M285K - course #15

IV. QUANTITATIVE THEORY: FLUCTUATIONS.

$$-\nabla \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon = \nabla \cdot f$$

3 questions: ⊛ macro observable $\int g \cdot \nabla u_\varepsilon \xrightarrow{a.s} \int g \cdot \nabla \bar{u}$ in terms of some homog eqn.

⊛ $a\left(\tfrac{\cdot}{\varepsilon}\right)$ oscillating on scale $O(\varepsilon)$

$\Rightarrow \nabla u_\varepsilon$ should also on scale $O(\varepsilon)$.

Intrinsic description: 2-scale expansion

$$\nabla u_\varepsilon - \left(\nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) + e_i\right) \nabla_i \bar{u} = O(\varepsilon) \quad d \geq 2$$

in $L^p(\mathbb{R}^d, L^q(\Omega))$

$1 < p, q < \infty$.

+ get a rate: $\left| \int g \cdot \nabla u_\varepsilon - \int g \cdot \nabla \bar{u} \right|$ in $L^q(\Omega)$, $q < \infty$

$$\le O(\varepsilon) + \underbrace{\left| \int g \cdot \nabla \varphi_i\left(\tfrac{\cdot}{\varepsilon}\right) \nabla_i \bar{u} \right|}_{O(\varepsilon^{d/2}) \text{ in } L^q(\Omega)\,!}$$

(cf CLT scaling for $\nabla \varphi_i$)

(*) a random $\Rightarrow$ ∇u_ε random

$$\varepsilon^{-d/2} \int g \cdot \left(a\left(\tfrac{\cdot}{\varepsilon}\right) - \mathbb{E} a \right) \xrightarrow[\text{(CLT)}]{\text{law}} \text{Gaussian } \int g \cdot K \xi$$

ξ white noise.

↳ expect: $\varepsilon^{-d/2} \int g \cdot \left(\nabla u_\varepsilon - \mathbb{E} \nabla u_\varepsilon \right) \xrightarrow{\text{law}}$ Gaussian?

should not be a white noise!

$\boxed{\text{Goal}}$: get some intrinsic description?

2-scale expansion? —

Observation.

$$\varepsilon^{-d/2}\int g\cdot(\nabla u_\varepsilon - \mathbb{E}\nabla u_\varepsilon) \xrightarrow{\text{law}} \text{Gaussian}$$

$$\varepsilon^{-d/2}\int g_i\, \nabla\varphi_i(\tfrac{\cdot}{\varepsilon})\, \nabla_i \bar u \xrightarrow{\text{law}} \text{Gaussian}$$

BUT the limits are different!

→ 2-scale expansion is NOT accurate in fluctuation scaling ...

Question: do we still have an intrinsic description in terms of correctors?

Key quantity: homogenization commutator } commutator between

- large-scale average

$$\Xi_\varepsilon[\nabla u_\varepsilon] := a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - \bar a \nabla u_\varepsilon \qquad (\quad \cdot \text{ field-flux relation}$$

Recall that this is a natural quantity:

$$\left[\begin{array}{ll} \Xi_\varepsilon[\nabla u_\varepsilon] \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^d)^d & \big\} \ d \text{ relation} \\ \Longleftrightarrow \nabla u_\varepsilon \rightharpoonup \nabla \bar u \ \& \ a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon \rightharpoonup \bar a \nabla \bar u \text{ in } L^2(\mathbb{R}^d)^d & \big\} \ 2d \text{ relation.} \end{array}\right.$$

<u>3 key principles</u> (to be stated later):

① $\begin{pmatrix} \nabla u_\varepsilon \\ a(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon \end{pmatrix}$ = deterministic projection of $\Xi_\varepsilon[\nabla u_\varepsilon]$

↪ all fluctuations reduce to fluctuations of $\Xi_\varepsilon[\nabla u_\varepsilon]$.

② 2-scale expansion of commutators:

$$\Xi_\varepsilon[\nabla u_\varepsilon] \sim \underbrace{\Xi_\varepsilon\Big[\big(\nabla\varphi_i(\tfrac{\cdot}{\varepsilon})+e_i\big)\nabla_i\bar u\Big]}_{\big[\underbrace{a(\nabla\varphi_i+e_i)-\bar a(\nabla\varphi_i+e_i)}_{\Xi_i^\circ \text{ "standard commutator"}}\big](\frac{\cdot}{\varepsilon})\nabla_i\bar u} + O(\varepsilon) \quad (d>2)$$

in fluctuation scaling!

(vs: $\nabla u_\varepsilon \sim \big(\nabla\varphi_i(\tfrac{\cdot}{\varepsilon})+e_i\big)\nabla_i\bar u + O(\varepsilon)$ in fluct scaling)

③ Ξ° is "almost" local wrt a.

$\Rightarrow \Xi^\circ \sim$ white noise on large scales —

Remark: role of commutator for fluctuations = improved locality.

$$-\nabla\cdot a\nabla u = \nabla\cdot f \qquad (\varepsilon=1)$$

⊛ ∇u : critical locality wrt a.

Malliavin: $-\nabla\cdot a\nabla D_z u = \nabla\cdot D_z a \nabla u$ $\qquad a(x) = a_0(G(x))$

$$D_z \nabla u(x) = \underbrace{\nabla\nabla G(x,z)}_{\sim |x-z|^{-d}}\, a_0'(G(z))\, \nabla u(z)$$

⊛ $\Xi[\nabla u]$: exactly local wrt a + small error.

<u>Lemma</u>.

$$D\,\Xi[\nabla u]_j = (\nabla\varphi_j^* + e_j)\cdot \overbrace{Da}^{\delta}\,\nabla u$$
$$- \underbrace{\nabla\cdot}_{\text{small on large scales!}} \Big((\varphi_j^* a + \sigma_j^*) \underbrace{\nabla D u}_{|\cdot|^{-d}} + \varphi_j^* \underbrace{Da}_{\delta}\, \nabla u \Big)$$

exact localit

↳ after ε-rescaling. [illegible]

$$\mathbb{D}\,\boxed{H}_\varepsilon[\nabla u_\varepsilon]_j = (\nabla\varphi_j^*(\tfrac{\cdot}{\varepsilon}) + e_j)\cdot \mathbb{D}a(\tfrac{\cdot}{\varepsilon})\,\nabla u_\varepsilon$$

$$- \nabla\cdot\Big(\underbrace{\varepsilon\,(\varphi_j^* a + \sigma_j^*)(\tfrac{\cdot}{\varepsilon})\,\nabla\mathbb{D}u_\varepsilon + \varepsilon\,\varphi_j^*(\tfrac{\cdot}{\varepsilon})\,\mathbb{D}a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon}_{O(\varepsilon),\ \text{critical locality}}\Big)$$

Proof. $\boxed{H}[\nabla u] = a\nabla u - \bar{a}\nabla u$

$$\begin{cases} -\nabla\cdot a^*(\nabla\varphi_j^* + e_j) = 0 \\ \nabla\cdot\sigma_j^* = a^*(\nabla\varphi_j^* + e_j) - \bar{a}^* e_j. \end{cases}$$

$$\leadsto \mathbb{D}\,\boxed{H}[\nabla u]_j = e_j\cdot\mathbb{D}a\,\nabla u + \underbrace{e_j\cdot(a-\bar{a})\,\nabla\mathbb{D}u}_{(a^*-\bar{a}^*)e_j\,\cdot\,\nabla\mathbb{D}u}$$

Decompose: $$(a^* - \bar{a}^*)e_j = \underbrace{-a^*\nabla\varphi_j}_{\text{grad like}} + \underbrace{\underbrace{\left(a^*(\nabla\varphi_j + e_j) - \bar{a}^* e_j\right)}_{\nabla\cdot\sigma_j^*}}_{\text{div free}}$$

"a-Helmholtz decomposition" of $a^* - \bar{a}^*$!

$$\hookrightarrow (a^* - \bar{a}^*)e_j \cdot \nabla Du = -\nabla\varphi_j^* \cdot a\nabla Du + (\nabla\cdot\sigma_j^*)\cdot\nabla Du$$

$$= -\nabla\cdot(\varphi_j^* a\nabla Du) + \varphi_j \nabla\cdot a\nabla Du$$

$$- \nabla\cdot(\sigma_j^*\nabla Du)$$

where $$(\nabla\cdot\sigma_j^*)\cdot\nabla Du = (\nabla_l \sigma^*_{jkl})(\nabla_k Du)$$

$$= \nabla_l(\sigma^*_{jkl}\nabla_k Du) - \overset{0}{\cancel{\sigma^*_{jkl}\nabla^2_{kl} Du}}$$

$$\underbrace{\quad}_{-\sigma_{jlk}} \qquad \underbrace{\quad}_{\text{skew}} \quad \underbrace{\quad}_{\text{sym}}$$

$$= -\nabla\cdot(\sigma_j^* \nabla D u).$$

$$\to \ (a^* - \bar{a}^*) e_j \cdot \nabla D u \ = -\nabla\cdot\left((\varphi_j^* a + \sigma_j^*) \nabla D u \right)$$

$$+ \underbrace{\varphi_j^* \underbrace{\nabla\cdot a \nabla D u}_{-\nabla\cdot D a \nabla u}}$$

$$- \nabla\cdot(\varphi_j^* D a \nabla u)$$

$$+ \nabla\varphi_j^* \cdot D a \nabla u .$$

IV.1 Pathwise structure of fluctuations

= intrinsic description of fluctuations in terms of [illegible]

Lemma (Principle 1 : reduction to commutators).

$$\int_{\mathbb{R}^d} g\cdot\left(\nabla u_\varepsilon - \nabla \bar u\right) = \int_{\mathbb{R}^d} \bar{\mathcal{H}}^*(g)\cdot \Xi_\varepsilon[\nabla u_\varepsilon]$$

$$\int_{\mathbb{R}^d} g\cdot\left(a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon - \bar a\nabla \bar u\right) = \int_{\mathbb{R}^d} \bar{\mathcal{L}}^*(g)\cdot \Xi_\varepsilon[\nabla u_\varepsilon]$$

where

$$\begin{cases} \bar{\mathcal{H}}(h) = \nabla(-\nabla\cdot\bar a\nabla)^{-1}\nabla\cdot h & \bar a\text{-Helmholtz projection (grad like)} \\ \bar{\mathcal{L}}(h) = h + \bar a\nabla(-\nabla\cdot\bar a\nabla)^{-1}\nabla\cdot h & \bar a\text{-Leray projection (div free)} \end{cases}$$

Proof. $-\nabla\cdot\bar a^*\nabla\bar v = \nabla\cdot g$

$$\nabla \bar{v} = \mathcal{H}^0 g$$

$$\rightsquigarrow \int g \cdot \nabla u_\varepsilon = -\int \nabla \bar{v} \cdot \bar{a} \, \nabla u_\varepsilon$$

$$= \int \nabla \bar{v} \cdot \mathcal{H}_\varepsilon[\nabla u_\varepsilon] \; \underbrace{- \int \nabla \bar{v} \cdot a\left(\tfrac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon}$$

$$= \int \nabla \bar{v} \cdot f$$

$$= -\int \nabla \bar{v} \cdot \bar{a} \, \nabla \bar{u}$$

$$= \int g \cdot \nabla \bar{u} \quad !$$

$\square$

REMARK:

Helmholtz proj $\mathcal{H}_0 g = \nabla \Delta^{-1} \nabla \cdot g$) grad like

Leray proj $\mathcal{L}_0 g = g - \nabla \Delta^{-1} \nabla \cdot g$; $\nabla \cdot \mathcal{L}_0 g = \nabla \cdot g - \nabla \cdot \nabla \Delta^{-1} \nabla \cdot g = 0$

div free.

$$L^2 \ni g = \underbrace{\mathcal{P}_0 g}_{\text{grad}} + \underbrace{\mathcal{L}_0 g}_{\text{div free}}$$