Please provide complete and well-written solutions to the following exercises.
Due on January 15th before noon.

Homework 1

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(i) Prove the following *union bound*: for all $A, B \in \mathcal{F}$,

$$\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B].$$

(ii) By induction, deduce from (i) that for all $m \geq 2$ and $A_1, \ldots, A_m \in \mathcal{F}$,

$$\mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} \mathbb{P}[A_i].$$

Further deduce, for all sequences $(A_i)_i \subset \mathcal{F}$,

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i].$$

(iii) Prove the following *inclusion-exclusion formula*: for all $m \geq 2$ and $A_1, \ldots, A_m \in \mathcal{F}$,

$$\mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] = \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \leq i_1 < \ldots < i_r \leq m} \mathbb{P}[A_{i_1} \cap \ldots \cap A_{i_r}].$$

(iv) Prove the Bonferroni inequality: for all $m \geq 2$ and $A_1, \ldots, A_m \in \mathcal{F}$,

$$\mathbb{P}\left[\bigcap_{i=1}^{m} A_i\right] \geq \left(\sum_{i=1}^{m} \mathbb{P}[A_i]\right) - (m-1).$$

Exercise 2. Let Ω be an uncountable set and let $\mathcal{F}$ be the collection of all subsets $A \subset \Omega$ such that either A or A^c is at most countable. Define

$$\mathbb{P}[A] := \begin{cases} 0 & : \text{ if } A \text{ is at most countable,} \\ 1 & : \text{ if } A^c \text{ is at most countable.} \end{cases}$$

Show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Exercise 3. Consider 52 standard playing cards, which are distributed at random between four players in such a way that each player receives 13 cards.

(i) What is the probability that each player receives one ace?

(ii) What is the probability that a player gets 6 spades, 3 hearts, 2 diamonds, and 2 clubs?

Exercise 4 (de Méré's paradox)**.** Consider 6-sided fair dice. Is it more probable to get at least one six with 4 throws of one die, or to get at least one double six with 24 throws of 2 dice?

Exercise 5. Suppose we roll three fair dice. Identify the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}^3$ as a $(6 \times 6 \times 6)$ 3-dimensional integer lattice and let X, Y, and Z denote the outcome of each die.

(i) For each $k \geq 1$, show that

$$\mathbb{P}[X+Y+Z=k] = 6^{-3}\big(\sharp \text{ intersections between plane } x+y+z=k \text{ and lattice } \Omega\big).$$

What are the minimum and maximum possible values for $X+Y+Z$?

(ii) Draw a cube for Ω and planes $x+y+z=k$ for $k=3,5,10,11,16,18$. Argue that the intersection gets larger as k increases from 3 to 10 and smaller as k goes from 11 to 18. Conclude that 10 and 11 are the most probable values for $X+Y+Z$.

(iii) Consider the following identity

$$\begin{aligned}&(x+x^2+x^3+x^4+x^5+x^6)^3\\ &\qquad = x^{18}+3x^{17}+6x^{16}+10x^{15}+15x^{14}+21x^{13}+25x^{12}+27x^{11}+27x^{10}\\ &\qquad\qquad +25x^9+21x^8+15x^7+10x^6+6x^5+3x^4+x^3.\end{aligned}$$

Show that the coefficient of x^k in the right-hand side equals the size of the intersection between the plane $x+y+z=k$ and the lattice Ω. Conclude that

$$\mathbb{P}[X+Y+Z=10] = \mathbb{P}[X+Y+Z=11] = \frac{27}{6^3} = \frac{1}{8}.$$

(This way of calculating probabilities is called the generating function method.)

Exercise 6. Consider n labeled keys, each supposed to be hung on its own hook.

(i) If the keys are hung independently at random on the different n hooks, assuming there is no limit to the number of keys that can be hung per hook, what is the probability that at least one key is hung on its own hook?

(ii) Now if the keys are hung independently at random on the different hooks but if one takes care to hang exactly one key on each hook, what is the corresponding probability.

(iii) Find the limit of both probabilities as $n \uparrow \infty$.

Exercise 7. Consider two urns. Urn A contains 3 white and 4 black balls, and Urn B contains 2 white and 6 black balls.

(i) You pick a ball at random from Urn A and place it in Urn B. Next, you pick a ball at random from Urn B. What is the probability that the ball is black?

(ii) You pick an urn at random. Next, you pick a ball at random from the chosen urn. Given the ball is black, what is the probability you picked Urn A?

Exercise 8. Suppose that you arrive at a bus stop at some time between 8am and 8.25am with uniform probability. Buses arrive at 8.03, 8.13, 8.23, 8.33, etc. What is the probability that you need to wait more than 2 minutes?

Exercise 9. Three friends, Pierre, Paul, and Jacques take turns flipping a fair coin in that order. The first one to flip tails wins the game.

(i) Write down a model for this experiment.

(ii) Describe the event that Pierre wins, the event that Paul wins, and the event that Jacques wins. Compute their probabilities.

Exercise 10. A coin is tossed repeatedly; on each toss a head is shown with probability p, or a tail with probability $1-p$. The outcomes of the tosses are independent. Let E denote the event that the first run of r successive heads occurs earlier that the first run of s successive tails. Let A denote the outcome of the first toss. Show that

$$\mathbb{P}[E|A = \text{head}] = p^{r-1} + (1 - p^{r-1})\mathbb{P}[E|A = \text{tail}].$$

Find a similar expression for $\mathbb{P}[E|A = \text{tail}]$, and deduce the value of $\mathbb{P}[E]$.

Exercise 11. Consider a die with a prime number p of faces and throw it once. Show that no two events A and B can be independent unless either A or B is the whole sample space or the empty set.

Exercise 12. There are n people standing in a line. Initially, the leftmost person is holding a potato. At each step, the person holding the potato passes it, with equal probability to one of their neighbors. Once the rightmost person receives the potato, the game ends.

(i) Describe the sample space.

(ii) If $n = 4$, explicitly describe the event that the game ends within 5 steps. What is its probability?

Addendum

(for personal consideration only, does **not** count for homework)

Exercise 13. Let $[0, 1) \subset \mathbb{R}$ be the unit interval. This exercise aims to prove that $[0, 1)$ is uncountable.

(i) Let $x \in [0, 1)$. Show that there exists a unique sequence $(x_n)_n \subset \{0, 1\}$ such that all x_n's are not all equal to 1 after some index and such that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Set $x = 0.x_1x_2x_3\cdots$, which is the binary representation of x.

(ii) Conversely, given a sequence $(x_n)_n \subset \{0, 1\}$ such that all x_n's are not all equal to 1 after some index, show that the series

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

converges in $[0, 1)$. Deduce a bijection between $[0, 1)$ and $2^{\mathbb{N}}$.

(iii) Suppose that $[0, 1)$ is countable. Then we can enumerate all of its elements $(a_n)_n$. Using (i), we can write each a_n by its unique binary representation,

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ a_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ a_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ a_4 &= 0.a_{41}a_{42}a_{43}a_{44}\cdots \\ & \quad \text{etc.} \end{aligned}$$

Now let $a \in [0, 1)$ be defined by

$$a = 0.\bar{a}_{11}\bar{a}_{22}\bar{a}_{33}\bar{a}_{44}\cdots$$

where we use the notation $\bar{0} = 1$ and $\bar{1} = 0$. Show that $a \neq a_n$ for all $n \geq 1$, which proves that we have found an element $a \in [0, 1)$ that is not among the list $(a_n)_n$. This is known as Cantor's diagonalization argument and proves that $[0, 1)$ is uncountable.

Exercise 14. Let $\Omega = [0, 1)$ and consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that the probability structure is uniform (or translation-invariant) on $[0, 1)$ if it satisfies $x + A \in \mathcal{F}$ and $\mathbb{P}[x + A] = \mathbb{P}[A]$ for all $A \in \mathcal{F}$ and all $x \in \mathbb{R}$, where the sum '+' on $[0, 1)$ stands for the sum modulo 1 (e.g. $x + y$ stands for the fractional part of the sum of $x, y \in [0, 1)$). This exercise aims to prove that Ω cannot be endowed with a uniform probability structure such that $\mathcal{F} = 2^{\Omega}$.

(i) Consider the relation $\sim$ on $[0, 1)$ defined by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q},$$

and show that $\sim$ is an equivalence relation on $[0, 1)$.

(ii) Consider the quotient space $\mathcal{A} = [0, 1)/\sim$ and use the axiom of choice to select a function $\psi : \mathcal{A} \to [0, 1)$ such that $\psi(\alpha) \in \alpha$ for all $\alpha \in \mathcal{A}$. Set $E := \psi(\mathcal{A})$ and show that

$$\{E + q : q \in \mathbb{Q}, |q| < 1\}$$

is a countable family of disjoint subsets whose union equals $[0, 1)$.

(iii) If $E \in \mathcal{F}$ and if the probability structure is uniform, show that it would lead to a contradiction.