Please provide complete and well-written solutions to the following exercises.
Due on March 5th before noon.

Homework 7

Exercise 1. Let $(X_n)_n$ be a sequence of independent random variables with Cauchy distribution, $f_{X_i}(t) = \frac{1}{\pi}(1+t^2)^{-1}$ for $t \in \mathbb{R}$. For all n, determine the distribution of $\frac{1}{n}(X_1 + \ldots + X_n)$.

Exercise 2. Let $(X_n)_n$ be a sequence of independent random variables, each with characteristic function ϕ.

(i) Given $a_n, b_n \in \mathbb{R}$, determine the characteristic function of $Y_n = a_n + b_n(X_1 + \ldots + X_n)$.

(ii) Suppose that $\phi(t) = e^{-|t|^\alpha}$ for some $\alpha \in (0, 2]$. Determine a_n, b_n such that Y_n has the same distribution as X_1 for all $n \geq 1$.

(iii) Explicitly find the probability density of X_1 when $\alpha = 1$ and $\alpha = 2$.

Exercise 3. The cumulant generating function K_X of the random variable X is defined by $K_X(t) = \log \mathbb{E}\left[e^{tX}\right]$. If the latter is finite in a neighborhood of the origin, then K_X has a convergent Taylor expansion

$$K_X(t) = \sum_{n=1}^{\infty} \tfrac{1}{n!}\kappa_n(X)t^n,$$

and $\kappa_n(X)$ is called the nth cumulant of X.

(i) Express $\kappa_1(X), \kappa_2(X), \kappa_3(X)$ in terms of moments of X.

(ii) If X and Y are independent random variables, show that $\kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y)$.

(iii) If X is a standard normal random variable, show that its cumulants are $\kappa_2(X) = 1$ and $\kappa_m(X) = 0$ for all $m \neq 2$.

Exercise 4. Let X, Y be independent standard normal random variables and let U, V be independent of X, Y. Show that

$$Z := \frac{UX + VY}{\sqrt{U^2 + V^2}}$$

is also a standard normal random variable.

Exercise 5. Is it possible for three random variables X, Y, Z to have the same distribution and to satisfy $X = U(Y + Z)$ where U is uniform on $[0, 1]$ and where Y, Z are independent of U and of one another?

Exercise 6. The random variable X is said to have a lattice distribution if there exist a, b such that X takes values in the set $a + b\mathbb{Z}$. The span of X is defined as the maximal value b for which there exists a such that X takes values in $a + b\mathbb{Z}$.

(i) Suppose that X has a lattice distribution with span b. Show that $|\phi_X(\frac{2\pi}{b})| = 1$ and that $|\phi_X(t)| < 1$ for $0 < t < \frac{2\pi}{b}$.

(ii) Suppose that $|\phi_X(\theta)| = 1$ for some $\theta \neq 0$. Show that X has a lattice distribution with span $\frac{2\pi k}{\theta}$ for some integer k.

Exercise 7. If X is an integer-valued random variable with characteristic function ϕ, show that

$$\mathbb{P}[X = k] = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk} \phi(t)\, dt.$$

What is the corresponding result for a random variable with a lattice distribution?

Exercise 8. Consider a sequence of Bernoulli trials with success probability p. Given $r \geq 1$, let N be the smallest value such that at least one run of length r occurs in the N first trials. Determine the moment generating function of N and deduce $\mathbb{E}[N]$.