

# M285K - course #22

### V.3 Large-scale & annealed $L^p$ regularity

$$\kappa_*^x [C_0] := \inf \{ \kappa > 0 : \forall R > \kappa, \frac{1}{R} \left( \int_{B_R(x)} |(\varphi, \sigma) - \bar{f}_{B_R(x}}(\varphi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{C_0} \}.$$

↑ stationary w.r.t  $x$

Theorem (large-scale  $L^p$  reg.).

$\exists \tilde{\kappa}_* \frac{1}{8}$ -Lipschitz stationary random field with  $\left( \kappa_* [C_0] \leq \tilde{\kappa}_* \leq \kappa_* [3^{d+2} C_0] \right)$

such that the following holds:

$\forall h \in C_c^\infty(\mathbb{R}^d, L^\infty(\Omega))^d$ , if  $\nabla u \in L^\infty(\Omega, L^2(\mathbb{R}^d)^d)$  satisfies

$$-\operatorname{Div} a \nabla u = \operatorname{Div} h,$$

Then  $\forall 1 < p < \infty$ :

$$\int_{\mathbb{R}^d} \left( \int_{B_*(x)} |Du|^2 \right)^{1/2} \leq_p \int_{\mathbb{R}^d} \left( \int_{B_*(x)} |h|^2 \right)^{1/2}$$

where  $B_*(x) = B(x, \tilde{r}_*(x))$ .

Proof. Based on Lip reg + dual C<sub>7</sub> lemma

Step 1: construction of  $\tilde{r}_*$ .

Choose  $\tilde{r}_* =$  largest  $\frac{1}{8}$ -Lip  $f_{ct} \leq r_*$

$$\tilde{r}_*^x = \inf_{y \in \mathbb{R}^d} \left( \frac{1}{8} |x-y| + r_*^y \right).$$

$$\forall R \geq \tilde{r}_*^x : \exists y : R \geq \frac{1}{8} |x-y| + r_*^y.$$

( $\Rightarrow R \geq \frac{1}{8} |x-y|$  &  $R \geq r_*^y$ )

$$\begin{aligned}
& C_0 \frac{1}{R} \left( \int_{B_R(x)} |f(y, \sigma) - f_{B_R(x)}(y, \sigma)|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{1}{R} \left( \int_{B_{R+|x-y|}(y)} |f(y, \sigma) - f_{B_{R+|x-y|}(y)}(y, \sigma)|^2 \right)^{\frac{1}{2}} \left( \frac{R+|x-y|}{R} \right)^{\frac{d}{2}} \\
& \leq \frac{1}{C_0} (R+|x-y|) \text{ by def of } \mu_\sigma^y \\
& \leq \frac{1}{C_0} \left( 1 + \frac{|x-y|}{R} \right)^{2+\frac{d}{2}} \\
& \leq 3^{2+\frac{d}{2}}.
\end{aligned}$$

Step 2.  $\forall D \text{ ball} \subseteq \mathbb{R}^d$ , decompose  $\nabla u = \nabla u_D^0 + \nabla u_D^1$ .

where  $\begin{cases} -\nabla \cdot \alpha \nabla u_D^0 = \nabla \cdot (h \mathbb{1}_D) \\ -\nabla \cdot \alpha \nabla u_D^1 = \nabla \cdot (h \mathbb{1}_{\mathbb{R}^d \setminus D}). \end{cases}$

$\int |\nabla u_D^0|^2 \leq \int |h|^2$

$\mathbb{R}^d$ 

& we prove  $\forall 2 \leq p < \infty$ :

$$\left[ \begin{aligned} \left( \int_{\mathcal{D}} \left( \int_{B_0(x)} |\nabla u_{\mathcal{D}}|^2 \right)^{p/2} \right)^{1/p} &\lesssim \left( \int_{\mathbb{R}^{2D}} \left( \int_{B_0(x)} |h|^2 \right)^{p/2} \right)^{1/2} \quad \textcircled{1} \\ \left( \int_{\frac{1}{2}\mathcal{D}} \left( \int_{B_0(x)} |\nabla u_{\mathcal{D}}|^2 \right)^{p/2} \right)^{1/p} &\lesssim \left( \int_{\mathcal{D}} \left( \int_{B_0(x)} |\nabla u_{\mathcal{D}}|^2 \right)^{p/2} \right)^{1/2} \quad \textcircled{2} \end{aligned} \right.$$

$\Rightarrow$  by dual CZ lemma:  $\forall 2 \leq p < \infty$ ,

$$\left[ \int_{\mathbb{R}^b} \left( \int_{B_0(x)} |\nabla u|^2 \right)^{p/2} \lesssim_p \int_{\mathbb{R}^d} \left( \int_{B_0(x)} |h|^2 \right)^{p/2} \right]$$

① (a) Assume  $\mathcal{D} = \mathcal{B}(x_{\mathcal{D}}, r_{\mathcal{D}})$ ,  $r_{\mathcal{D}} \leq \frac{1}{4} \tilde{r}_{*}(x_{\mathcal{D}})$ .

$$\begin{aligned}
C \forall x_0 \in D: \int_{B_\delta(x_0)} |\nabla u_D|^2 &\leq |B_\delta(x_0)|^{-1} \int_{\mathbb{R}^d} |\nabla u_D|^2 \\
&\lesssim |B_\delta(x_0)|^{-1} \int_D |h|^2 \quad \text{by energy estimate.} \\
&\leq |B_\delta(x_0)|^{-1} \int_{B_\delta(x_0)} |h|^2 \\
&\lesssim \int_{B_\delta(x_0)} |h|^2 \quad \text{by } \frac{1}{\delta}\text{-Lip property.} \\
&\lesssim \int_{\frac{1}{2}D} \left( \int_{B_\delta(x)} |h|^2 \right) dx \quad \text{because } r_D \text{ small.} \\
&\quad \sim
\end{aligned}$$

(b) Assume  $r_D \geq \frac{1}{4} \tilde{r}_\delta(x_D)$ :

$$\int ( \int |\nabla u_D|^2 ) dx \leq |D|^{-1} \int ( \int |\nabla u_D|^2 ) dx$$

$$\begin{aligned}
& \int_{\mathbb{D}} |B_\delta(x)|^{-1} |\nabla u_{\mathbb{D}}|^2 \\
& \leq |\mathbb{D}|^{-1} \int_{\mathbb{R}^d} |\nabla u_{\mathbb{D}}|^2 && \text{by } \frac{1}{\delta} \text{-Lip prop.} \\
& \leq \int_{\mathbb{D}} |h|^2 && \text{by energy est.} \\
& \leq \int_{\mathbb{D}} \left( \int_{B_\delta(x)} |h|^2 \right) dx
\end{aligned}$$

② Use that  $u_{\mathbb{D}}^{\wedge}$  is  $\alpha$ -harmonic in  $\mathbb{D}$ .

We prove:  $\forall x_0 \in \frac{1}{12}\mathbb{D}$ ,  $\int_{B_\delta(x_0)} |\nabla u_{\mathbb{D}}^{\wedge}|^2 \leq \int_{\mathbb{D}} \left( \int_{B_\delta(x)} |\nabla u_{\mathbb{D}}^{\wedge}|^2 \right) dx.$

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(a) Assume  $\rho_{\mathbb{D}} \leq 3\rho_{x_0}$  : done.

(b) Assume  $r_D \geq 3r_0(x_0)$ :

$$\int_{B_0(x_0)} |\nabla u_D|^2 \lesssim \int_{B(x_0, r_0(x_0) + \frac{1}{12}r_D)} |\nabla u_D|^2 \quad \text{by large-scale Poincaré}$$

$$\subset B(x_D, \underbrace{\frac{1}{12}r_D + r_0(x_0) + \frac{1}{12}r_D}_{\leq \frac{1}{3}r_D})$$

$$\underbrace{\hspace{10em}}_{\leq \frac{1}{2}r_D}$$

$$\subset \frac{1}{2}D.$$

$$\int_D |\nabla u_D|^2 \lesssim \int_D \left( \int_{B_0(x)} |\nabla u_D|^2 \right) dx$$



Step 3: duality to go from  $p \geq 2$  to  $p \leq 2$  -

Disjointify the  $\{B_\delta(x)\}_{x \in \mathbb{R}^d}$  : can define  $\mathcal{P}$  partition of  $\mathbb{R}^d$   
s.t.  $\forall Q \in \mathcal{P}$ :  $\tilde{r}_Q \approx \text{diam } Q$   
inside  $Q$ .

$$\hookrightarrow \int_{\mathbb{R}^d} \left( \int_{B_\delta(x)} |f| |h|^2 \right)^{p/2} \approx \sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |f| |h|^2 \right)^{p/2}$$

$$= \sup \left\{ \int_{\mathbb{R}^d} h \cdot g : \sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |g|^2 \right)^{p'/2} = 1 \right\}.$$

□

To (D.1) (P.1) (P.2) (P.3) (P.4) (P.5) (P.6) (P.7) (P.8) (P.9) (P.10) (P.11) (P.12) (P.13) (P.14) (P.15) (P.16) (P.17) (P.18) (P.19) (P.20) (P.21) (P.22) (P.23) (P.24) (P.25) (P.26) (P.27) (P.28) (P.29) (P.30) (P.31) (P.32) (P.33) (P.34) (P.35) (P.36) (P.37) (P.38) (P.39) (P.40) (P.41) (P.42) (P.43) (P.44) (P.45) (P.46) (P.47) (P.48) (P.49) (P.50) (P.51) (P.52) (P.53) (P.54) (P.55) (P.56) (P.57) (P.58) (P.59) (P.60) (P.61) (P.62) (P.63) (P.64) (P.65) (P.66) (P.67) (P.68) (P.69) (P.70) (P.71) (P.72) (P.73) (P.74) (P.75) (P.76) (P.77) (P.78) (P.79) (P.80) (P.81) (P.82) (P.83) (P.84) (P.85) (P.86) (P.87) (P.88) (P.89) (P.90) (P.91) (P.92) (P.93) (P.94) (P.95) (P.96) (P.97) (P.98) (P.99) (P.100)

Theorem (commuted  $\nabla$  and divergence).

$\forall h \in C_c^\infty(\mathbb{R}^d, L^\infty(\Omega))^d$ , if  $\nabla u \in L^\infty(\Omega, L^2(\mathbb{R}^d)^d)$  satisfies  
 $-\operatorname{div} \nabla u = \operatorname{div} h = \mathbb{R}^d$ ,

then,  $\forall 1 < p, q < \infty$ ,  $\forall \delta > 0$ ,

$$\| [\nabla u] \|_{L^p(\mathbb{R}^d, L^q(\Omega))} \lesssim_{p, q, \delta} \| [h] \|_{L^p(\mathbb{R}^d, L^{q+\delta}(\Omega))}$$

where  $[\nabla u](x) = \begin{pmatrix} f & |\nabla u|^2 \\ \beta(x) \end{pmatrix}^{\frac{1}{2}}$ .

Proof. Use quenched long-scale  $L^p$  reg + Lip reg + dual CZ lemma.

Step 1.  $\forall D$  ball  $\subset \mathbb{R}^d$ , some decomposition  $\nabla u = \nabla u_D^0 + \nabla u_D^1$

&  $\forall 1 < q < \infty, \forall 1 \leq p \leq \infty,$

large-scale  $\rightarrow$   $\int_D \mathbb{E} \left[ \int_{B_*(x)} (f |Du_D^0|^2)^{\frac{q}{2}} dx \right] \lesssim \int_{\partial D} \mathbb{E} \left[ \int_{B_*(x)} (f |h|^2)^{\frac{q}{2}} \right]$

large-scale  $\rightarrow$   $\left( \int_{\frac{2}{24}D} \mathbb{E} \left[ \int_{B_*(x)} (f |Du_D^1|^2)^{\frac{q}{2}} dx \right]^{\frac{p}{q}} \right)^{\frac{2}{p}} \lesssim \left( \int_D \mathbb{E} \left[ \int_{B_*(x)} (f |Du_D^1|^2)^{\frac{q}{2}} dx \right]^{\frac{2}{q}} \right)^{\frac{2}{p}}$

to apply dual CZ-lemma:  $\forall 1 < p, q < \infty,$

$$\int_{\mathbb{R}^d} \mathbb{E} \left[ \int_{B_*(x)} (f |Du|^2)^{\frac{q}{2}} \right]^{\frac{p}{q}} \lesssim_{p,q} \int_{\mathbb{R}^d} \mathbb{E} \left[ \int_{B_*(x)} (f |h|^2)^{\frac{q}{2}} \right]^{\frac{p}{q}}$$

