M285K - course #22

V.3 Large-scale & annealed L^p regularity

$$r_*^x[C_0] := \inf\left\{ r>0 : \forall R>r,\ \frac{1}{R}\left(\fint_{B_R(x)} \left|(\varphi,\sigma) - \fint_{B_R(x)}(\varphi,\sigma)\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{C_0} \right\}.$$

stationary wrt x

Theorem (large-scale L^p reg.).

$\exists\ \tilde{r}_*$ $\frac{1}{8}$-Lipschitz stationary random field with $\left[r_*[C_0] \leq \tilde{r}_* \leq r_*[3^{d+2}C_0] \right]$

such that the following holds:

$\forall h \in C_c^\infty(\mathbb{R}^d, L^\infty(\Omega))^d$, if $\nabla u \in L^\infty(\Omega, L^2(\mathbb{R}^d)^d)$ satisfies

$$-\nabla\cdot a\nabla u = \nabla\cdot h,$$

Then $\forall\ 1<p<\infty$:

$$\int_{\mathbb{R}^d} \left(\fint_{B_*(x)} |\nabla u|^2 \right)^{p/2} \lesssim_p \int_{\mathbb{R}^d} \left(\fint_{B_*(x)} |h|^2 \right)^{p/2}$$

where $B_*(x) = B(x, \tilde{r}_*(x))$.

Proof. Based on Lip reg + dual CZ lemma

Step 1: construction of $\tilde{r}_*$.

Choose $\tilde{r}_* =$ largest $\frac{1}{8}$-Lip fct $\leq r_*$

$$\tilde{r}_*^x = \inf_{y \in \mathbb{R}^d} \left(\frac{1}{8}|x-y| + r_*^y \right).$$

$\forall R > \tilde{r}_*^x$: $\exists y$: $R \geq \frac{1}{8}|x-y| + r_*^y$.

$\left(\to R \geq \frac{1}{8}|x-y| \;\&\; R \geq r_*^y \right)$

$$C_0\,\frac{1}{R}\left(\fint_{B_R(x)}\left|(\varphi,\sigma)-\fint_{B_R(x)}(\varphi,\sigma)\right|^2\right)^{1/2}$$

$$\le \underbrace{\frac{1}{R}\underbrace{\left(\fint_{B_{R+|x-y|}(y)}\left|(\varphi,\sigma)-\fint_{B_{R+|x-y|}(y)}(\varphi,\sigma)\right|^2\right)^{1/2}}\left(\frac{R+|x-y|}{R}\right)^{d/2}}_{\le \frac{1}{C_0}(R+|x-y|) \text{ by def of } r_*^y}$$

$$\le \frac{1}{C_0}\underbrace{\left(1+\frac{|x-y|}{R}\right)^{1+d/2}}_{\le 8^{1+d/2}}.$$

Step 2. $\forall D$ ball $\subseteq \mathbb{R}^d$, decompose $\nabla u = \nabla u_D^{0} + \nabla \hat{u}_D$.

where
$$\begin{cases} -\nabla\cdot a\nabla u_D^{0} = \nabla\cdot(h\,\mathbb{1}_D) \\ -\nabla\cdot a\nabla \hat{u}_D = \nabla\cdot(h\,\mathbb{1}_{\mathbb{R}^d\setminus D}). \end{cases}$$

$$\int|\nabla u_D^{0}|^2 \le \int|h|^2$$

$\mathbb{R}^d$

& we prove $\forall\, 2 \leq p < \infty$:

$$\begin{cases} \left(\fint_{D} \left(\fint_{B_\delta(x)} |\nabla u_D^0|^2 \right) \right)^{\frac{1}{2}} \lesssim \left(\fint_{2D} \left(\fint_{B_\delta(x)} |h|^2 \right) \right)^{\frac{1}{2}} & (1) \\ \left(\fint_{\frac{1}{12}D} \left(\fint_{B_\delta(x)} |\nabla u_D^1|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \lesssim \left(\fint_{D} \left(\fint_{B_\delta(x)} |\nabla u_D^1|^2 \right) \right)^{\frac{1}{2}} . & (2) \end{cases}$$

$\Rightarrow$ [by dual CZ lemma: $\forall\, 2 \leq p < \infty$,

$$\int_{\mathbb{R}^d} \left(\fint_{B_\delta(x)} |\nabla u|^2 \right)^{p/2} \lesssim_p \int_{\mathbb{R}^d} \left(\fint_{B_\delta(x)} |h|^2 \right)^{p/2} .$$
]

(1) (a) Assume $D = B(x_D, r_D)$, $r_D \leq \frac{1}{4} \tilde{r}_*(x_D)$.

$\hookrightarrow \forall x_0 \in D$: $\fint_{B_\delta(x_0)} |\nabla u_D^0|^2 \leq |B_\delta(x_0)|^{-1} \int_{\mathbb{R}^d} |\nabla u_D^0|^2$

$\lesssim |B_\delta(x_0)|^{-1} \int_D |h|^2$ by energy estimate.

$\leq |B_\delta(x_0)|^{-1} \int_{B_\delta(x_D)} |h|^2$

$\lesssim \fint_{B_\delta(x_D)} |h|^2$ by $\frac{1}{8}$-Lips property.

$\lesssim \fint_{\frac{1}{2}D} \left(\fint_{B_\delta(x)} |h|^2 \right) dx$ because r_D small.

(b) Assume $r_D \geq \frac{1}{4} \tilde{r}_\delta(x_D)$:

$\fint \left(\fint |\nabla u^0|^2 \right) dx \leq |D|^{-1} \int \left(\fint |\nabla u_D^0|^2 \right) dx$

$$\fint_{D}\left(\fint_{B_\delta(x)} |\nabla u_D|^2\right)dx \lesssim \int_{\mathbb{R}^d}\left(\fint_{B_\delta(x)} \dots\right)$$

$$\lesssim |D|^{-1}\int_{\mathbb{R}^d} |\nabla \mathring{u}_D|^2 \quad \text{by } \tfrac{1}{8}\text{-Lip prop.}$$

$$\lesssim \fint_{D} |h|^2 \quad \text{by energy est.}$$

$$\lesssim \fint_{2D}\left(\fint_{B_\delta(x)} |h|^2\right)dx$$

② Use that $\hat{u}_D$ is a-harm in D.

<u>We prove</u>: $\forall x_0 \in \frac{1}{12}D$, $\underline{\fint_{B_\delta(x_0)} |\nabla \hat{u}_D|^2 \lesssim \fint_{D}\left(\fint_{B_\delta(x)} |\nabla \hat{u}_D|^2\right)dx.}$

(a) Assume $r_D \leq 3r_\delta(x_0)$: clear.

(b) Assume $r_D \geq 3 r_\delta(x_0)$:

$$\fint_{B_\delta(x_0)} |\nabla u_D^1|^2 \lesssim \fint_{B(x_0,\, r_\delta(x_0) + \frac{1}{12} r_D)} |\nabla u_D^1|^2 \quad \text{by large-scale Lipschitz reg.}$$

$$B(x_0,\, r_\delta(x_0) + \tfrac{1}{12} r_D) \subseteq B\Big(x_D,\, \underbrace{\underbrace{\tfrac{1}{12} r_D + \underbrace{r_\delta(x_0)}_{\leq \frac{1}{3} r_D} + \tfrac{1}{12} r_D}_{\leq \frac{1}{2} r_D}\Big)}_{\subseteq \frac{1}{2} D}.$$

$$\lesssim \fint_{\frac{1}{2} D} |\nabla u_D^1|^2$$

$$\lesssim \fint_D \Big(\fint_{B_\delta(x)} |\nabla u_D^1|^2 \Big) dx$$

Step 3: duality to go from $p \geq 2$ to $p \leq 2$ -

Disjointify the $\{B_\delta(x)\}_{x\in\mathbb{R}^d}$: can define P partition of $\mathbb{R}^d$

s.t. $\forall Q \in P$: $\tilde{r}_\delta \simeq \operatorname{diam} Q$ inside Q -

$$\hookrightarrow \int_{\mathbb{R}^d} \left(\fint_{B_\delta(x)} |h|^2 \right)^{p/2} \simeq \sum_{Q\in P} |Q| \left(\fint_Q |h|^2 \right)^{p/2}$$

$$= \sup\left\{ \int_{\mathbb{R}^d} h\cdot g \;:\; \sum_{Q\in P} |Q| \left(\fint_Q |g|^2 \right)^{p'/2} = 1 \right\}.$$

$\square$

Th (... | (P regularity)

Theorem (Annealed L^p regularity).

$\forall h \in C_c^\infty(\mathbb{R}^d, L^\infty(\Omega))^d$, if $\nabla u \in L^\infty(\Omega, L^2(\mathbb{R}^d)^d)$ satisfies

$$-\nabla\cdot a\nabla u = \nabla\cdot h \quad \text{in } \mathbb{R}^d,$$

Then, $\forall\, 1 < p, q < \infty$, $\forall \delta > 0$,

$$\big\| [\nabla u] \big\|_{L^p(\mathbb{R}^d, L^q(\Omega))} \lesssim_{p,q,\delta} \big\| [h] \big\|_{L^p(\mathbb{R}^d, L^{q+\delta}(\Omega))}$$

where $[\nabla u](x) = \left(⨍_{B(x)} |\nabla u|^2 \right)^{\frac{1}{2}}$.

Proof. Use quenched large-scale L^p reg + Lip reg + dual CZ lemma.

Step 1. $\forall D$ ball $\subset \mathbb{R}^d$, some decomposition $\nabla u = \nabla u_D^0 + \nabla u_D^1$

& $\forall\, 1 < q < \infty$, $\forall\, 1 \le p \le \infty$,

large-scale L^1 reg.

$$\fint_D \mathbb{E}\left[\left(\fint_{B_*(x)} |Du_D^0|^2\right)^{\frac{q}{2}}\right] dx \lesssim \fint_{6D} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |h|^2\right)^{\frac{q}{2}}\right].$$

large-scale Lip.

$$\left(\fint_{\frac{1}{24}D} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |Du_D^1|^2\right)^{\frac{q}{2}}\right]^{p/q} dx\right)^{\frac{1}{p}} \lesssim \left(\fint_D \mathbb{E}\left[\left(\fint_{B_\delta(x)} |Du_D^1|^2\right)^{q/2}\right] dx\right)^{\frac{1}{q}}.$$

$\hookrightarrow$ apply dual CZ-lemma: $\forall\, 1 < p, q < \infty$,

$$\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |Du|^2\right)^{\frac{q}{2}}\right]^{p/q} \lesssim_{p,q} \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_\delta(x)} |h|^2\right)^{q/2}\right]^{p/q}.$$