Please provide complete and well-written solutions to the following exercises.
Due on January 29nd before noon.

Homework 3

Exercise 1. Let X be a nonnegative discrete random variable.

(i) Prove that the identity $\mathbb{E}\left[\frac{1}{X}\right] = \frac{1}{\mathbb{E}[X]}$ holds if and only if $\mathrm{Var}\left[X\right] = 0$.

(ii) What about the identity $\mathbb{E}\left[e^X\right] = e^{\mathbb{E}[X]}$?

Exercise 2. A discrete random variable X with nonnegative integer values is said to be memoryless if it satisfies for all $k, j \geq 0$,

$$\mathbb{P}\left[X > k + j | X > k\right] = \mathbb{P}\left[X > j\right].$$

(In other words, if we are given that $X > k$, then the distribution of $X - k$ is the same as the original distribution of X.)

(i) If X has a geometric distribution, show that it is memoryless.

(ii) Show that geometric distributions are the only distributions on nonnegative integers with this property.

Exercise 3. Let X have a Poisson distribution with parameter λ.

(i) Find the pmf of $Y = |\sin(\frac{\pi}{2}X)|$.

(ii) For all $k \geq 1$, compute $\mathbb{E}\left[X(X-1)\ldots(X-k+1)\right]$ (so-called kth factorial moment of X).

(iii) For all $k \geq 1$, deduce $\mathbb{E}\left[X^k\right]$ (so-called kth moment of X).

(iv) For all $k \geq 1$, deduce $\mathbb{E}\left[(X - \mathbb{E}\left[X\right])^k\right]$ (so-called kth centered moment of X).

Exercise 4. The probability of obtaining a head when a certain coin is tossed is p. The coin is tossed repeatedly until n heads occur in a row. Compute the expectation and the variance of the total number of tosses required for this to happen.

Exercise 5. You toss n coins, where each one shows heads with probability p. Each coin which shows heads is tossed again.

(i) What is the distribution of the number of heads resulting from the second round of tosses? What about the third round?

(ii) Compute their expectations and variances.

Exercise 6.

(i) Let X be a discrete random variable with nonnegative integer values. Show that

$$\mathbb{E}\left[X\right] = \sum_{k=0}^{\infty} \mathbb{P}\left[X > k\right].$$

(ii) A fair die having two faces colored blue, two red, and two green, is thrown repeatedly. Find the probability that not all colors occur in the first k throws. Using (i), compute the expectation of the first time at which all three colors have occurred at least once previously.

Exercise 7. Every package of some commodity includes a small and exciting object. There are c different types of object, and each package is equally likely to contain any given type. You buy one package each day.

(i) Find the probability that the first n objects which you collect do not form a complete set.

(ii) Find the mean number of days which elapse before you have a full set of objects.

(iii) Find the mean number of days which elapse between the acquisitions of the jth new type of object and the $(j+1)$th new type.

Exercise 8. Consider a simple random walk $(S_n)_n$ starting at $S_0 = 0$: at each integer time $n \geq 1$, the walk S_n goes either up (i.e. $S_n := S_{n-1} + 1$) or down (i.e. $S_n := S_{n-1} - 1$), both with equal probabilities. Let $T = \min\{n \geq 1 : S_n = 0\}$ be the time of the first return of the walk to its starting point.

(i) Compute $\mathbb{P}[T = 2n]$.

(ii) Deduce that $\mathbb{E}[T^\alpha] < \infty$ if and only if $\alpha < \frac{1}{2}$.

Exercise 9. An urn contains N balls, K of which are blue and $N - K$ red. A random sample of n balls is withdrawn without replacement from the urn. Let X be the number of blue balls in this sample. By definition, X has a hypergeometric distribution with parameters N, K, n.

(i) Compute $\mathbb{E}[X]$.

(ii) Compute $\mathbb{E}[X(X-1)]$ and deduce that

$$\mathrm{Var}[X] = n\frac{K}{N}\left(1 - \frac{K}{N}\right)\frac{N-n}{N-1}.$$

(iii) Show that, if $N, K \uparrow \infty$ such that $K/N \to p$, then

$$\mathbb{P}[X = k] \to \binom{n}{k} p^k (1-p)^{n-k}.$$

Argue that this shows that, for small n and large N, the distribution of X barely depends on whether or not the balls are replaced in the urn immediately after their withdrawal.