M285K - course #9

III.1 Malliavin calculus (cont'd)

Theorem.

(i) Poincaré's inequality:
$\forall F \in R(\Omega)$:
$$\mathrm{Var}[F] \leq \mathbb{E}[\|DF\|_h^2]$$

$\mathbb{E}[|\tilde{F - \mathbb{E}F}|^2] \leq \mathbb{E}[F \mathcal{L} F]$
Spectral gap for $\mathcal{L}$
* $\mathrm{Ker}\,\mathcal{L} = \mathbb{R}$ constants
* $\mathcal{L}|_{(\mathrm{Ker}\,\mathcal{L})^\perp} \geq 1$.

(ii) Helffer-Sjöstrand identity:
$\forall F, F' \in R(\Omega)$:
$$\mathrm{Cov}[F, F'] = \mathbb{E}\left[\langle DF, (\mathcal{L}+1)^{-1} DF' \rangle_h\right]$$

(iii) Log-Sobolev inequality:
$\forall F \in R(\Omega)$:
$$\mathrm{Ent}[F^2] := \mathbb{E}\left[F^2 \log \frac{F^2}{\mathbb{E}F^2}\right] \leq 2\,\mathbb{E}[\|DF\|_h^2].$$

Rem: Sobolev ineque.
$\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$
$n = \infty$: $\frac{2n}{n-2} = 2 \ldots$
$\to$ Poincaré.

(iv) Higher moment bounds:

$\forall F \in \mathcal{R}(\Omega)$:

$$\mathbb{E}\left[|F-\mathbb{E}F|^{2p}\right] \leq \underbrace{(2p+1)^{p}}_{\text{optimal : } \mathbb{E}|G|^{2p} \sim (2p+1)^{p}.} \mathbb{E}\left[\|DF\|_h^{2p}\right]$$

Lemma (Mehler's formula). $\forall F = F(G) \in \mathcal{R}(\Omega)$:

$$e^{-t\mathcal{L}}F = \mathbb{E}'\left[F\left(e^{-t}G + \sqrt{1-e^{-2t}}\,G'\right)\right]$$

where $\begin{cases} G' = \text{iid copy of } G. \\ \mathbb{E}' = \text{expectation wrt. } G'. \end{cases}$

Rem: interpolation $G_t = e^{-t}G + \sqrt{1-e^{-2t}}\,G'$ have same law as G $\forall t$.

Proof. Consider $U^t F := \mathbb{E}'[F(G_t)]$

Clearly $\{U^t\}_t$ is C_0-semigroup on $L^q(\Omega)$, $q < \infty$.

& Generator: $\forall F \in R(\Omega)$, $F = g(G(\xi_1) \dots G(\xi_n))$

$$\frac{d}{dt} U^t F \Big|_{t=0} = (\text{explicit}) = \underbrace{-\mathcal{L}}\, F \quad \underline{\text{exercise!}} \qquad \square$$

Proof of theorem:

(i) Poincaré inequality

$$\mathrm{Var}[F] = \underbrace{\mathbb{E} F^2} - \underbrace{(\mathbb{E}F)^2} = -\int_0^\infty \frac{d}{dt}\left[\mathbb{E}\left[(e^{-t\mathcal{L}}F)^2\right]\right] dt$$

cf Mehler: $e^{-t\mathcal{L}}F \xrightarrow[t\to\infty]{} \mathbb{E}F$ in L^2

$$= +2\int_0^\infty \mathbb{E}\left[\left(e^{-t\mathcal{L}}F\right)\left(\mathcal{L}e^{-t\mathcal{L}}F\right)\right] dt$$

$(\mathcal{L} = D^*D)$

$$= 2\int_0^\infty \mathbb{E}\left[\|\underbrace{De^{-t\mathcal{L}}F}\|_h^2\right] dt$$

$= \underbrace{e^{-t}}\, e^{-t\mathcal{L}}(DF)$ by Mehler

[otherwise: $\underline{D\mathcal{L} = (\mathcal{L}+1)D}$]

$$= 2\underbrace{\int_0^\infty e^{-2t}}\ \underbrace{\mathbb{E}\left[\|e^{-t\mathcal{L}}(DF)\|_h^2\right]}_{\le \mathbb{E}[\|DF\|_h^2]} dt$$

(e.g. via Mehler)

(or $\mathcal{L} \geq 0$)

$$\leq \mathbb{E}\left[\|DF\|_h^2\right].$$

(ii) Helffer–Sjöstrand identity.

Note that (i) gives
$$\left|\begin{array}{l} \ker \mathcal{L} = \mathbb{R} \\ \mathcal{L}|_{(\ker \mathcal{L})^\perp} \geq 1 \\ \mathcal{L}| : (\ker \mathcal{L})^\perp \to (\ker \mathcal{L})^\perp \text{ invertible} \end{array}\right.$$

$$\left(\begin{array}{l} \operatorname{Im} \mathcal{L} \perp \mathbb{R} \\ \mathbb{E}[\mathcal{L} F] = 0 \\ = \mathbb{E}\langle D1, DF\rangle. \end{array}\right.$$

$\forall F, F' \in R(\Omega)$, $\mathbb{E}F = \mathbb{E}F' = 0$.

In part: $F' \in (\ker \mathcal{L})^\perp \Rightarrow [F' = \mathcal{L} Z$ for some $Z \in \mathbb{D}^{2,2}(\Omega)$.

$$\operatorname{Cov}(F, F') = \mathbb{E}\, F F' = \mathbb{E}\, F \mathcal{L} Z$$
$$= \mathbb{E}\left[\langle DF, DZ\rangle_h\right]$$

$$\& \quad DF' = D\mathcal{L} Z = (\mathcal{L}+1) DZ$$
$$\Rightarrow DZ = (\mathcal{L}+1)^{-1} DF'.$$

$$= \mathbb{E}\left[\langle DF, (\mathcal{L}+1)^{-1} DF' \rangle_h \right].$$

(iii) Log-Sobolev:

$$\mathrm{Ent}\, F^2 = \mathbb{E}\left[F^2 \log F^2\right] - \mathbb{E}[F^2] \log \mathbb{E}[F^2].$$

$$= - \int_0^\infty \frac{d}{dt} \left(\mathbb{E}\left[e^{-t\mathcal{L}}(F^2) \log e^{-t\mathcal{L}}(F^2) \right] \right) dt$$

$$= + \int_0^\infty \mathbb{E}\left[\left(\mathcal{L} e^{-t\mathcal{L}}(F^2) \right) \left[\cancel{1+} \log e^{-t\mathcal{L}}(F^2) \right] \right] dt$$

$$= \int_0^\infty \mathbb{E}\left[\frac{\| D e^{-t\mathcal{L}}(F^2) \|_h^2}{e^{-t\mathcal{L}}(F^2)} \right] dt \quad \ast$$

where $\|D e^{-tL}(F^2)\|_h^2 = 4e^{-2t} \|e^{-tL}(F\,DF)\|_h^2$

$$\overset{\text{exercise}}{\underset{\text{Mehler + CS}}{\leq}} 4e^{-2t} \underbrace{e^{-tL}(F^2)} \, e^{-tL}(\|DF\|_h^2).$$

$$\to \mathrm{Ent}(F^2) \leq \underbrace{4\int_0^\infty e^{-2t}}_{2} \mathbb{E}\left[\cancel{e^{-tL}} \|DF\|_h^2\right] dt$$

(iv) <u>Higher moment bounds.</u> (usual consequence of log Sob.)

Proof from HS: $\quad F \in R(\Omega),\ \mathbb{E}F = 0.$

$$\begin{aligned} \mathbb{E}\, F^{2p} &= \mathbb{E}\, F\, F^{2p-1} \\ &= \mathrm{Cov}(F, F^{2p-1}) \end{aligned}$$

$$= \mathbb{E} \langle DF, (\mathcal{L}+1) DF \rangle_h$$

$$\overset{HS}{=} (2p-1)\, \mathbb{E}\left[F^{2p-2} \langle DF, (\mathcal{L}+1)^{-1} DF \rangle_h\right]$$

$$\overset{\text{Hölder}}{\leq} (2p-1)\, \mathbb{E}\left[F^{2p}\right]^{1-\frac{1}{p}} \underbrace{\mathbb{E}\left[\| (\mathcal{L}+1)^{-\frac{1}{2}} DF\|_h^{2p}\right]^{\frac{1}{p}}}_{\leq\, \mathbb{E}\left[\|DF\|_h^{2p}\right]^{\frac{1}{p}} \text{ (e.g. Mehler)}}$$

$$\mathbb{E}\left[F^{2p}\right]^{\frac{1}{p}} \leq (2p-1)\, \mathbb{E}\left[\|DF\|_h^{2p}\right]^{\frac{1}{p}}. \qquad \square$$

<u>Rem</u>: $\left.\begin{array}{l}\text{Poisson setting} \\ \text{iid setting}\end{array}\right\} \to \exists$ adapted version of Malliavin.

<u>III.2 Large-scale regularity.</u>

$$-\nabla \cdot a \nabla w = \nabla \cdot h \qquad (\varepsilon = 1)$$

a) Classical regularity results.

① Energy estimate: $\|\nabla w\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{\alpha} \|h\|_{L^2(\mathbb{R}^d)}$

Rem: $-\Delta v = \nabla \cdot g \Rightarrow$ maximal L^p regularity:

$\|\nabla v\|_{L^p(\mathbb{R}^d)} \leq C(p) \|g\|_{L^p(\mathbb{R}^d)} \quad \forall\, 1 < p < \infty.$

(very optimal cst)

② Perturbative improvement:

Lemma (Meyers' argument). $\exists\, C = C(d, \alpha, \beta) > 0$

s.t. $\|\nabla w\|_{L^p(\mathbb{R}^d)} \leq C \|h\|_{L^p(\mathbb{R}^d)}$

$\forall\, |p-2| \leq \frac{1}{C}.$

Proof. $\to \left(-\nabla\cdot a \nabla w = \nabla \cdot h \right) \times \frac{2}{\alpha+\beta}$

$$\Rightarrow -\Delta w = \nabla\cdot\left(\frac{2a}{\alpha+\beta} - I_d\right)\nabla w + \nabla\cdot\left(\frac{2}{\alpha+\beta}h\right).$$

$\Rightarrow$ maximal L^p reg for Δ: $\forall 1<p<\infty$,

$$\|\nabla w\|_{L^p(\mathbb{R}^d)} \leq C(p) \Big\| \underbrace{\left(\frac{2a}{\alpha+\beta} - I_d\right)}_{\substack{|\cdot| \leq \frac{\beta-\alpha}{\alpha+\beta} \\ \text{because } \alpha\leq a\leq\beta.}} \nabla w \Big\|_{L^p(\mathbb{R}^d)} + \frac{2}{(\alpha+\beta)} C(p)\|h\|_{L^p}.$$

$$\leq \overbrace{\frac{\beta-\alpha}{\alpha+\beta}\,\boxed{C(p)}}^{<1} \underbrace{\|\nabla w\|_{L^p}}_{\text{absorb!}} + \frac{2}{\alpha+\beta} C(p) \|h\|_{L^p}.$$

Riesz-Thorin interpolation: $C(p) \leq C(2)^{\frac{4}{p}-1} C(4)^{2-\frac{4}{p}}$, $2<p<4$

$$\to \overline{\lim}\, C(p) \leq C(2) = 1$$

$p \to 2$

$$-\Delta v = \nabla \cdot g$$
$$\int |\nabla v|^2 \leq \int |g|^2.$$

$\Rightarrow \exists\, C_0$ large: $\forall\, |p-2| \leq \frac{1}{C_0}$

$$\frac{\beta-\alpha}{\alpha+\beta} C(p) < 1. \quad \Rightarrow \text{con desired!} \qquad \square$$

③ C^r regularity: $\qquad -\nabla \cdot a \nabla w = \nabla \cdot h.$

• De Giorgi-Nash-Moser theory:

Theorem. $\exists\, \gamma = \gamma(d, \alpha, \beta) > 0, \quad \forall q > d.$

s.t. $\displaystyle \sup_{B(x)} |w| + \sup_{\substack{y,z \in B(x) \\ y \neq z}} \frac{|w(y) - w(z)|}{|y-z|^\gamma} \leq \|w\|_{L^2(B_2(x))} + \|h\|_{L^q(B_2(x))}.$

(• If a smooth, then get higher order reg.)

~~That's all! ∃ counterexamples: L^p & C^r regularity fail in general~~
$|p-2| \geq \frac{1}{C_0}$, $r > \gamma_0$

b) Large-scale regularity. $\quad -\nabla \cdot a \nabla w = \nabla \cdot h$

Homogenization $\quad (-\nabla \cdot a \nabla) \approx \underbrace{(-\nabla \cdot \bar{a} \nabla)}_{\text{all standard cont-coeff regularity.}}$ on large scales