M285K - course #17

IV.1 Pathwise theory of fluctuations.

<u>Theorem</u>. $\forall f, g \in C_c^\infty(\mathbb{R}^d)^d$, $\forall q < \infty$,

$$\mathbb{E}\left[\left| \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} g \cdot \left(\nabla u_\varepsilon - \mathbb{E}[\nabla u_\varepsilon]\right) - \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} \bar{\mathcal{H}}^{*}(g) \cdot \Xi_\varepsilon^\circ[\nabla \bar{u}] \right|^q\right]^{\frac{1}{q}}$$

$$\lesssim_{f,g,q} \varepsilon\, \mu_d\left(\tfrac{1}{\varepsilon}\right).$$

IV.2 Scaling limit of Ξ°.

(matrix-valued) white noise $\quad \left(\mathbb{E}\, \xi_{ij}(x)\, \xi_{kl}(0) = \delta_{ik}\delta_{jl}\, \delta(x).\right)$

Recall : $\varepsilon^{-\frac{d}{2}} \int h : \left(a\left(\tfrac{\cdot}{\varepsilon}\right) - \mathbb{E}\, a\right) \xrightarrow[\text{CLT}]{\text{law}} \int h : C_0\, \xi$ for some cont C_0

$$C_0 : C_0 = \int_{\mathbb{R}^d} \mathrm{Cov}\left(a(x), a(0)\right) dx$$

$\ldots\ \varepsilon^{-\frac{d}{2}} \int \ldots\ \Xi^\circ[\nabla \ldots] \xrightarrow{\text{law}} \int \nabla \ldots : \bar{K}\, \xi$

Expect : $\varepsilon^{-1/2}\int g\cdot [A]_\varepsilon[\nabla u] \to \int \nabla u\otimes g : \nabla_0 S$

2 steps: * convergence of variance

* asymptotic normality

Theorem (convergence of covariance).

$\forall f,g \in C_c^\infty(\mathbb{R}^d)^d,$

$$\left| \mathrm{Var}\left[\varepsilon^{-1/2}\int g\cdot [A]^0_\varepsilon[D\bar u]\right] - \int \nabla\bar u\otimes g : \bar K\, \nabla \bar u\otimes g \right| \lesssim_{f,g} \varepsilon\, \mu_d\left(\tfrac{1}{\varepsilon}\right).$$

for some 4-tensor $\bar K$ $(= \bar K_0 : \bar K_0)$ (typically non-degenerate).

Exercise: Green-Kubo formula for variance:

$$\bar{K}_{ijkl} = \lim_{L\uparrow\infty} \int_{\mathbb{R}^d} \chi\left(\frac{x}{L}\right) \underbrace{\mathrm{Cov}\Big(e_j\cdot(a-\bar{a})(\nabla\phi_i+e_i)(x) \,;\, e_l\cdot(a-\bar{a})(\nabla\phi_k+e_k)(0)\Big)}_{\text{a priori} \sim |x|^{-d} \text{ not summable}} dx$$

for any $\chi\in C^\infty_c(\mathbb{R})$, $\chi(0)=1$.

Proof. Starting point is Helffer-Sjöstrand formula

$$\mathrm{Var}\, X = \mathbb{E}\left[\langle DX, (1+\mathcal{L})^{-1} DX\rangle_{\mathfrak{H}}\right]$$

$$= \mathbb{E}\left[\| (1+\mathcal{L})^{-\frac{1}{2}} DX\|^2_{\mathfrak{H}}\right].$$

Recall $$D\, \Xi^{\circ}_{\varepsilon}[\nabla\bar{u}]_j = \Big[\overbrace{(\nabla\phi^*_j+e_j)(\tfrac{\cdot}{\varepsilon})\cdot \underline{Da}(\tfrac{\cdot}{\varepsilon})\,(\nabla\phi_i+e_i)(\tfrac{\cdot}{\varepsilon})}^{\equiv Q_\varepsilon}\, \nabla_i\bar{u}\Big]$$

$$
\left.\begin{aligned}
&- (\nabla_i \bar{u})\, \nabla\cdot \left[\varepsilon \,(a\varphi_j^* + \sigma_j^*)(\tfrac{\cdot}{\varepsilon})\, \cancel{\nabla}\mathcal{D}\varphi_i(\tfrac{\cdot}{\varepsilon}) \right] \\
&- (\nabla_i \bar{u})\, \nabla\cdot \left[\varepsilon\, \varphi_j^*(\tfrac{\cdot}{\varepsilon})\, \mathcal{D}a(\tfrac{\cdot}{\varepsilon})\, (\nabla\varphi_i + e_i)(\tfrac{\cdot}{\varepsilon}) \right]
\end{aligned}\right\} \text{error } O\!\left(\varepsilon\, \mu_d(\tfrac{1}{\varepsilon})\right)
$$

<u>Exercise</u>:
$$
\left| \mathrm{Var}\left[\varepsilon^{-d/2} \int g \cdot \Xi_\varepsilon^{\circ}[\nabla \bar{u}]\right] - \underbrace{\mathbb{E}\left[\left\| (1+\mathcal{L})^{-1/2}\, \varepsilon^{-d/2} \int g\cdot Q_\varepsilon \right\|_h^2 \right]}_{\equiv K_\varepsilon(f,g)} \right| \lesssim_{f,g} \varepsilon\, \mu_d(\tfrac{1}{\varepsilon}).
$$

Let's compute K_ε!

Recall $D_z a(\tfrac{x}{\varepsilon}) = a_0'(G(z))\, \delta(\tfrac{x}{\varepsilon} - z) = \varepsilon^d\, a_0'(G(z))\, \delta(x - \varepsilon z)$

$$
\Rightarrow \varepsilon^{-d/2} \int g \cdot Q_\varepsilon = \varepsilon^{d/2}\, g_j(\varepsilon z)\, \nabla_i \bar{u}(\varepsilon z)\, (\nabla\varphi_j^* + e_j)(z) \cdot a_0'(G(z))\, (\nabla\varphi_i + e_i)(z).
$$

$$
\Rightarrow K_\varepsilon(f,g) = \mathbb{E}\left[\left\| (1+\mathcal{L})^{-1/2}\, \varepsilon^{d/2}\, (g_j \nabla_i \bar{u})(\varepsilon\,\cdot) \left[\underbrace{(\nabla\varphi_j^* + e_j)\cdot a_0'(G)\,(\nabla\varphi_i + e_i)} \right] \right\|_h^2 \right]
$$

$$\equiv T_{ij}$$

$$= \varepsilon^{d} \iint dx\,dy\; C(x-y)\; (\varrho_j D_i \bar{u})(\varepsilon x)\; (\varrho_\ell D_k \bar{u})(\varepsilon y)$$

$$\underbrace{\mathbb{E}\left[T_{ij}(x)\, (1+\mathcal{L})^{-1} T_{k\ell}(y) \right]}_{= \mathbb{E}\left[T_{ij}(x-y)\, (1+\mathcal{L})^{-1} T_{k\ell}(0)\right] \text{ by stationarity}}$$

$$= \iint dx\,dy\; C(x)\; (\varrho_j D_i \bar{u})(\varepsilon x + y)\; (\varrho_\ell D_k \bar{u})(y)$$

$$\mathbb{E}\left[T_{ij}(x)\, (1+\mathcal{L})^{-1} T_{k\ell}(0) \right]$$

$$= \Big(\underbrace{\int_{\mathbb{R}^d} C(x)\; \mathbb{E}\left[T_{ij}(x) (1+\mathcal{L})^{-1} T_{k\ell}(0) \right] dx}_{\bar{k}_{ijk\ell}\ !} \Big) \int_{\mathbb{R}^d} (\varrho_j D_i \bar{u})(y)\; (\varrho_\ell D_k \bar{u})(y)\, dy$$

$$+\, O_{f,g}(\varepsilon^2) \leftarrow \text{exercise}$$

$\square$

<u>Theorem</u> (asymptotic normality).

$\forall f, g \in C_c^\infty(\mathbb{R}^d)^d$,

usual rate for CLT for i.i.d structure

correction due to borderline nonlocality

$$\delta\left[\varepsilon^{-\frac{d}{2}} \int g \cdot \boxed{\Psi}_\varepsilon^\circ[\nabla \bar{u}]\right] \lesssim_{f,g} \varepsilon^{\frac{d}{2}} |\log \varepsilon| \qquad (\text{optimal !})$$

where $\delta(X) := d_{TV}\left(\frac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}\,X}}, \mathcal{N}\right) + W_2\left(\frac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}\,X}}, \mathcal{N}\right).$

<u>Proof</u>. Set $\varepsilon = 1$.

$$\delta(X) \le \frac{1}{\ldots} \mathbb{E}\left[\|DX\|^4\right]^{\frac{1}{4}} \mathbb{E}\left[\|D^2X\|^4\right]^{\frac{1}{4}}$$

Malliavin - Stein method: $d(X) \lesssim \frac{1}{\mathrm{Var}\, X} \mathbb{E}\left[\|DX\|_h\right] \cdot \mathbb{E}\left[\|D^2 X\|_{h\to h}\right]$.

Step 1.

$$D \int g \cdot \underbrace{H}^{\partial}[\nabla \bar{u}] = \dots$$

$$D^2 \int g \cdot \underbrace{\varphi}^{\partial}[\nabla \bar{u}] = U_1 + U_2 + U_3 \quad \text{(viewed as operators } h \to h\text{)}$$

$$U_1(x,y) = \delta(x-y)\Big((g_j \nabla \bar{u})\,(\nabla\varphi_j^* + e_j) \cdot a_0''(G)\,(\nabla\varphi_i + e_i)\Big)(x)$$

exactly local

$\underbrace{// \; D^2_{xy} a = a_0''(G)\, \delta(x-y)}$

$$U_2(x,y) = \Big((g_j \nabla_i \bar{u})\,(\nabla\varphi_j^* + e_j) \cdot a_0'(G)\, \nabla D_y \varphi_i \Big)(x) + \mathrm{sym}(x,y)$$

field nonlocality.

(responsible for $|\log \varepsilon|$ in the rate)

$$U_3(x,y) = \delta(x-y)\Big(\varphi_j^* \,\nabla(g_j \nabla_i \bar{u}) \cdot a_0''(G)(\nabla\varphi_i + e_i)\Big)(x)$$
$$+ \Big(\varphi_j^* \,\nabla(g_j \nabla_i \bar{u}) \cdot a_0'(G)\, \nabla D_y \varphi_i \Big)(x) + \mathrm{sym}$$
$$+ \int \nabla(g_j \nabla_i \bar{u}) \cdot (a\varphi_j^* + \sigma_j^*)\, D^2_{xy} \nabla\varphi_i$$

nonlocal errors

(of gradient on test fct!)

Indeed: $D \int \rho \cdot \boxed{H}^{\circ}[\nabla \bar u] = \int \rho \cdot \underbrace{Da}\,(\nabla\varphi_i + e_i)\nabla_i \bar u + \int \rho \cdot (a - \bar a)\nabla D\varphi_i \, \nabla_i \bar u \,.$

$$D^2 \int \rho \cdot \boxed{H}^{\circ}[\nabla \bar u] = \int \rho \cdot D^2 a \,(\nabla\varphi_i + e_i)\nabla_i \bar u \;\Big\}\to U_1$$

$$+ 2\int \rho \cdot \underbrace{Da}\, \nabla D\varphi_i \, \nabla_i \bar u \;\Big\}\to U_2$$

$$+ \int \rho \cdot \underbrace{(a - \bar a)}_{\circ} \nabla D^2 \varphi_i \, \nabla_i \bar u$$

Recall $(a^{\circ} - \bar a^{\circ})e_j = -a^{\circ}\nabla\varphi_j^{\circ} + \nabla\cdot\sigma_j^{\circ}$

$\leadsto \int \rho \cdot (a - \bar a)\nabla D^2\varphi_i \, \nabla_i \bar u = \int \rho_j \nabla_i \bar u \left(-a^{\circ}\nabla\varphi_j^{\circ} + \nabla\cdot\sigma_j^{\circ}\right)\cdot \nabla D^2 \varphi_i$

$$\begin{cases} -\nabla\cdot a \nabla D\varphi_i = \nabla\cdot Da(\nabla\varphi_i + e_i) \\ -\nabla\cdot a \nabla D^2\varphi_i = 2\nabla\cdot Da \nabla D\varphi_i \end{cases}$$

→ exercise : integration by parts & recognize U_1, U_2, U_3.

$+ D \cdot a D D^2 \varphi_i$

Step 2 : $\mathbb{E}\left[\left\| D \int g \cdot \boxed{\Xi}^\circ [D\bar{u}] \right\|_h^4\right]^{1/4} \lesssim \|f\|_{L^4} \|g\|_{L^4}$

as in proof of CLT scaling. ✓

(using corrector estimates + annealed reg)

Step 3. $\mathbb{E}\left[\left\| \overbrace{D^2 \int g \cdot \boxed{\Xi}^\circ [D\bar{u}]}^{U_1 + U_2 + U_3} \right\|_{h \to h}^4\right]^{1/4} \lesssim p \, \|g D\bar{u}\|_{L^p(\mathbb{R}^d)} + C(q) \, \|\mu_d(|\cdot|) \, D(g D\bar{u})\|_{L^q(\mathbb{R}^d)}$

$\forall p, q \geq 4$

(→ rescale : $(f, g) \to (\varepsilon^{d/4} f(\varepsilon \cdot), \ \varepsilon^{d/4} g(\varepsilon \cdot))$

anyway smaller !

$\Rightarrow$ right-hand side : $\underbrace{\rho\, \varepsilon^{\frac{d}{2}-\frac{d}{\rho}}}_{\circ} + C(q)\, \varepsilon\, \mu_d\left(\frac{1}{\varepsilon}\right)\, \varepsilon^{\frac{d}{2}-\frac{d}{q}}$

optimize in $\rho = |\ell_\varepsilon| \Rightarrow \underbrace{|\ell_\varepsilon|}\, \varepsilon^{d/2}$ correct rate !!

- U_1 : local - easier - } exercise
- U_3 : error - easier - }

Difficult part : U_2.

$$\| U_2 \|_{h\to h} = \sup_{\substack{\|\rho\|_h = 1 \\ \|\rho'\|_h = 1}} \langle \rho', U_2 \rho \rangle_h$$

$$= \sup_{\rho, \rho'} \iint \overbrace{C * \rho'(x)}^{\zeta'}\, U_2(x,y)\, \overbrace{C * \rho(y)}^{\zeta}\, dx\, dy$$

& $\|\rho\|_h = \| C_0 * \rho \|_{L^2}$.

$$\lesssim \sup_{\substack{\zeta,\,\zeta' \\ \|[\zeta]_\infty\|_{L^2} = \|[\zeta']_\infty\|_{L^2} = 1}} \iint \zeta'(x)\, U_2(x,y)\, \zeta(y) .$$

$$\leq \sup_{\zeta :\, \|[\zeta_\infty]\|_{L^2} = 1} \int dy \left[\int dx\, U_2(x,y)\, \zeta(x) \right]^2$$

$$\mathbb{E}\left[\|U_2\|^4_{h\to h} \right]^{1/4} = \sup_{\|X\|_{L^4(\Omega)} = 1} \mathbb{E}\left[X^2 \|U_2\|^2_{L\to h} \right]^{1/2}$$

we have replaced $X\zeta$ by "ζ".

$$\lesssim \sup_{\|[\zeta]_\infty\|_{L^4(\Omega, L^2(\mathbb{R}^d))} = 1} \mathbb{E}\left[\int dy \left[\int dx\, U_2(x,y)\, \zeta(x) \right]^2 \right]^{1/2} .$$

(i.e : this duality argument has allowed to permute $\mathbb{E}$ & sup !)

Let ζ be fixed with $\|[\zeta]_\infty\|_{L^4(\Omega, L^2(\mathbb{R}^d))} = 1$.

Compute $\displaystyle\int dx\, U_2(x,y)\,\zeta(x) = \int \zeta\, \rho_j \partial_i \bar{u}\, (\nabla \varphi_j^* + e_j) \cdot a_0'(G)\, \nabla D_y \varphi_i$

auxiliary problem:

$$\begin{cases} -\nabla \cdot a^* \nabla T_i \\ \quad = \nabla \cdot \left[\zeta \rho_j \partial_i \bar{u}\; a_0'(G)^* (\nabla \varphi_j^* + e_j)\right]. \end{cases}$$

$$= -\int \nabla T_i \cdot a \nabla D_y \varphi_i$$

$$= \int \nabla T_i \cdot D_y a\, (\nabla \varphi_i + e_i)$$

$$= \nabla T_i(y) \cdot a_0'(G(y))\, (\nabla \varphi_i + e_i)(y).$$

$$\mathbb{E}\left[\int \int [\ldots]^2\right] \leq \mathbb{E} \int [\nabla T_i]^2 [\nabla \varphi_i + e_i]^2$$

$$\mathbb{E}\left[\int [\zeta v_2]\right] \lesssim \mathbb{E}\int [\nabla T_i]\,[\nabla\varphi_i + e_i]$$

$$\le \underbrace{\|[\nabla\varphi_i+e_i]\|^2_{L^p(\Omega)}}_{\substack{\le \rho^c \\ \text{corrector estimates}}} \; \underbrace{\|[\nabla T]\|^{\sim 2}_{L^2(\mathbb{R}^d,\, L^{\frac{2p}{p-2}}(\Omega))}}$$

$$\le \|[\zeta \rho_i \nabla_i \bar u\, (\nabla\varphi_i + e_i)]\|_{L^2(\mathbb{R}^d;\, L^{\frac{2p}{p-2}}(\Omega))}$$

by perturbative annealed regularity!

$$\le \|[\nabla\varphi_j + e_j]\|_{L^{2p}(\Omega)}\, \|[\rho\nabla\bar u]_\infty\|_{L^p(\mathbb{R}^d)}\, \|[\zeta]_\infty\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^d,\, L^{\frac{2p}{p-2}}(\Omega))}.$$

& $\|[\zeta]_\infty\|_{L^4(\Omega,\, L^2(\mathbb{R}^d))} = 1$

$\Rightarrow \|[\zeta]\|_{L^r(\mathbb{R}^d;\, L^r(\Omega))} \lesssim 1 \quad \forall\, 2 \le r \le 4$ by interpolation

$$\Rightarrow \mathbb{E}\left[\int \ldots\right] \lesssim \rho^{2C}\, \|[\rho \ldots]\|^2$$

$$\to \mathbb{E}\left[\int \left[\int \xi\, U_2 \int\right]\right] \lesssim p \quad \|[g \nabla u]_\infty\|_{L^p}$$

imp. $\to$
$$\mathbb{E}\left[\|U_2\|^4_{h\to h}\right]^{1/4} \lesssim p^C \, \|[g \nabla \bar{u}]_\infty\|_{L^p} .$$

In fact: we can get p!

$$\|[\nabla \varphi]\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}}! \quad \text{(Gaussian moments)}$$

(we don't prove this here...)

$\square$