Please provide complete and well-written solutions to the following exercises.
Due on February 26th before noon.

Homework 6

Exercise 1. Let X, Y be either discrete or jointly absolutely continuous random variables.

(i) If the map $x \mapsto \mathbb{E}[Y|X = x]$ is linear, show that it necessarily coincides with the least square regression line

$$\mathbb{E}[Y|X = x] \;=\; \mathbb{E}[Y] + \rho(X, Y)\frac{\sigma(Y)}{\sigma(X)}(x - \mathbb{E}[X]).$$

(ii) If $\mathbb{E}[Y|X = x] = ax + b$ and $\mathbb{E}[X|Y = y] = cy + d$ for all x, y, for some $a, b, c, d \in \mathbb{R}$, deduce that

$$\rho(X, Y)^2 = ac \qquad \text{and} \qquad \operatorname{sgn}(\rho(X, Y)) = \operatorname{sgn}(a) = \operatorname{sgn}(b).$$

(iii) In a class of n students, assume each student gets independently a grade A with probability p_A, a grade B with probability p_B, and a grade C with probability $1 - p_A - p_B$. Let X and Y be the number of students getting a grade A and a grade B, respectively. Determine the joint density of X, Y, their marginal distributions, and the conditional distribution of $[Y|X = x]$ and $[X|Y = y]$. Using (ii), deduce $\rho(X, Y)$ without making any further computation.

Exercise 2. Let X and Y be independent random variables, each uniformly distributed on $[0, 1]$.

(i) Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Show that $\mathbb{E}[U] = \frac{1}{3}$, and deduce the covariance of U and V.

(ii) Show that $\mathbb{P}\left[V^2 > U > x\right] = \frac{1}{3} - x + \frac{2}{3}x^{\frac{3}{2}}$ for $x \in (0, 1)$.

(iii) What is the probability that the random quadratic equation $x^2 + 2Vx + U = 0$ has real roots?

(iv) Given that the two roots R_1, R_2 of the above quadratic are real, what is the probability that both $|R_1| \leq 1$ and $|R_2| \leq 1$?

Exercise 3. Let $(X_n)_n$ be a sequence of uncorrelated random variables, each having mean μ and variance σ^2. If $\bar{X}_n := \frac{1}{n}(X_1 + \ldots + X_n)$, show that

$$\mathbb{E}\Big[\tfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\Big] = \sigma^2.$$

Exercise 4. Two players A and B play a series of independent games. The probability that A wins any particular game is p^2, that B wins is q^2, and that the game is a draw is $2pq$, where $p + q = 1$. The winner of a game scores 2 points, the loser none; if a game is drawn, each player scores 1 point. Let X and Y be the number of points scored by A and B, respectively, in a series of n games. Prove that $\operatorname{Cov}[X; Y] = -2npq$.

Exercise 5.

(i) A discrete random variable X with nonnegative integer values satisfies $\mathbb{E}[X^r] = \lambda$ for all $r \geq 1$. Compute its moment generating function and deduce the law of X.

(ii) If a discrete random variable X satisfies $\mathbb{E}[X^r] = 5^r$ for all $r = 1, 2, \ldots$, find the moment-generating function of X. What is its distribution?

Exercise 6. Let $(X_n)_n$ be a sequence of independent and identically distributed random variables, and let N be a random variable with positive integer values and independent of the X_n's. Find the moment generating function of $S_N := X_1 + \ldots + X_N$ in terms of the moment generating functions of X_1 and N, provided these exist.

Exercise 7. Let X, Y be random variables with equal variance.

(i) Show that $X+Y$ and $X-Y$ are uncorrelated. Give an example to show that $X+Y$ and $X-Y$ need not be independent even if in addition X and Y are independent.

(ii) Further assume that X, Y are independent and identically distributed, with mean 0 and variance 1, and with finite moment generating M. If $X+Y$ and $X-Y$ are independent, show that $M(2t) = M(t)^3 M(-t)$ and deduce that X and Y have the normal distribution with mean 0 and variance 1.

Exercise 8. There are c different types of coupon, and each coupon obtained is equally likely to be any one of the c types. Find the moment generating function of the total number N of coupons which you must collect in order to obtain a complete set. Deduce its expectation.

Exercise 9. There is a random number N of foreign objects in your soup, with moment generating function M_N. Each object is a fly with probability p, and otherwise a spider; different objects have different types. Let F be the number of flies and S the number of spiders.

(i) Show that F has moment generating function $M_F(t) = M_N(\log(pe^t + 1 - p))$.

(ii) Suppose that N has the Poisson distribution with parameter λ. Show that F has the Poisson distribution with parameter λp and that F ans S are independent.

(iii) Let $p = \frac{1}{2}$ and suppose that F and S are independent. Show that $M_N(t) = M_N(\log(\frac{1}{2}(1+e^t)))^2$ and deduce that N has a Poisson distribution.