M285K - course #13

III.3 Stochastic corrector estimates.

Theorem.

(i) $\mathbb{E}\left[[D(\varphi,\sigma)]^{2p}\right]^{\frac{1}{2p}} \le (Cp)^C$ (stretched exp. moments)

(ii) $\mathbb{E}\left[\left(\int_{\mathbb{R}^d} g \, D(\varphi,\sigma)\right)^{2p}\right]^{\frac{1}{2p}} \lesssim (Cp)^C \|g\|_{L^2(\mathbb{R}^d)}$ $\quad \forall g \in C_c^\infty(\mathbb{R}^d)$.

(iii) $\mathbb{E}\left[\left[(\varphi,\sigma) - \fint_B (\varphi,\sigma)\right]^{2p}(x)\right]^{\frac{1}{2p}} \lesssim (Cp)^C \times \begin{cases} 1, & d>2 \\ \log^{\frac{1}{2}}(2+|x|), & d=2 \\ \langle x\rangle^{\frac{1}{2}}, & d=1 \end{cases}$

Proof:

* Steps 1, 2, 3: proof of (i)-(ii) by buckling.
* Step 4: proof of (iii) from (ii) by integrating

} focus on $\varphi = \varphi_i$

* Step 4: proof ... $e = e_i$

* Step 5: some results for σ. $(-\Delta\sigma = \nabla \times q)$

Step 1. $\forall\, 1 < p < \infty \quad \forall g \in C_c^\infty(\mathbb{R}^d)^d$

$$\left\| \int_{\mathbb{R}^d} g \cdot \nabla\varphi \right\|_{L^{2p}(\Omega)}^2 \lesssim p \, \|g\|_{L^2(\mathbb{R}^d)}^2 \, \| [\nabla\varphi + e] \|_{L^{2p}(\Omega)}^2$$

& we'll upgrade this as follows: $\forall R, q \geq 1$,

$$\left\| \int_{\mathbb{R}^d} g \cdot \nabla\varphi \right\|_{L^{2p}(\Omega)}^2 \lesssim p \, \|g\|_{L^2(\mathbb{R}^d)}^2 \, \left\| \left(\int_{B_R} [\nabla\varphi + e]^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)}$$

Starting point is Malliavin calculus in form of moment bounds:

$\| \ldots \|^2 \ldots \| \ldots \|^2$

$$\left\| \int_{\mathbb{R}^d} g \cdot \nabla\varphi \right\|_{L^{2p}(\Omega)} \lesssim p \left\| D \int_{\mathbb{R}^d} g \cdot \nabla\varphi \right\|_{L^{2p}(\Omega, h)}$$

Let's compute Malliavin derivative:

$$-\nabla \cdot a(\nabla\varphi + e) = 0.$$

$$\hookrightarrow -\nabla \cdot a \nabla D_z \varphi = \nabla \cdot D_z a (\nabla\varphi + e).$$

Formally: $D_z \nabla\varphi(x) \approx \nabla\nabla G(x,z) (\nabla\varphi(z) + e)$

Consider auxiliary problem: $-\nabla \cdot a^* \nabla v = \nabla \cdot g$ in $\mathbb{R}^d$.

$$\to D_z \int_{\mathbb{R}^d} g \cdot \nabla\varphi = \int_{\mathbb{R}^d} g \cdot \nabla D_z \varphi = -\int_{\mathbb{R}^d} \nabla v \cdot a \nabla D_z \varphi \quad (\text{cf eqn for } \nabla v)$$

$$= \int_{\mathbb{R}^d} \nabla v \cdot D_z a (\nabla\varphi + e)$$

$$a(x) = \overset{C^1_b}{\overset{\Psi}{a}} (G(x))$$

Remember that $a(x) = a_0(G(x))$

$$\leadsto D_z a(x) = a_0'(G(x))\,\delta(z-x).$$

$$\Rightarrow D_z \int_{\mathbb{R}^d} g \cdot \nabla\varphi = \nabla v(z) \cdot a_0'(G(z))\,(\nabla\varphi(z) + e).$$

$$\Rightarrow \left\| \int_{\mathbb{R}^d} g \cdot \nabla\varphi \right\|^2_{L^{2p}(\Omega)} \lesssim p\, \left\| \nabla v \otimes (\nabla\varphi + e) \right\|^2_{L^{2p}(\Omega;\, \underset{\smile}{h})}.$$

Remember $\|f\|_h^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x) f(y)\, C(x-y)\, dx\, dy$

$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \underbrace{\left(\fint_{B(x)} |f|\right)\left(\fint_{B(y)} |f|\right)}_{} \left(\sup_{|z| \leq 2} |C(x-y+z)|\right) dx\, dy$$

$$\leq \frac{1}{2}\left(\fint |f|\right)^2 + \frac{1}{2}\left(\fint |f|\right)^2.$$

$$2\left(\fint_{B(x)}\right) + 2\left(\fint_{B(y)}\right)$$

$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\fint_{B(x)}|f|\right)^2\left(\sup_{|z|\le 2}|C(x-y+z)|\right)dx\,dy$$

& recall $\int_{\mathbb{R}^d}\left(\sup_{B(x)}|C|\right)dx<\infty.$

$$\to \|f\|_2^2 \lesssim \int_{\mathbb{R}^d}\left(\fint_{B(x)}|f|\right)^2 dx.$$

Get: $\left\|\int g\cdot\nabla\varphi\right\|_{L^{2p}(\Omega)}^2 \lesssim p\,\left\|[\nabla g]\,[\nabla\varphi+e]\right\|^2_{L^{2p}(\Omega;\,L^2(\mathbb{R}^d))}$

$$= p\,\mathbb{E}\left[\left(\int_{\mathbb{R}^d}[\nabla g]^2\,[\nabla\varphi+e]^2\right)^p\right]^{\frac{1}{p}}$$

$$= p \sup_{\|F\|_{L^{2p'}(\Omega)}=1} \mathbb{E}\left[F^2 \int_{\mathbb{R}^d} [D z]^2 [\nabla\varphi + e]^2 \right].$$

$$= \mathbb{E}\left[\int_{\mathbb{R}^d} [D(Fz)]^2 [\nabla\varphi + e]^2 \right]$$

$$\leq \mathbb{E}\left[\int_{\mathbb{R}^d} \left(\sup_{B_R(x)} [\nabla\varphi + e]^2 \right) \left(⨍_{B_R(x)} [D(Fz)]^2 \right) dx \right]$$

$$\stackrel{\text{Hölder}}{\leq} \int_{\mathbb{R}^d} \left\| \sup_{B_R(x)} [\nabla\varphi + e]^2 \right\|_{L^p(\Omega)} \left\| ⨍_{B_R(x)} [D(Fz)]^2 \right\|_{L^{p'}(\Omega)} dx$$

$$\stackrel{\text{stat.}}{=} \underbrace{\left\| \sup_{B_R} [\nabla\varphi + e]^2 \right\|_{L^p(\Omega)}}_{(\alpha)} \times \int_{\mathbb{R}^d} \| [D(Fz)] \|^2_{L^{2p'}(\Omega)}$$

(b)

(a): $$\sup_{B_R} [\nabla\varphi + e]^2 \approx \sup_{z \in \frac{1}{C}\mathbb{Z}^d \cap B_R} [\nabla\varphi + e]^2(z)$$

$(R \geq 1)$

$$\lesssim \left(\sum_{z \in \frac{1}{C}\mathbb{Z}^d \cap B_R} [\nabla\varphi + e]^{2q} \right)^{\frac{1}{q}}$$

$$\approx \left(\int_{B_{2R}} [\nabla\varphi + e]^{2q} \right)^{\frac{1}{q}}.$$

(b): $$\int_{\mathbb{R}^d} \| [\nabla(F v)] \|^2_{L^{2p'}(\Omega)}$$

$p \gg 1 \Rightarrow |p' - 1| \ll 1.$

$|2p' - 2| \ll 2.$

→ can use perturbative annealed regularity.

$$-\nabla \cdot a^* \nabla [F v] = \nabla \cdot (F g) \quad \text{in } \mathbb{R}^d,$$

$$\lesssim \int_{\mathbb{R}^d} \underbrace{\| [F g] \|^2_{L^{2p'}(\Omega)}}_{= \underbrace{\|F\|^2_{L^{2p'}(\Omega)}}_{=1} [g]^2} = \int_{\mathbb{R}^d} |g|^2 .$$

$\Rightarrow$ conclude:

$$\left\| \int g \cdot \nabla \varphi \right\|^2_{L^{2p}(\Omega)} \lesssim p \, \|g\|^2_{L^2(\mathbb{R}^d)} \left\| \left(\int_{B_R} [\nabla \varphi + e]^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)} .$$

Step 2: $\exists C_0 > 0$

$\forall 1 < p < \infty, \forall |q-1| \le \frac{1}{C_0}, \quad \forall 1 \le r \le \frac{1}{C_0} R$

$$\left\| \left(\fint \ |\nabla \varphi + e|^{2q} \right)^{\frac{1}{q}} \right\| \quad \le C \left\| \ \fint (\nabla \varphi + e) \right\|^2$$

$$\left\| \left(\int_{B_R} |\underbrace{\nabla \varphi + e}_{\nabla u}|^p \right)^{\frac{1}{p}} \right\|_{L^p(\Omega)} \lesssim C_0 \left\| \int_{B_{2R}} |\underbrace{\nabla \varphi + e}_{\nabla u}| \right\|_{L^{2p}(\Omega)}.$$

$$u = \varphi + e \cdot x, \qquad -\nabla \cdot a \nabla u = 0.$$

* Caccioppoli inequality: $\forall \ell > 0$, $\forall C \in \mathbb{R}$, $\displaystyle \fint_{B_\ell} |\nabla u|^2 \lesssim \frac{1}{\ell^2} \fint_{B_{2\ell}} |u - C|^2.$

* Poincaré-Sobolev: $\displaystyle C = \fint_{B_{2\ell}} u.$

$$C_0 \ \forall \ell > 0: \left(\fint_{B_\ell} |\nabla u|^2 \right)^{\frac{1}{2}} \lesssim \left(\fint_{B_{2\ell}} |\nabla u|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2d}}.$$

* Gehring's lemma: $\exists C_0 > 0$ large, $\forall |q - 1| \leq \frac{1}{C_0}$,

$$\forall \ell \quad \left(\fint |\nabla u|^{2q} \right)^{\frac{1}{2q}} \lesssim \left(\fint |\nabla u|^2 \right)^{\frac{1}{2}}$$

$$\forall R>0: \left(\fint_{B_R} |\nabla u|^{2q}\right)^{\frac{1}{2q}} \lesssim \left(\fint_{B_{2R}} |\nabla u|^2\right)^{\frac12}$$

$$\lesssim \frac{1}{R}\left(\fint_{B_{4R}} |u-c|^2\right)^{\frac12} \quad \forall c.$$

Smuggle in $\fint_{B_r(x)} u$, $1 \le r \le R$.

$$c = \fint_{B_{4R}}\left(\fint_{B_r(x)} u\right)dx.$$

$$\to \left(\fint_{B_R} |\nabla u|^{2q}\right)^{\frac{1}{2q}} \lesssim \underbrace{\frac{1}{R}\left(\fint_{B_{4R}} \left|u(x) - \fint_{B_r(x)} u\right|^2 dx\right)^{\frac12}}_{\le \frac{r}{R}\left(\fint_{B_{4R}} |\nabla u|^2\right)^{\frac12}} + \underbrace{\frac{1}{R}\left(\fint_{B_{4R}} \left|\fint_{B_r(x)} u - c\right|^2 dx\right)^{\frac12}}_{\text{Poincaré: } \le \left(\fint_{B_{4R}} \left|\fint_{B_r(x)} \nabla u\right|^2\right)^{\frac12}}.$$

Take $\|(\cdot)^2\|_{L^p(\Omega)}$:

$$\left\| \left(\fint_{B_R} |Du|^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)} \lesssim \underbrace{\left(\frac{r}{R}\right)^2 \left\| \fint_{B_{4R}} |Du|^2 \right\|_{L^p(\Omega)}} + \underbrace{\left\| \fint_{B_{4R}} \left| \fint_{B_r(x)} Du \right|^2 dx \right\|_{L^p(\Omega)}}$$

$$\lesssim \left(\frac{r}{R}\right)^2 \left\| \fint_{B_R} |Du|^2 \right\|_{L^p(\Omega)}$$

by stationarity!

$$\leq \underbrace{\left(\frac{r}{R}\right)^2 \left\| \left(\fint_{B_R} |Du|^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)}}$$

(Jensen)

absorb for $r \ll R$.

$$\lesssim \underbrace{\left\| \fint_{B_r} Du \right\|_{L^{2p}(\Omega)}^2}$$

by stationarity!

$\to$ $\forall\, 1 \leq r \ll R$:

$$\left\| \left(\fint_{B_R} |Du|^{2q} \right)^{\frac{1}{q}} \right\|_{L^p(\Omega)} \lesssim \left\| \fint_{B_r} Du \right\|_{L^{2p}(\Omega)}^2 .$$

[illegible] //