M285K - course #18

Chapters III & IV : fine description of oscillations & fluctuations with optimal rates.

$$(*) \quad \left\| \nabla u_\varepsilon - (\nabla\varphi_i(\tfrac{\cdot}{\varepsilon}) + e_i)\, \nabla_i \bar{u} \right\|_{L^p(\mathbb{R}^d;\, L^q(\Omega))} \lesssim \varepsilon\, \mu_d(\tfrac{1}{\varepsilon}) \quad \forall\, 1 < p, q < \infty.$$

$$(\circledast) \quad d_{TV}\left(\frac{\varepsilon^{-d/2} \int g \cdot (\nabla u_\varepsilon - \mathbb{E}\nabla u_\varepsilon)}{\left(\int (\nabla\bar{u} \otimes \bar{\mathcal{H}}^*_g) : \bar{k}\, (\nabla\bar{u} \otimes \bar{\mathcal{H}}^*_g)\right)^{\frac{1}{2}}},\ \mathcal{N} \right) \lesssim \underbrace{\varepsilon\, \mu_d(\tfrac{1}{\varepsilon})}_{\text{conv. of covariance structure}} + \varepsilon^{d/2} |\log \varepsilon|$$

that is: $\varepsilon^{-d/2} \int g \cdot (\nabla u_\varepsilon - \mathbb{E}\nabla u_\varepsilon) \xrightarrow{\text{law}} \int (\nabla\bar{u} \otimes \bar{\mathcal{H}}^*_g) : \bar{K}_0\, \xi$

$$(\bar{K}_0 : \bar{K}_0 = \bar{k}).$$

<u>2 ingredients</u>: – Malliavin calculus: linearize wrt a

[illegible] L^p regularity

– annealed C regularity .

V. LARGE-SCALE REGULARITY THEORY.

$$-\nabla\cdot \underline{a}\nabla u = \nabla\cdot h, \qquad h\in C_c^\infty(\mathbb{R}^d; L^\infty(\Omega)).$$

<u>Heuristics</u> : * De Giorgi counterexamples should not be typical for a stationary random ensemble

→ expect better regularity after averaging

"annealed regularity"

* on large scales $\left[\, (-\nabla\cdot a\nabla)^{-1} \sim \underset{\bar a \text{ cst coeff}}{(-\nabla\cdot \bar a\nabla)^{-1}} \,\right]$

→ expect same regularity as for Δ on large scales "quenched large-scale regularity"

V.1 "Prehistory": annealed Green function estimates.

Recall: $-\nabla \cdot a \nabla G(\cdot, y) = \delta(\cdot - y)$

$G(\cdot, y) \in W^{1,1}_{loc}(\mathbb{R}^d) \cap H^1_{loc}(\mathbb{R}^d \setminus \{y\})$

Green representation:
$$\begin{cases} -\nabla \cdot a \nabla u = \nabla \cdot h \\ \nabla u(x) = \displaystyle\int_{\mathbb{R}^d} \nabla\nabla G(x,y)\, h(y)\, dy . \end{cases}$$

Well-known: $|G(x,y)| \lesssim |x-y|^{2-d}$ pointwise

Theorem (Delmotte-Deuschel '05).

Let a be stat., let G Green's fct.

Then $\forall x,y \in \mathbb{R}^d$: $\begin{cases} \mathbb{E}\left[|\nabla G(x,y)|^2\right]^{\frac{1}{2}} \lesssim |x-y|^{1-d}. \\ \mathbb{E}\left[|\nabla\nabla G(x,y)|\right] \lesssim |x-y|^{-d}. \end{cases}$

Consequence : $-\nabla\cdot a\nabla u = \nabla\cdot h$

$$\|u\|_{L^p(\mathbb{R}^d, L^1(\Omega))} \overset{\text{above}}{\lesssim} \left\| |\cdot|^{1-d} * h \right\|_{L^p(\mathbb{R}^d)} \overset{\text{HLS}}{\lesssim} \|h\|_{L^{\frac{dp}{d+p}}(\mathbb{R}^d)} \quad \left(p > \frac{d}{d-1}\right)$$

Proof. Focus on ∇G.

On $\mathbb{R}^d \setminus \{y\}$: $-\nabla\cdot a\nabla G(\cdot,y) = 0$

$\hookrightarrow -\nabla\cdot a\nabla \partial_y G(\cdot,y) = 0$

De Giorgi-Nash-Moser: $|\partial_y G(0,y)|^2 \leq \sup_{B_{|y|/6}} |\partial_y G(\cdot,y)|$

$$\overset{\text{DGNM}}{\lesssim} \fint_{B_{|y|/3}} |\partial_y G(x,y)|^2 dx$$

Stationarity: $G(x,y) \overset{\text{law}}{=} G(x+z, y+z) \quad \forall z.$

$$C_0 \mathbb{E}|\nabla_2 G(0,y)|^2 \lesssim \fint_{B_{|y|/3}} \mathbb{E}|\nabla_2 G(x,y)|^2 dx = \fint_{B_{|y|/3}} \mathbb{E}|\nabla_2 G(-y,-x)|^2 dx$$

Caccioppoli inequality: $-\nabla \cdot a^* \nabla \; G(\cdot y, \cdot) = 0$ on $\mathbb{R}^d \setminus \{y\}$.

$$C_0 \fint_{B_{|y|/3}} |\nabla_2 G(-y,-x)|^2 dx \lesssim |y|^{-d} \int_{\frac{1}{3}|y| \leq |x-y| \leq \frac{4}{3}|y|} |\nabla_2 G(-y,-x)|^2 dx$$

$$\overset{\text{Cacci}}{\lesssim} |y|^{-d-2} \int \quad |G(-y,-x)|^2 dx$$

$$\lesssim |y|^{-d-2} \int_{\frac{1}{3}|y| \le |x-y| \le \frac{5}{3}|y|} |\mathcal{G}(\ldots)| \, dx$$

$$\text{So } |\nabla_2 \mathcal{G}(0,y)|^2 \lesssim \underbrace{\underbrace{|y|^{-d-2}}_{|y|^{-d-2}} \underbrace{\int_{\frac{1}{3}|y| \le |x-y| \le \frac{5}{3}|y|}}_{|y|^{d}} \underbrace{\underbrace{|\mathcal{G}(-y,-x)|^2}_{|x-y|^{2(2-d)}}}_{|y|^{2(2-d)}}}_{|y|^{2(1-d)}} dx$$

pointwise bound on $\mathcal{G}$.

$\square$

V.2. Large-scale $C^{1,\alpha}$ regularity (Armstrong–Smart '14, ...)

Goal: $\left[-\Delta v = \nabla \cdot h \right.$

$$[\nabla v]_{C^\alpha} \lesssim [h]_{C^\alpha} \quad \forall \alpha \in (0,1).$$

Use Campanato version of Hölder spaces

Theorem (Campanato). If $g \in L^2(U)$ satisfies $\fint_{B_r(x)} |g - \fint_{B_r(x)} g|^2 \leq M^2 r^{2\alpha}$ $\forall B_r(x) \subseteq U$

for some exponent α, and some M,

Then $g \in C^\alpha(U)$ & $\forall U' \subset\subset U$:

$$\|g\|_{C^\alpha(U')} = \sup_{U'} |g| + \sup_{\substack{x,y \in U' \\ x \neq y}} \frac{|g(x) - g(y)|}{|x-y|^\alpha}$$

$$\lesssim_{\alpha, U', U} M + \|g\|_{L^2(U)}.$$

In heterogeneous setting:

$$\mathcal{E}_{a}(g, D) := \inf_{e \in \mathbb{R}^d} \fint_D |g - (e + \nabla\varphi_e)|^2$$

"excess"

L^2-deviation from gradient of a-harm coordinate.

Control will be restricted to sufficiently large scales: for some constant $C_0 < \infty$,

$$r_* = r_*[C_0] := \inf\left\{ r \geq 0 \;:\; \forall R \geq r, \quad \frac{1}{R^2} \fint_{B_R} \left| (\varphi, \sigma) - \fint_{B_R} (\varphi, \sigma) \right|^2 \leq \frac{1}{C_0} \right.$$

"minimal radius"

<u>Lemma</u> (control on r_*).

- If a stat. ergodic, then $r_* < \infty$ a.s.
- If $a = a_0(G)$, G Gaussian stat, cov $\mathcal{C} = C_0 * C_0$, $\int [C_0]_\infty < \infty$
  (same assumption as in Chapter III-IV).

  Then $\mathbb{E}\left[|r_*|^q\right]^{\frac{1}{q}} \lesssim q^C$, $\forall q < \infty$.
  (stretched exp. moment)

Proof: ① $\frac{1}{R^2} \fint_{B_R} \left|(\varphi,\sigma) - \fint_{B_R} (\varphi,\sigma)\right|^2 = \fint_{B} \left| \frac{1}{R}(\varphi,\sigma)(R\cdot) - \fint_{B} \frac{1}{R}(\varphi,\sigma)(R\cdot)\right|^2$

$\to 0$ a.s by qualitative sublinearity

$\Rightarrow r_* < \infty$ a.s.

② $\left\| \frac{1}{R^2} \fint_{B_R} \left|(\varphi,\sigma) - \fint_{B_R}(\varphi,\sigma)\right|^2 \right\|_{L^q(\Omega)} \lesssim \frac{1}{R^2} \fint_{B_R} \mu_d(|x|)^2 \, dx$

$$\lesssim \frac{\mu_d(R)^2}{R^2} = \begin{cases} 1/R^2 & d>2 \\ \log R/R^2 & d=2 \\ 1/R & d=1 \end{cases}$$

& use some union bound to estimate $\mathbb{P}[r_* > \ell]$. (exercise). $\square$

$$\begin{cases} -\nabla\cdot a\nabla u = 0 \quad \text{on } B_R \\ \mathcal{E}(\nabla u, B_r) \le \left(\frac{r}{R}\right)^{2\alpha} \mathcal{E}(\nabla u, B_R) \quad \forall r_* \le r \le R \end{cases}$$

Main result: $\left(\mathrm{Cc}(V_n, D_r) = \left(\frac{r}{R}\right) \mathrm{Cc}(V_n, D_R) \right.$