he fishes for 2.5 hours?

Exercise 2. Let $(S_t)_t$ be the price of stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable X_i with mean μ and variance σ^2. In other words, we consider $S_t = S_0 \prod_{i=1}^{N(t)} X_i$, where by convention $S_t = S_0$ if $N(t) = 0$. Find $\mathbb{E}[S(t)]$ and $\mathrm{Var}[S(t)]$.

Exercise 3. Two copy editors read a 300-page manuscript. The first found 100 errors, the second found 120, and their lists contain 80 errors in common. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page, while the two copy editors catch errors with unknown probabilities of success p_1 and p_2. Let X_0 be the number of typos that neither found. Let X_1 and X_2 be the number of typos found only by 1 and only by 2, respectively, and let X_3 be the number of typos found by both.

(i) Find the joint distribution of (X_0, X_1, X_2, X_3).

(ii) Use the previous answer to infer an estimate of p_1, p_2, and then on the number of undiscovered typos.

Exercise 4. A light bulb has a lifetime that is exponential with a mean of 200 days. When it burns out a janitor replaces it immediately. In addition, there is a handyman who comes at times of a Poisson process at rate .01 and replaces the bulb as "preventive maintenance".

(i) How often is the bulb replaced?

(ii) In the long run, what fraction of the replacements are due to failure?

Exercise 5. Consider two independent Poisson processes $(N_1(t))_t$ and $(N_2(t))_t$ with rates λ_1 and λ_2. What is the probability that the two-dimensional process $(N_1(t), N_2(t))_t$ ever visits the point (m, n)?

Exercise 6. Let $(N(t))_{t\geq 0}$ be a Poisson process with rate λ. Let T be a stopping time for $(N(t))_t$, that is, a nonnegative random variable such that for all $s \geq 0$ the event $\{T = s\}$ is independent of the process $(N(t+s) - N(s))_{t\geq 0}$.

(i) Conditioning on $\{T < \infty\}$, show that $(N(t+T) - N(T))_{t \geq 0}$ is a Poisson process with rate λ and is independent of $(N(t))_{t \leq T}$.

(ii) Let S_1 be the first time that we see three arrivals during a unit of time, that is,

$$S_1 := \inf\{t \geq 0 : N(t) - N(t-1) = 3\}.$$

Show that S_1 is a stopping time.

(iii) Let S_2 be the first time t such that there is no arrival in the interval $[t, t+1]$. Is S_2 a stopping time? Is $(N(t+S_2) - N(S_2))_{t \geq 0}$ a Poisson process with rate λ?