M285K - course #10

Some further comment on Malliavin — Bjoern's question.

Q1/ key relation $D\mathcal{L} = (\mathcal{L}+1)D$, intuition? creation / annihilation?
$[D, \mathcal{L}] = D$

Q2/ intuition on Mehler's formula.

Consider finite-dim setting (1D setting):

- G_0 standard Gaussian random variable (vs G Gaussian field)
- $(\Omega, \mathbb{P})$ generated by G_0
- $L^2(\Omega) = \{ f(G_0) : f \in L^2(\gamma) \} \cong L^2(\gamma)$.

$$\mathbb{E}[|f(G_0)|^2] = \int |f(x)|^2 \underbrace{\frac{e^{-x^2/2}}{\sqrt{2\pi}}}_{\gamma(x)} dx$$

- $\mathcal{R}(\Omega) = \{ f(G_0) : f \in C_c^\infty(\mathbb{R}^d) \}$.

$h \in \mathbb{R}$

Finite-dim Malliavin derivative $\partial f(G_0) := f'(G_0)$, $f(G_0) \in R[\Omega]$.

Divergence operator $\partial^* f(G_0) = (-f' + x f)(G_0)$.

$$\mathbb{E}\, f(G_0)\, \partial h(G_0) = \mathbb{E}\, f(G_0)\, h'(G_0)$$
$$= \int f(x)\, h'(x)\, \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$
$$\overset{IBP}{=} \int \underbrace{(-f'(x) + x f(x))}_{\partial^* f(x)}\, h(x)\, \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx.$$

Ornstein-Uhlenbeck operator: $L = \partial^* \partial$

$$L f(G_0) = (-f'' + x f')(G_0)$$

Q1| Notice $[\partial, \partial^*] = 1$

$L\, H_p(G_0) = p\, H_p(G_0)$

→ indeed creation/annihilation operators:

$$\begin{cases} \partial H_p(G_0) = (p-1) H_{p-1}(G_0) \\ \partial^* H_p(G_0) = H_{p+1}(G_0) \end{cases}$$

Link to harmonic oscillator:

$$e^{x^2/2}(-\Delta + x^2)e^{-x^2/2} = L + 1.$$

Infinite-dim setting: $[D, D^*] = DD^* - D^*D + Id$

But: define $\partial_\xi = \langle \xi, D \rangle$, ∂_ξ^* ($\xi \in h$)

↳ $[\partial_\xi, \partial_\xi^*] = \|\xi\|_h^2 \, Id.$

What remains: key relation $\begin{cases} [D, L] = D \\ [D^*, L] = -D^* \end{cases}$ **I**

Rem: $\forall \xi \in \mathfrak{h}$, $\quad \mathcal{L}(D^*)^p[\xi^{\otimes p}] = (D^*)^p(\mathcal{L}+p)[\xi^{\otimes p}]$

$$= p\,(D^*)^p[\xi^{\otimes p}].$$

Easy to check: span $\{(D^*)^p(\xi^{\otimes p}) : p \geq 1,\ \xi \in \mathfrak{h}\}$ is dense in $L^2(\Omega)$.

$\to$ $\sigma(\mathcal{L}) = \mathbb{N}_\infty$

$$L^2(\Omega) = \bigoplus_{p=0}^{\infty} \mathcal{H}_p, \quad \mathcal{H}_p = \text{span}\{(D^*)^p(\xi^{\otimes p}) : \xi \in \mathfrak{h}\}.$$

$\mathcal{H}_p$: p-th chaos; $(D^*)^p(\xi^{\otimes p}) = H_p(G(\xi))$. //

Q2/ Mehler's formula.

In 1D setting: $\quad e^{-tL} f(G_0) = (P_t f)(G_0).$

$$\begin{cases} \partial_t P_t f = \Delta(P_t f) - x \cdot \nabla(P_t f) \\ P_t f\big|_{t=0} = f \end{cases}$$

→ want explicit Green's fct for O-U operator.

SDE method: $P_t f(x) = \mathbb{E}\left[f(X_x^t)\right]$

where
$$\begin{cases} dX_x^t = \sqrt{2}\, dB_x^t - X_x^t\, dt \\ X_x^t\big|_{t=0} = x \end{cases}$$

"Ornstein-Uhlenbeck process".

Explicit solution: $d(e^t X_x^t) = \sqrt{2}\, e^t\, dB_x^t$

$$X_x^t = e^{-t} x + \sqrt{2} \int_0^t e^{-(t-s)}\, dB_x^s.$$

centered Gaussian r.v.

$\mathrm{Var} = 1 - e^{-2t}$.

$\hookrightarrow P_t f(x) = \mathbb{E}[f(X_x^t)]$

$$= \int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

"Mehler's formula.

$$= \mathbb{E}'\left[f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, G_0'\right)\right]$$

(G_0': iid copy of G_0.)

$\square$

III.2 Large-scale regularity.

a) Classical regularity for $-\nabla\cdot a\nabla w = \nabla\cdot f$.

M (1, ..., 2 < 1) DG-N-M (C^{σ}, σ small) energy, Meyers (L^p, $|p-2| \le c_0$), ...

$\to$ there are only *deterministic* regularity estimates.

With positive proba, $\forall R \geq 0$, a can resemble a counterexample on B_R.

b) Large-scale regularity

Homog: $\quad (-\nabla \cdot a \nabla) \quad \simeq \quad (-\nabla \cdot \bar{a} \nabla)$ on large scales

maximal L^p reg, Schauder theory, etc. $\to$ inherit better regularity.

(Armstrong - Smart '14).

Chapter V: $\exists$ random field r_* ("minimal radius") $\quad$ $\mathbb{E}\, |r_*|^p < \infty \quad \forall p < \infty$

s.t. get same regularity as for Δ on scales $\geq r_*$.

e.g. we'll prove:

<u>Theorem</u> (quenched large-scale L^p regularity).

$\forall\, 1 < p < \infty$: $\quad -\nabla \cdot a \nabla w = \nabla \cdot g$

$$\int_{\mathbb{R}^d} \Big(\underbrace{\fint_{B_{r_*(x)}(x)} |\nabla w|^2}_{L^2 \text{ estimate on scale} \leq r_*} \Big)^{p/2} \lesssim_p \int_{\mathbb{R}^d} \Big(\underbrace{\fint_{B_{r_*(x)}(x)} |g|^2} \Big)^{p/2}$$

L^p estimate on scale $\geq r_*$.

c) <u>Annealed estimates</u>.

Useful variant: up to averaging wrt stationary ensemble, should get some regularity as for $-\Delta$.

<u>Theorem</u> (Annealed L^p regularity). $[h \in C_c^\infty(\mathbb{R}^d; L^\infty(\Omega))$

$-\nabla\cdot a\nabla w = \nabla\cdot h$

$$\left[\begin{aligned} &\forall 1<p,q<\infty, \ \forall \delta>0, \\ &\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B(x)} |\nabla w|^2\right)^{\frac{q}{2}}\right]^{\frac{p}{q}} \lesssim_{p,q,\delta} \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B(x)} |h|^2\right)^{\frac{q+\delta}{2}}\right]^{\frac{p}{q+\delta}}. \end{aligned}\right.$$

<u>Notation</u>: $[h](x) = \left(\fint_{B(x)} |h|^2\right)^{\frac{1}{2}}$ quadratic averages.

$$\to \left[\ \|[\nabla w]\|_{L^p(\mathbb{R}^d; L^q(\Omega))} \lesssim_{p,q,\delta} \|[h]\|_{L^p(\mathbb{R}^d; L^{q+\delta}(\Omega))}\right.$$

loss some

loose some stoch. integrability

$\leadsto \sigma_\varepsilon$ is not unif bounded.

see: Chapter V.

d) Perturbative version : only thing needed in this chapter.

Theorem (perturbative annealed L^p regularity).

$\exists C_0 = C_0(d, \alpha, \beta) > 0$ large.

s.t. $\forall |p-2|, |q-2| \leq \frac{1}{C_0}$:

$$\| [\nabla w] \|_{L^p(\mathbb{R}^d; L^q(\Omega))} \lesssim \| [h] \|_{L^p(\mathbb{R}^d; L^q(\Omega))}$$

$h \in C_c^\infty(\mathbb{R}^d, L^\infty(\Omega))$.

$-\nabla \cdot a \nabla w = \nabla \cdot h$

↑ no loss in this perturbative setting.

View this as upgrading Meyers' estimate:

$$\left(\left[\| [Dw] \|_{L^p(\mathbb{R}^d)} \leq \| [h] \|_{L^p(\mathbb{R}^d)} \right. \text{ a.s.} \right.$$
$$\text{for } |p-2| \leq \frac{1}{C_0}.$$

Key ingredient for the proof:

Lemma ($\sim$ dual version of Calderón-Zygmund).

(Caffarelli-Peral '98, Shen '07).

Given $1 \leq p_0 < p_1 \leq \infty$, $F, G \in L^{p_0} \cap L^{p_1}(\mathbb{R}^d)$, $F, G \geq 0$.

Assume that $\forall$ ball $D \subseteq \mathbb{R}^d$ $\exists$ measurable $F_D^0, F_D^1 \geq 0$.

$$\begin{cases} F \leq F_D^0 + F_D^1 \\ F_D^1 \leq F + F_D^0 \end{cases}$$

$D = B(x, r)$
$CD = B(x, Cr)$

s.t

$$\begin{cases} \left(\fint_{D} |F_D^0|^{p_0} \right)^{1/p_0} \leq C_0 \left(\fint_{C_0 D} |G|^{p_0} \right)^{1/p_0} & \text{(local dependence)} \\ \left(\fint_{\frac{1}{C_0} D} |F_D^1|^{p_1} \right)^{1/p_1} \leq C_0 \left(\fint_{D} |F_D^1|^{p_0} \right)^{1/p_0} & \text{(reverse Jensen)} \end{cases}$$

Then, $\forall\, p_0 < p < p_1$:

$$\int_{\mathbb{R}^d} |F|^p \lesssim_{C_0, p_0, p_1, p} \int_{\mathbb{R}^d} |G|^p .$$

Idea for application: $-\nabla \cdot a \nabla w = \nabla \cdot h$

$\forall$ ball D: $\nabla w = \nabla w_D^0 + \nabla w_D^1$

$$\{ -\nabla \cdot a \nabla w_D^0 = \nabla \cdot (h \mathbb{1}_D) \}$$

$$\left(-\nabla \cdot a \nabla w_D^1 = \nabla \cdot \left(h\, \mathbb{1}_{\mathbb{R}^d \setminus D} \right) = 0 \text{ in } D \right)$$

Wednesday 3/2: proof of dual CZ lemma.

Friday 5/2: proof of perturbative L^p reg.