M285K - course #8

III. QUANTITATIVE THEORY: OSCILLATIONS.

Solution $\nabla(-\nabla\cdot a\nabla)^{-1}\nabla\cdot$

2 difficulties: a) nonlinear wrt a $\Rightarrow$ simplification: Gaussian setting

b) nonlocal wrt a

↳ use Malliavin calculus:

reduce stoch information on X

to decay info on DX "$= \frac{\partial X}{\partial a}$"

e.g. $-\nabla\cdot a\nabla u = \nabla\cdot f$

$\Rightarrow -\nabla\cdot a\,\boxed{\nabla D u} = \nabla\cdot(Da\,\nabla u)$

$\nabla D u = \boxed{\nabla(-\nabla\cdot a\nabla)^{-1}\nabla\cdot}\,(Da\,\nabla u)$

$\| \nabla D u \|_{L^p} \le \| D a \nabla u \|_{L^p}$?

need large-scale regularity

Plan of this chapter: ① Malliavin

② Large-scale regularity: perturbative version.

③ Stochastic corrector estimates

④ Quantitative corrector results.]

Goal of the course: simplest Gaussian setting.

BUT full results, most key ideas.

References : * Otto, Gloria, Neukamm, ...

* Armstrong, Smart, Mourrat, Kuusi, ...

III.1 Gaussian setting & Malliavin calculus.

GAUSSIAN SETTING.

Coefficient field $a(x,\omega) = a_0(G(x,\omega))$

where * $a_0 \in C^2_b(\mathbb{R})^{d\times d}$ st $\begin{cases} e\cdot a_0(G)e \geq \alpha |e|^2 & \forall e \in \mathbb{R}^d \\ |a_0(G)e| \leq \beta |e| & \forall G \in \mathbb{R} \end{cases}$

* $G : \mathbb{R}^d \times \Omega \to \mathbb{R}$ centered stationary Gaussian random field

i.e. $G = \{G(x)\}_{x \in \mathbb{R}^d}$ collection of random variables

s.t. $\begin{cases} \mathbb{E}\, G(x) = 0 \quad \forall x \\ \mathbb{E}\, G(x) G(y) = K(x,y) \end{cases}$

of the form $K(x,y) = C(x-y)$

for some "covariance fct" $C \in L^\infty(\mathbb{R}^d)$

positive definite

$C(x) = C(-x)$.

Mixing assumption: G has integrable correlations

i.e. $\displaystyle\int_{\mathbb{R}^d} \left(\sup_{B(x)} |C| \right) dx < \infty$.

($\to G$ α-mixing)

<u>More precisely</u>: we can represent $G(x) = \int_{\mathbb{R}^d} C_0(x-y)\, d\xi(y)$

where $\begin{cases} C_0 * C_0 = C \\ d\xi \text{ white noise.} \end{cases}$

& we assume $\int_{\mathbb{R}^d} \left(\sup_{B(x)} |C_0| \right) dx < \infty$.

<u>Remark</u>: $C \in L^1 \cap L^\infty(\mathbb{R}^d)$, $\hat{C} \geq 0 \Rightarrow C$ unif continuous

$\Rightarrow G$ stoch continuous!

$\Rightarrow G$ jointly measurable.

(in particular $x \mapsto G(x)$ meas. a.s)

CRASH COURSE ON MALLIAVIN.

- Isonormal Gaussian process.

G viewed as a random element in $\mathcal{S}'(\mathbb{R}^d)$

$\forall \xi \in C_c^\infty(\mathbb{R}^d)$, set $\mathcal{G}(\xi) = \int_{\mathbb{R}^d} G\xi$

$\equiv$ centered Gaussian random variables

with $\mathbb{E}\left[\mathcal{G}(\xi_1)\mathcal{G}(\xi_2)\right] = \iint_{\mathbb{R}^d\,\mathbb{R}^d} \xi_1(x)\,\xi_2(y)\,C(x-y)\,dx\,dy$.

$\equiv \langle \xi_1, \xi_2 \rangle_{\mathfrak{h}}$

Define $\mathfrak{h}$ = closure of $C_c^\infty(\mathbb{R}^d)$ for $\|\cdot\|_{\mathfrak{h}}$, $\|\xi\|_{\mathfrak{h}}^2 = \langle \xi, \xi \rangle_{\mathfrak{h}}$.

(quotiented wrt kernel of $\|\cdot\|_{\mathfrak{h}}$)

& $\mathcal{G}$ extends as a centered Gaussian process on $\underline{h}$ (Hilbert)

$$\text{"\underline{isonormal}"} \quad \mathbb{E}\left[\mathcal{G}(\xi_1)\,\mathcal{G}(\xi_2)\right] = \langle \xi_1 , \xi_2 \rangle_h .$$

$$\mathcal{G} : h \to L^2(\Omega)$$

* <u>"Model" dense subspace</u>.

WLOG assume $(\Omega, \mathbb{P})$ endowed with σ-algebra generated by $\mathcal{G}$

(i.e. by $\{ \underline{\mathcal{G}(\xi)} : \xi \in C_c^\infty \}$)

Then $\mathcal{R}(\Omega) = \left\{ g\left(\widehat{\mathcal{G}(\xi_1)}, \dots, \widehat{\mathcal{G}(\xi_n)}\right) : n \geq 1,\ g \in C_b^\infty(\mathbb{R}^n),\ \xi_1 \dots \xi_n \in C_c^\infty(\mathbb{R}^d) \right\}$.

is dense in $L^2(\Omega)$. (view it as C_c^∞ dense in $L^2(\mathbb{R}^d)$).

Also define $R(\Omega, h) = \left\{ \sum_{i=1}^{n} \xi_i F_i \ : \ F_i \in R(\Omega), \xi_i \in h \right\}$

is dense in $L^2(\Omega, h)$.

* <u>Malliavin derivative</u>.

$\forall F \in R(\Omega)$, say $F = g\left(\overbrace{G(\xi_1)}^{\int_{\mathbb{R}^d} G \xi_1}, \dots, G(\xi_n) \right)$,

define $DF \in L^2(\Omega; h)$

$$DF = \sum_{j=1}^{n} \xi_j \, (\partial_j g)\left(G(\xi_1), \dots, G(\xi_n) \right).$$

formally: $DF = \text{"} \frac{\partial F}{\partial G} \text{"}$

Lemme. $\left[D : \mathcal{R}(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega, h) \right.$ is closable!

$\Rightarrow$ denote by D its closure "Malliavin derivative".

Proof. Idea: if D^* is densely defined then D is closable.

Step 1: $\left[\begin{array}{l} \mathrm{dom}\, D^* \supseteq \mathcal{R}(\Omega; h) \\ \& \; D^*(\xi F) = G(\xi) F - \langle \xi, DF \rangle_h . \\ \quad \forall \xi \in h,\ F \in \mathcal{R}(\Omega) \end{array} \right.$

Suffices to prove $\left[\mathbb{E}\left[\langle DH, \xi \rangle_h F \right] \right.$ $\forall H \in \mathcal{R}(\Omega)]$

$\mathbb{E}\left[G(\xi) F H \right] - \mathbb{E}\left[\langle \xi, DF \rangle H \right]$

$$= \mathbb{E}[G(\zeta) FH] - \mathbb{E}[\langle \zeta, \mathcal{D}H \rangle_h]$$

Note that $D(FH) = FDH + HDF$. (Leibniz)

Suffices to prove $\boxed{\begin{array}{l} \mathbb{E}[\langle DF, \zeta \rangle_h] = \mathbb{E}[G(\zeta) F] \\ \forall F \in \mathcal{R}(\Omega),\ \zeta \in C_c^\infty(\mathbb{R}^d). \end{array}}$

Let $F \in \mathcal{R}(\Omega)$, say $\left[\begin{array}{l} F = g(G(\zeta_1), \dots, G(\zeta_n)) \\ \text{WLOG assume } \{\zeta_1 \dots \zeta_n\} \text{ orthogonal in } h. \\ \& \ \zeta_1 = \zeta. \qquad \& \ \|\zeta_j\|_h = 1 \end{array} \right.$
$\zeta \in C_c^\infty(\mathbb{R}^d)$,

iid Gaussian standard

Compute $\mathbb{E}[\langle DF, \zeta \rangle_h] = \mathbb{E}[(\partial_1 g)(G(\zeta_1), \dots, G(\zeta_n))]$

$$= \int (\partial_1 g)(z_1, \dots z_n)\, e^{-\frac{1}{2}(z_1^2 + \dots + z_n^2)}\, dz_1 \dots$$

$$= \int_{(\mathbb{R}^d)^m} \left((\partial_1 g)(z_1 \dots z_m)\right) \frac{e^{\dots}}{(\sqrt{2\pi})^m} dz_1 \dots dz_m$$

$$\overset{IBY}{=} \int_{(\mathbb{R}^d)^m} z_1 \, g(z_1 \dots z_m) \, \frac{e^{-\frac{1}{2}(z_1^2 + \dots)}}{\sqrt{2\pi}^m} dz_1 \dots dz_m$$

$$= \mathbb{E}\left[\mathcal{G}(\xi_1) \underbrace{g(\mathcal{G}(\xi_1) \dots \mathcal{G}(\xi_n))}_{F} \right]$$

$$= \mathbb{E}\left[\mathcal{G}(\xi) F\right].$$

<u>Step 2</u>. Give $\begin{cases} F_n \in R(\Omega) \to 0 \text{ in } L^2(\Omega) \\ DF_n \to T \text{ in } L^2(\Omega; h) \end{cases}$.

Want to prove $T = 0$!

$\forall H \in R(\Omega), \xi \in h$: write $\mathbb{E}[H \langle \xi, DF_n \rangle_h] \to \mathbb{E}[H \langle \xi, T \rangle_h]$

$$= \mathbb{E}\left[F_n D^*(\xi H)\right]$$

$$= \mathbb{E}\left[G(\xi) F_n H\right] - \mathbb{E}\left[F_n \langle \xi, DH \rangle_h\right]$$

$$\to 0 \qquad \Rightarrow T = 0 ! \qquad \square$$

Define: $\|F\|^2_{\mathbb{D}^{1,2}(\Omega)} = \|F\|^2_{L^2(\Omega)} + \|DF\|^2_{L^2(\Omega, h)}$

$$\langle F, H \rangle_{\mathbb{D}^{1,2}(\Omega)} = \mathbb{E}[FH] + \mathbb{E}\left[\langle DF, DH \rangle\right]_{L^2(\Omega, h)}$$

& define $\mathbb{D}^{1,2}(\Omega)$ = closure of $\mathcal{R}(\Omega)$ wrt this norm.

= domain of D.

(first) Malliavin-Sobolev space.

⊛ <u>Divergence operator</u> D^* (= adjoint of D.)

<u>Recall</u>:

$$\begin{cases} \operatorname{dom} D^* \supseteq R(\Omega, h) \\ D^* : R(\Omega, h) \subseteq L^2(\Omega, h) \to L^2(\Omega) \\ D^*(\xi H) = G(\xi) H - \langle \xi, DH \rangle_h . \\ \xi \in h, \ H \in R(\Omega) \end{cases}$$

Note: $D^*(\xi) = G(\xi) = \int \xi \, G$

"Skorokhod integral".

<u>Theorem</u> (Meyer inequalities).

$$\begin{cases} \| D^* X \|_{L^2(\Omega)} \lesssim \| X \|_{\mathbb{D}^{1,2}(\Omega, h)} \\ \text{In particular } \operatorname{dom} D^* \supseteq \mathbb{D}^{1,2}(\Omega, h) \end{cases}$$

skip the proof. $\left[\ldots\right.$ approximation, ... $\ldots$ $\ldots$

* Ornstein-Uhlenbeck operator

Infinite-dim Laplacian $\mathcal{L} = \mathbb{D}^*\mathbb{D} : \mathcal{R}(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega)$

Meyer: $\operatorname{dom} \mathcal{L} = \mathbb{D}^{2,2}(\Omega)$. & $\mathcal{L}$ is self-adjoint on $\mathbb{D}^{2,2}$!

Lemma. (exercise)

$$\left[\begin{aligned} &\forall F \in \mathcal{R}(\Omega), \text{ say } F = g(G(\xi_1) \ldots G(\xi_n)), \\ &\mathcal{L}F = \sum_{j=1}^{n} G(\xi_j)(\partial_j g)(G(\xi_1) \ldots G(\xi_n)) \\ &\qquad - \sum_{j,h=1}^{n} \langle \xi_j, \xi_h \rangle_H (\partial^2_{jh} g)(G(\xi_1), \ldots, G(\xi_n)) \end{aligned}\right.$$

Lemma. (exercise)

$$\begin{cases} \text{(i)} \quad D\mathcal{L} = (\mathcal{L}+1)D \quad \text{on } \mathcal{R}(\cdot) . \\ \text{(ii)} \quad \mathcal{L}T_x = T_x \mathcal{L} \end{cases}$$

[Main result to be used: $\boxed{\mathrm{Var}[F] \leq \|DF\|_{L^2(\Omega, h)}}$;]