M285K - course #5

Corrector: $\exists!\, \varphi \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega))$

s.t. $\begin{cases} -\nabla\cdot a(\nabla\varphi_i + e_i) = 0 \quad \text{in } \mathbb{R}^d \text{ a.s.} \\ \left(-\underbrace{\nabla\cdot a\nabla}(\varphi_i + x_i) = 0\right) \\ \nabla\varphi \text{ stat.}, \ \mathbb{E}[\nabla\varphi] = 0 \\ \int_B \varphi = 0 \end{cases}$

Corollary (qualitative sublinearity)

If $v \in H^1_{loc}(\mathbb{R}^d, L^2(\Omega))$ s.t. $\begin{cases} \nabla v \text{ stat.} \\ \mathbb{E}[\nabla v] = 0 \end{cases}$

Then, $\varepsilon v(\frac{\cdot}{\varepsilon}) \to 0$ in $L^2_{loc}(\mathbb{R}^d)$ a.s.

$\left(L^q_{loc},\ q < \frac{2d}{d-2}\right)$.

(In particular, $\varepsilon\varphi(\frac{\cdot}{\varepsilon}) \rightharpoonup 0$ in L^2_{loc})

Proof. ① $\left(\frac{1}{R^2} \fint_{B_R} \left| v - \fint_{B_R} v \right|^2\right) \xrightarrow{R\uparrow\infty} 0$ a.s.

② $\frac{1}{R} \fint_{B_R} v \longrightarrow 0$ a.s.

$\Rightarrow \frac{1}{R^2} \fint_{B_R} |v|^2 \longrightarrow 0$

$= \fint_B \left| \frac{1}{R} v(R\cdot) \right|^2$

Assume WLOG: $\fint_B v = 0$.

Step 1. $v_R = \frac{1}{R}\left(v(R\cdot) - \fint_B v(R\cdot) \right) \in H^1(B)$

Ergodic thm $\nabla v_R = (\nabla v)(R\cdot) \longrightarrow \mathbb{E}[\nabla v] = 0$ in $L^2(B)$ a.s.

Poincaré v_R bnd in $H^1(B)$

Weak compactness: $v_R \rightharpoonup l$ in $H^1(B)$ a.s.

$$\nabla \ell = 0$$

$$\fint_B v_R = 0 \implies \fint_B \ell = 0 \quad \Big\} \quad \ell \equiv 0.$$

Rellich: $v_R \to 0$ in $L^2(B)$.

Step 2. $$\frac{1}{R}\fint_{B_R} v = \frac{1}{R}\left(\fint_{B_R} v - \fint_B v\right)$$

$$= \frac{1}{R}\int_1^R \left(\frac{d}{dt}\fint_{B_t} v\right) dt$$

Write $$\fint_{B_t}\big(v(y) - v(0)\big)\,dy = \fint_{B_t}\left(\int_0^1 y\cdot\nabla v(sy)\,ds\right)dy$$

$$= \fint_B \left(\int_0^1 ty\cdot\nabla v(sty)\,ds\right)dy$$

$$= \fint_B \left(\int_0^t y\cdot\nabla v(sy)\,ds\right)dy$$

$$= \int_0^{\cdots} \left(\fint_B y \cdot \nabla v(sy)\, dy \right) ds$$

$$\Rightarrow \frac{1}{R} \fint_{B_R} v = \underbrace{\frac{1}{R} \int_1^R}\ \underbrace{\left(\fint_B \tilde{y} \cdot \nabla v(sy)\, dy \right)}_{\to 0 \quad s\uparrow\infty \quad a.s. \ \text{by ergodic thm.}} ds$$

$$\xrightarrow{R\uparrow\infty} 0 \quad a.s.$$

(Cesàro)

$\square$

Second proof of existence of correctors

Massive approximation: $\underbrace{\frac{1}{T}\varphi_{T,i}} - \nabla \cdot \hat{a}\left(\nabla \varphi_{T,i} + e_i\right) = 0 \quad \text{in } \mathbb{R}^d$

$S \ldots \quad \forall T \ldots \quad \forall \ldots \quad \exists!$ solution $\varphi \in H^1_{\ldots}(\mathbb{R}^d)$

Lemma. $\forall\, T < \infty$, $\forall\, a$ $\exists!$ solution $\varphi_{T,i} \in H^1_{uloc}(\mathbb{R}^d)$

$$= \Big\{ v \in H^1_{loc}(\mathbb{R}^d), \text{ s.t. } \sup_z \int_{B(z)} |v|^2 + |\nabla v|^2 < \infty \Big\}.$$

By uniqueness, φ_T stationary.

Moreover, $\mathbb{E}\left[|\nabla\varphi_T - \nabla\varphi|^2 + \frac{1}{T}|\varphi_T|^2 \right] \xrightarrow{T\uparrow\infty} 0$.

Proof. Step 1: existence of φ_T.

First solve $\begin{cases} \frac{1}{T}\varphi_{T,R} - \nabla\cdot a(\nabla\varphi_{T,R} + e) = 0 & \text{in } B_R \\ \varphi_{T,R} = 0 & \text{on } \partial B_R. \end{cases}$

$\exists!\ \varphi_{T,R} \in H^1_0(B_R)$.

Test the eqn with $\varphi_{T,R}\,\eta^2_{T,z}$, $\eta_{T,z}(x) = e^{-\frac{c}{\sqrt{T}}|x-z|}$, $c>0$.

$$\int_{B_R} \eta_{T,z}^2 \left(\frac{1}{T} |\varphi_{T,R}|^2 + a |\nabla \varphi_{T,R}|^2 \right)$$

$$|\cdot| \leq \eta_{T,z} \frac{c}{\sqrt{T}}$$

$$\leq -2 \int_{B_R} \eta_{T,z}\, \varphi_{T,R}\, \nabla \eta_{T,z} \cdot a \left(\nabla \varphi_{T,R} + e \right)$$

$$- \int \eta_{T,z}^2 \underbrace{\nabla \varphi_{T,R}}_{\text{absorbed!}} \cdot a e$$

Exercise: choose $c > 0$ small enough

$$\to \underbrace{\int_{B_R} \eta_{T,z}^2 \left(\frac{1}{T} |\varphi_{T,R}|^2 + |\nabla \varphi_{T,R}|^2 \right)}_{\geq \ \frac{1}{C(T)} \int_{B(z)} \left(|\varphi_{T,R}|^2 + |\nabla \varphi_{T,R}|^2 \right)} \lesssim \int_{B_R} \eta_{T,z}^2$$

Exercise : $R/\infty \Rightarrow \exists$ solution φ_T to the eqn in H^1_{uloc}.

Step 2 : uniqueness in H^1_{uloc}.

Step 3 : $\begin{cases} \nabla\varphi_T \rightharpoonup \nabla\varphi & \text{in } L^2_{loc}(\mathbb{R}^d; L^2(\Omega)) \\ \nabla\varphi_T^{st\,0} \rightharpoonup (\nabla\varphi)^0 & \text{in } L^2(\Omega). \end{cases}$

Exercise : as in Step 1, prove that $\sup_z \int_{B(z)} \left(\left|\frac{1}{\sqrt{T}}\varphi_T\right|^2 + |\nabla\varphi_T|^2 \right) \lesssim 1$

Weak compactness : $\begin{cases} \frac{1}{\sqrt{T}}\varphi_T \rightharpoonup \varphi \\ \nabla\varphi_T \rightharpoonup \nabla\tilde{\varphi} \end{cases}$

Pass to limit in eqn : $\begin{cases} -\nabla\cdot a(\nabla\tilde{\varphi}+e) = 0 & \text{in } \mathbb{R}^d \text{ a.s} \\ \nabla\tilde{\varphi} \text{ stat}, \ \mathbb{E}\,\nabla\tilde{\varphi} = 0 \end{cases}$ $\Rightarrow$ conclude.

Step 4 : Strong convergence.

Idea: $\begin{cases} F_n \rightharpoonup F & \text{in } L^2(\Omega) \\ \mathbb{E}|F_n|^2 \to \mathbb{E}|F|^2 \end{cases}$ $\Rightarrow F_n \to F$ in $L^2(\Omega)$

because $\mathbb{E}|F_n - F|^2$

$$= \mathbb{E}|F_n|^2 + \mathbb{E}|F|^2 - 2\,\mathbb{E}F_nF$$

$$\to 0.$$

Note that $\mathbb{E}\left[e\cdot a(\nabla\varphi + e)\right] = \mathbb{E}\left[(\nabla\varphi+e)\cdot a(\nabla\varphi+e)\right]$

(corrector eqn: $\mathbb{E}\left[\nabla\varphi\cdot a(\nabla\varphi+e)\right]=0$)

$$\Big\uparrow\, T\uparrow\infty$$

$$\mathbb{E}\left[e\cdot a(\nabla\varphi_T+e)\right] = \mathbb{E}\left[\frac{1}{T}|\varphi_T|^2 + (\nabla\varphi_T+e)\cdot a(\nabla\varphi_T+e)\right]$$

& that $\lim\limits_{T\uparrow\infty} \mathbb{E}\left[(\nabla\varphi_T+e)\cdot a(\nabla\varphi_T+e)\right]$

$$\geq \mathbb{E}\left[(\nabla\varphi+e)\cdot a(\nabla\varphi+e)\right].$$

$$\Rightarrow \begin{cases} \mathbb{E}\, \frac{1}{T}|\varphi_T|^2 \to 0 \\ \mathbb{E}\left[(\nabla\varphi_T + e)\cdot a(\nabla\varphi_T + e)\right] \to \mathbb{E}\left[(\nabla\varphi + e)\cdot a(\nabla\varphi + e)\right] \end{cases}$$

$\Rightarrow$ deduce strong convergence. $\square$

II.4 Qualitative homogenization result.

Let a be stationary and ergodic.

Lemma. Recall $\bar{a} e_i = \mathbb{E}\left[a(\nabla\varphi_i + e_i)\right]$.

There holds $\begin{cases} e\cdot \bar{a} e \geq \alpha |e|^2 \\ |\bar{a} e| \leq \frac{\beta^2}{\alpha}|e| \end{cases}$ $\forall e \in \mathbb{R}^d$.

($\to$ remains unif. elliptic)

In addition, $\overline{a'} = (\bar{a})'$
(in particular, $\bar{a}$ is symmetric whenever a is.)

Proof. (1) $e \cdot \bar{a} e = \mathbb{E}[e \cdot a(\nabla\varphi_e + e)]$

$$\left\{\begin{array}{l} \varphi = (\varphi_1, \dots, \varphi_d) \\ \varphi_e = \sum_{i=1}^{d} e_i \varphi_i \end{array}\right.$$

$= \mathbb{E}[(\nabla\varphi_e + e) \cdot a(\nabla\varphi_e + e)]$

by corrector eqn
$(\mathbb{E}[\nabla\psi \cdot a(\nabla\varphi_e + e)] = 0 \quad \forall \psi \in H^1(\Omega))$

$\geq \alpha \underbrace{\mathbb{E}[|\nabla\varphi_e + e|^2]}$

$\underbrace{\underbrace{\mathbb{E}[|\nabla\varphi_e|^2]}_{\geq 0} + |e|^2 + 2\cancel{\mathbb{E}[e \cdot \nabla\varphi_e]}^{0}}_{\geq |e|^2}$.

(2) $|\bar{a} e| = |\mathbb{E}[a(\nabla\varphi_e + e)]|$

$\leq \beta\, \mathbb{E}[|\nabla\varphi_e + e|^2]^{\frac{1}{2}}$

$$\leq \beta \left(\frac{1}{\alpha} e \cdot \bar{a} e \right)^{\frac{1}{2}} \Rightarrow \text{conclude.}$$

③ $$e_j \cdot \bar{a} e_i = \mathbb{E}\left[e_j \cdot a (\nabla \varphi_i + e_i) \right]$$

$$= \mathbb{E}\left[(\nabla \varphi'_j + e_j) \cdot a (\nabla \varphi_i + e_i) \right] \qquad (\varphi' = \text{corrector associated with } a')$$

$$= \mathbb{E}\left[(\nabla \varphi_i + e_i) \cdot a' (\nabla \varphi'_j + e_j) \right]$$

$$= \mathbb{E}\left[e_i \cdot a' (\nabla \varphi'_j + e_j) \right]$$

$$= e_i \cdot \overline{a'} e_j$$

□

<u>Theorem</u> (qual homog result; Papanicolaou-Varadhan '79, Kozlov '79).

For any $f \in L^2(\mathbb{R}^d)^d$,

let $u_\varepsilon \in \dot{H}^1(\mathbb{R}^d)$ be unique solution of $-\nabla \cdot a(\frac{\cdot}{\varepsilon}) \nabla u_\varepsilon = \nabla \cdot f$ in $\mathbb{R}^d$

Let $u_\varepsilon \in \dot{H}^1(\mathbb{R}^d)$ be unique solution of $\ldots$

& let $\bar{u} \in \dot{H}^1(\mathbb{R}^d)$ be unique solution of $\underbrace{-\nabla\cdot \bar{a}\nabla\bar{u} = \nabla\cdot f}$ in $\mathbb{R}^d$,

Then, a.s. $\begin{cases} \overset{\text{field}}{\nabla u_\varepsilon} \rightharpoonup \nabla \bar{u} & \text{in } L^2(\mathbb{R}^d) \\ \underset{\text{flux}}{a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon} \rightharpoonup \bar{a}\nabla\bar{u} & \text{in } L^2(\mathbb{R}^d). \end{cases}$

[$a\left(\frac{\cdot}{\varepsilon}\right) \xrightarrow{H} \bar{a}$
H - convergence
= convergence of all fields & fluxes.]

Remark. Structure = Maxwell's eqns $\left[\begin{cases} \nabla\cdot(\text{flux}) = \text{charge } [-\nabla\cdot f] \\ \text{field} = \nabla u_\varepsilon \end{cases}\right]$

& constitutive relation: $\text{flux} = a\left(\frac{\cdot}{\varepsilon}\right) \times \text{field}.$

homogenization $\Big($ $\bar{a}\,\times$

Observation. | Define the "homogenization commutator"

$$\Xi_\varepsilon[\nabla u_\varepsilon] := \underbrace{a\left(\tfrac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon} - \underbrace{\bar{a}\nabla u_\varepsilon}$$

The following are equivalent:

$$\text{(i)} \quad \boxed{H}_\varepsilon[\nabla u_\varepsilon] \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^d)$$

$$\text{(ii)} \quad \begin{cases} \nabla u_\varepsilon \rightharpoonup \nabla \bar{u} \\ a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon \rightharpoonup \bar{a}\nabla\bar{u} \end{cases} \quad \text{in } L^2(\mathbb{R}^d).$$

Proof. (ii) $\Rightarrow$ (i) trivial

(i) $\Rightarrow$ (ii) $\quad -\nabla\cdot \bar{a}^t \nabla \bar{v} = \nabla\cdot g \quad \text{in } \mathbb{R}^d, \quad g \in L^2(\mathbb{R}^d)$

$$\text{Get } \int \nabla\bar{v} \cdot \boxed{H}_\varepsilon[\nabla u_\varepsilon] \xrightarrow{(i)} 0$$

$$= \int \nabla\bar{v}\cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon - \int \nabla\bar{v}\cdot\bar{a}\nabla u_\varepsilon$$

$$= -\int \nabla\bar{v}\cdot g + \int g\cdot\nabla u_\varepsilon$$

(eqn for u_ε) (eqn for $\bar{v}$)

$$= -\underbrace{\int g \cdot \nabla \bar{u}}_{\text{(eqns for } \bar{u}, \bar{v})} + \underbrace{\int g \cdot \nabla u_\varepsilon} \to \underbrace{\nabla u_\varepsilon \rightharpoonup \nabla \bar{u}}$$

& similarly $a(\tfrac{\cdot}{\varepsilon}) \nabla u_\varepsilon \rightharpoonup \bar{a} \nabla \bar{u}$. $\square$

$$-\nabla \cdot a_\varepsilon \nabla u_\varepsilon = \nabla \cdot f \quad \text{in } \mathbb{R}^d$$

$$f \in L^2 \to \nabla u_\varepsilon \in L^2$$

$$-\nabla \cdot a_\varepsilon \nabla u_\varepsilon = h \quad \text{in } \mathbb{R}^d$$

$$h \in \text{Sob} \to \nabla u_\varepsilon \in L^2$$

$\parallel$ embedding

$$\nabla \cdot [\nabla \Delta^{-1} h]$$