M285K - course #1

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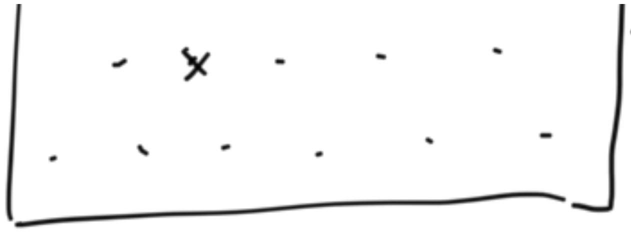
I. INTRODUCTION

I.1 Goal of the course

Stochastic homogenization theory: effective behavior in heterog. media



impurities,
microstructures } statistical description

Micro: small-scale variations

Macro: averaging on large scales
effective ("homogenized") behavior

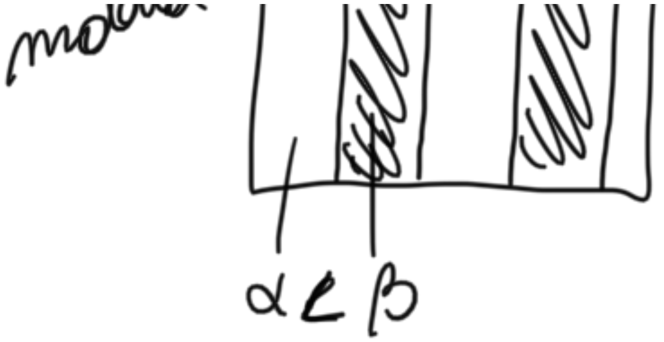
} "LLN"

Difficulty: crucial dependence on geometry.

conducting
fluid

$\to$ resistance in series

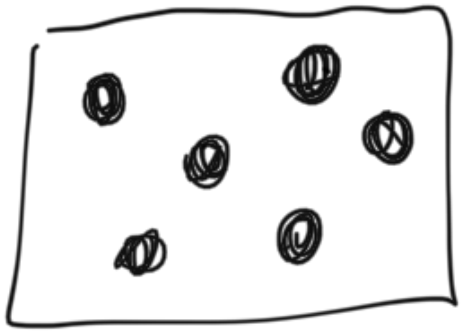
$\to$ with ... $\alpha+\beta$



→ arithmetic average $\frac{\ldots}{2}$

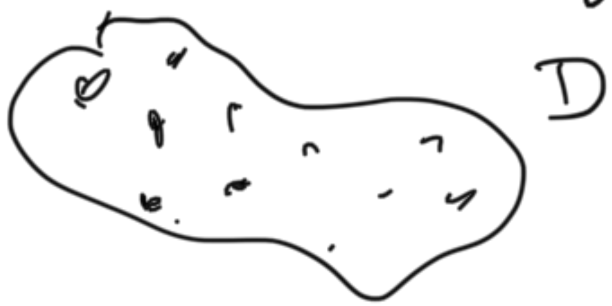
↑ resistance in parallel

→ geometric average $2\frac{\alpha\beta}{\alpha+\beta}$



Focus in this course: electrostatics (linear elliptic PDEs)

- material described by its conductivity $a^\omega(\frac{\cdot}{\varepsilon})$ (ε = microscale)
  ($\omega \in \Omega$) realization of some random field.



- h charge distribution.

→ electric field $\nabla u_\varepsilon^\omega$: $-\nabla \cdot \left(a^\omega(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon^\omega \right) = h$ in D.

$$\left\{ u_\varepsilon^w |_{\partial D} = 0 \right.$$

<u>Aim of the theory</u>: ∇u_ε^w has small-scale variation $O(\varepsilon)$

BUT $\nabla u_\varepsilon^w \xrightarrow{L^2} \nabla \bar{u}$ (a.s)

with $\left\{ \begin{array}{l} -\nabla \cdot \bar{a} \nabla \bar{u} = h \\ \bar{u}|_{\partial D} = 0 \end{array} \right.$ $\bar{a}$ = effective conductivity

<u>Rem</u>: $\left\{ \begin{array}{l} a^w(\frac{\cdot}{\varepsilon}) \rightharpoonup \mathbb{E}\, a \quad (\text{if } a \text{ ergodic}) \\ \nabla u_\varepsilon^w \rightharpoonup \nabla \bar{u} \end{array} \right.$

$$a^w(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon^w \rightharpoonup \bar{a} \nabla \bar{u} \neq (\mathbb{E}\, a) \nabla \bar{u}$$

$\Rightarrow$ question: compute $\bar{a}$, $\mathbb{E}[a^{-1}]^{-1} \leq \bar{a} \leq \mathbb{E}\, a$

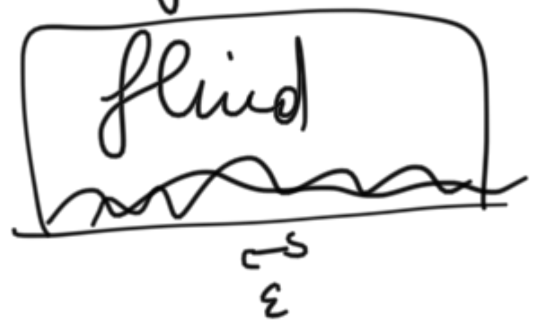
Other settings: waves in heterogeneous media (large wavelength)

__Other settings__: * waves in heterog medium ([illegible])

* nonlinear elasticity

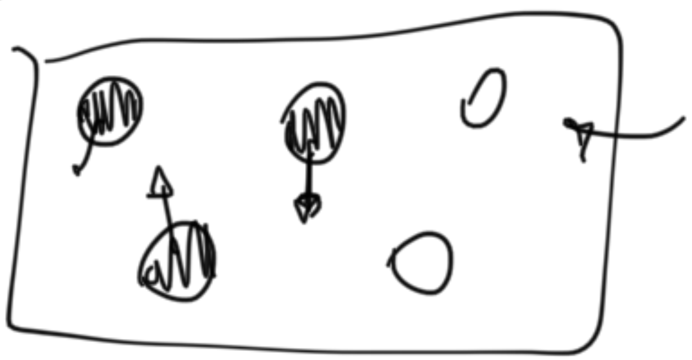
polymer network model
> rubber hyperelasticity

* fluid flow with rough botton



→ effective BC

* rigid particles in a viscous fluid



Stokes fluid

→ effective viscosity
sedimentation

* interface motion in heterog media.

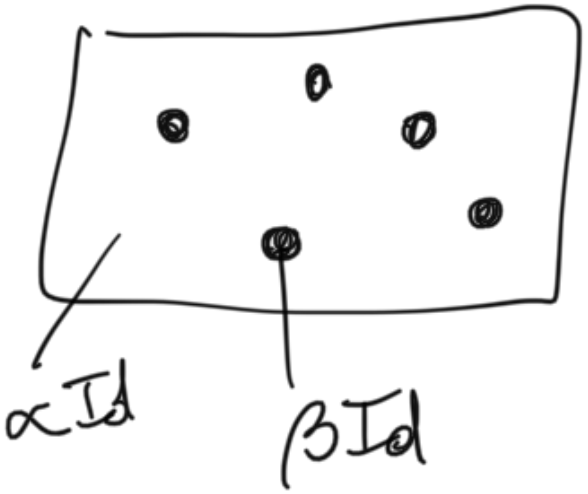
Goals of the theory:

① qualitative theory:
- existence of an effective behavior (LLN) $\nabla u_\varepsilon^\omega \rightharpoonup \nabla \bar{u}$
  $-\nabla\cdot\bar{a}\nabla u = h$
- representation formula for $\bar{a}$
+ general question: do we remain in the same PDE class?
  (elliptic, hamiltonian, conservation law)

  do we keep non-degeneracy?

② rep formula for $\bar{a}$ is often very costly numerically (auxiliary problems)
- get general bounds on $\bar{a}$: $\mathbb{E}[a^{-1}]^{-1} \le \bar{a} \le \mathbb{E}\, a$
  + refined

- effective medium answer: 2-phase mixtures



(vol fraction $\lambda = \lim_{R\uparrow\infty} \dfrac{|B_R \cap \bigcup_n I_n|}{|B_R|}$)

$<< 1$

Clausius-Mossotti formula:

$$\bar{a} = \alpha \, \mathrm{Id} \left(1 + \lambda \, \frac{(\beta-\alpha)d}{\beta + \alpha(d-1)} + O(\lambda^2) \right).$$

③ quantitative theory:

- error estimates $\| u_\varepsilon - \bar{u} \|_{L^2} \lesssim \varepsilon$?
- describe local oscillations of ∇u_ε ($\rightharpoonup \nabla \bar{u}$)

$$\| u_\varepsilon - \bar{u} - \varepsilon u_1(\cdot, \tfrac{\cdot}{\varepsilon}) \|_{L^2} \lesssim \varepsilon^2 \ ?$$

$$\| \nabla u_\varepsilon - \nabla \bar{u} - \underbrace{\nabla_2 u_1(\cdot, \tfrac{\cdot}{\varepsilon})} \|_{L^2} \lesssim \varepsilon$$

- describe random fluctuations

$$\varepsilon^{-d/2} \left(\int g \cdot \nabla u_\varepsilon - \int g \cdot \nabla \bar{u} \right) \xrightarrow{\text{law}} \mathcal{N}(\sigma_{g,f}) \ ?$$

Rem. $-\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla\cdot f$ in $\mathbb{R}^d$.

$$\nabla u_\varepsilon = \underbrace{\nabla\left(-\nabla\cdot a(\tfrac{\cdot}{\varepsilon})\nabla\right)^{-1}\nabla\cdot}\, f$$

complicated transformation of a nonlinear & nonlocal via elliptic solution operator

→ mixture of PDE & proba techniques

I.2 History.

- early 19th century: Navier, Cauchy, Poisson, Faraday — discrete models for macromolecules → continuum mechanics

- mid 19th century: relate micro & macro properties

Mossotti, Clausius, Maxwell, Rayleigh, ...

conductivity

Bruggeman: elasticity ...

Einstein PhD thesis 1905: effective viscosity

($\to$ evaluate Avogadro number)

- late 19th century: pointillist paintings

  Seurat, Signac

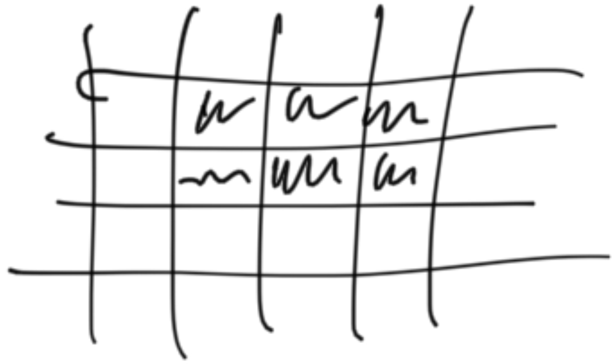
- early 20th century: rigorous averaging methods for ODEs (Poincaré ...)

- mid 20th century: LOT of motivation from engineering.

  * composite materials: reinforced concrete, rubber ...

  need math theory to do in silico prediction

* optimal design of microstructures.
  (planes ...)
* oil extraction.



- starting in 1970s : Tartar, Murat, ...
  qualitative theory in periodic setting
- starting in 1983 : random setting
  Papanicolaou, Varadhan, Kozlov, ...
- recent years 2010-2020 : optimal quantitative theory. [for linear ell eqns]

(Otto, Gloria, Neukamm, Fischer]

2 groups $\left\{\begin{array}{l}\text{[illegible]} \\ \text{Armstrong, Smart, Mourrat, Kuusi, ...}\end{array}\right.$

Main technical ingredient: large-scale regularity.

Wednesday 12 pm: 1D setting.

Friday: No lecture!

Next week: qualitative theory.