EFFECTIVE SEDIMENTATION SPEED OF RANDOM SUSPENSIONS

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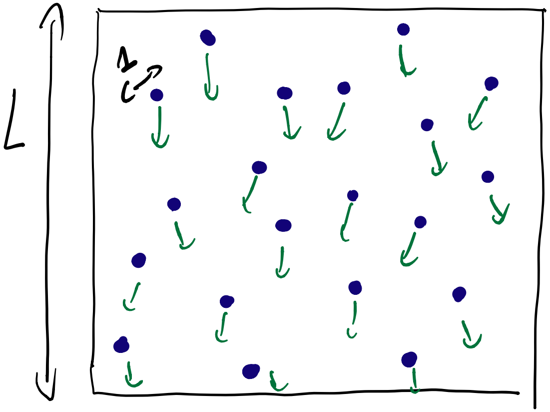
Collective behavior of particles in fluids, Dec 14-17, 2020

Introduction

torus Θ_L, $L \gg 1$



Model : heavy rigid particles fall by gravity in a Stokes fluid

size 1

Setting : no inertia (steady Stokes)

Question : analysis of the map

$$\{x_i\} \longmapsto \{V_i\}$$

(position of particles) (velocity of particles)

Interested in statistical rather than pointwise properties

Assume $\{x_i\}_i$ are distributed according to $\mathbb{P}_L$

what can we say on the distribution of $\{V_i\}_i$?

Observables : $\bar{V}_L = \mathbb{E}_L(V_i)$, $\sigma_L^2 = \mathbb{E}_L(|V_i - \bar{V}_L|^2)$

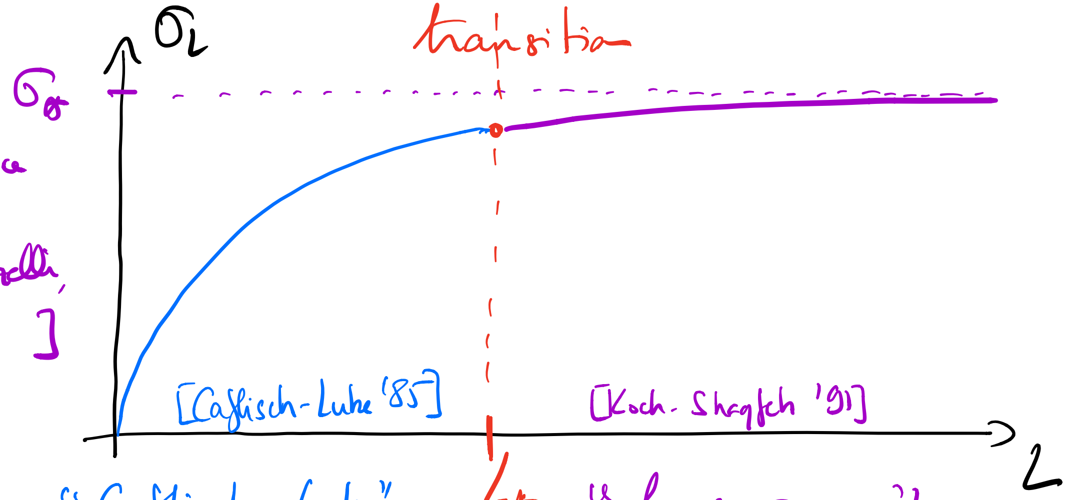
effective sedimentation speed

variance of the sedimentation speed

Physical insight from experiments (dimension $d = 3$)

- Effective sedimentation speed: $\bar{V}_L \to \bar{V}$ as $L \uparrow \infty$
- Variance of the sedimentation speed σ_L^2:

[cf. "divergence paradox" with E. Guazzelli, $L_* \sim 20\,\lambda^{-1/3}$]



"Caflisch-Luke" regime $\sigma_L \sim \sqrt{L}$

"long range" screening of interactions

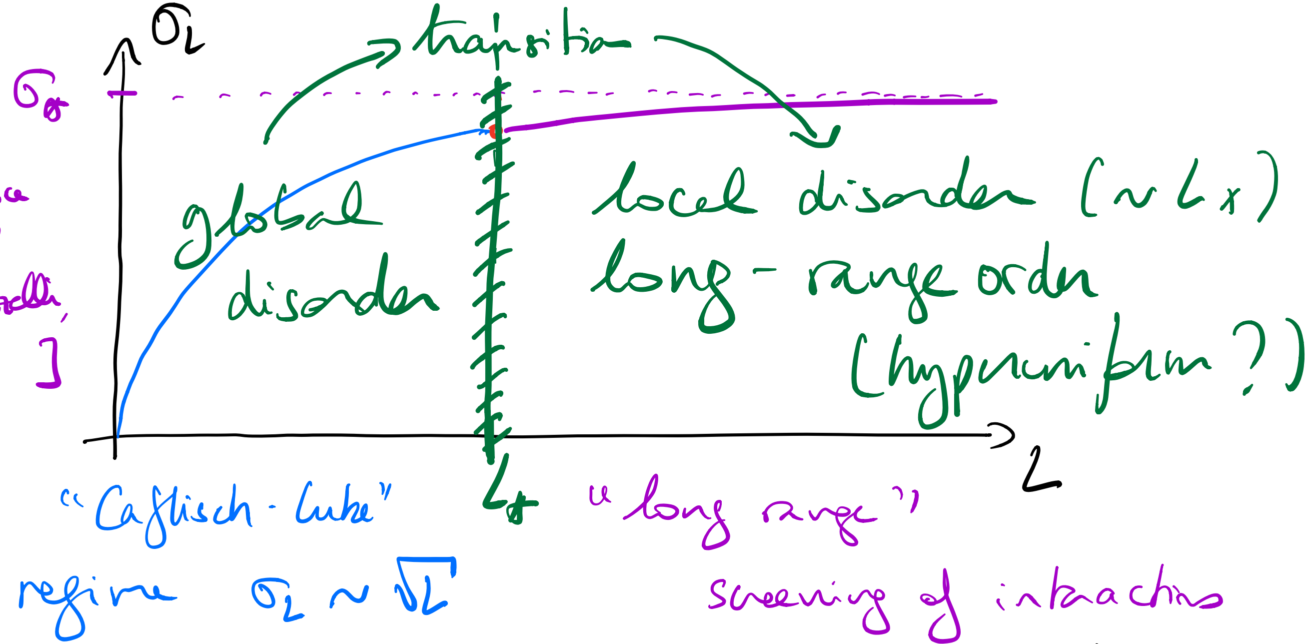
Aim of the talk: 1 Identify a measure $\mathbb{P}_L$ likely to yield such a transition

2 Rigorously prove this behavior

Physical insight from experiments (dimension $d = 3$)

- Effective sedimentation speed: $\bar{V}_L \to \bar{V}$ as $L \uparrow \infty$
- Variance of the sedimentation speed:

[cf. "divergence paradox" with E. Guazzelli, $L_* \sim 20\, \lambda^{-1/3}$]



Aim of the talk: ① Identify a measure $\mathbb{P}_L$ likely to yield such a transition

② Rigorously prove this behavior

Physical relevance?

Assumptions
- $\inf_{n \neq m} d(I_n, I_m) > 0$: positive minimal distance
- any shape, polydisperse etc (crucial input: $\mathbb{P}_L$)

E. Guazzelli's lecture

1941 Burgers: problem!

1972 Batchelor $\bar{V}$ well-defined

1985 Caflisch-Luke $\mathrm{Var}[V_L] \sim L$

1991 Koch-Shaqfeh $\mathrm{Var}[V_L] \sim 1$ with long-range order

1997 Segré-Herzhaft-Chaikin

2017 Goldfriend-Diamant-Witten

...

Based on linear analysis and various stochastic cancellations

Aim of the talk

Show why the linear analysis yields a good description of the nonlinear long-range multibody interactions, even for hyperuniform statistics

Outline of the talk

(I) Equations & difficulties

(II) Linear analysis : CL versus screening

a matter of stochastic cancellations

(III) (Hyperuniform) functional calculus

CLT scaling, and more

(IV) nonlinear analysis : a matter of locality

(V) Main results and main steps of the proof

+ (hyperuniform) variance inequality

+ sensitivity calculus and duality

+ annealed Calderón - Zygmund estimates
for the "colloidal Stokes equation"

(I) Equations and difficulties

(I) Equations & difficulties

$$\begin{cases} -\Delta \varphi_L + \nabla \Pi_L = -\alpha_L e & \mathcal{O}_L \setminus I_L \\ \operatorname{div} \varphi_L = 0 & \mathcal{O}_L \setminus I_L \\ D(\varphi_L) = 0 & I_L = \cup I_n \\ e|I_n| + \int_{\partial I_n} \sigma(\varphi_L, \Pi_L)\nu = 0 \quad (1) & \forall n \\ \int_{\partial I_n} \Theta \nu \cdot \sigma(\varphi_L, \Pi_L)\nu = 0 \quad (2) & \forall n, \forall \Theta \in \mathbb{M}^{skew} \end{cases}$$

$\varphi_L(x) = \Theta_n(x - x_n) + V_n$

φ_L : $\begin{cases} \text{velocity of the fluid} & \mathcal{O}_L \setminus I_L \\ \text{velocity of the particles } I_n & (\text{rigid body} : D(\varphi_L) = 0) \end{cases}$

symmetrized gradient

Π_L : pressure (fluid is incompressible $\operatorname{div} \varphi_L = 0$)

$-\alpha_L e$: backflow (pressure in experiments), $\sigma = 2D(\varphi_L) - \Pi_L \, Id$

(cf. talk by E. Guazzelli) stress tensor

(1) : force equilibrium on particles (gravity & fluid)

(2) : momentum equilibrium on particles (fluid)

(I) Equations & difficulties

Well-posedness is standard (with $\alpha_L = \frac{\lambda_L}{1-\lambda_L}$, $\lambda_L = L^{-d}\sum_n |I_n|$)

Reformulation of the equation in Q_L:

$$-\Delta \varphi_L + \nabla(\Pi_L \mathbb{1}_{Q_L \setminus I_L}) = -\alpha_L e \mathbb{1}_{Q_L \setminus I_L} - \sum_n \delta_{\partial I_n} \sigma(\varphi_L, \Pi_L)\nu$$

Test with φ_L and use BC + rigid body motion

$$\int_{Q_L} |\nabla \varphi_L|^2 = -\alpha_L e \cdot \int_{Q_L \setminus I_L} \varphi_L - \sum_n \int_{\partial I_n} \varphi_L \cdot \sigma(\varphi_L, \Pi_L)\nu$$

$$= -\alpha_L e \int_{Q_L \setminus I_L} \varphi_L + \sum_n e \cdot \int_{I_n} \varphi_L$$

$$= (1+\alpha_L) \sum_n e \cdot \int_{I_n} \varphi_L$$

$$\Rightarrow \quad V_L := \frac{1}{N} \sum_n e \cdot \int_{I_n} \varphi_L = \frac{1}{\alpha_L} \fint_{Q_L} |\nabla \varphi_L|^2$$

By Poincaré's inequality: $|V_L| \lesssim L^2$ (pb encountered by Burgers '41)

NEED OF STOCHASTIC CANCELLATIONS

(II) Linear analysis: Caflisch-Luke versus screening

(cf. talk by G. Guazzelli

and the papers by | Batchelor '72
| Caflisch-Luke '85
| Koch-Shaqfeh '91)

(II) Linear analysis

Other formulation: $\varphi_L = \frac{1}{1-\lambda_L} P_L \varphi_L^\circ$
(cf. R. Höfer)

+ P_L orthogonal projection from $H_{div}(Q_L) \to \{\varphi \in H_{div} \mid D(\varphi) = 0 \text{ on } I_L\}$.

+ φ_L° solves $-\Delta\varphi_L^\circ + \nabla\pi_L^\circ = (\mathbb{1}_{I_L} - \lambda_L) e, \ \mathrm{div}\, \varphi_L^\circ = 0 \quad Q_L$.

$\longrightarrow$ equation for φ_L° is linear wrt $\mathcal{P}_L$

$\longrightarrow$ P_L: multibody nonlinear hydrodyn. interactions

$$\int_{Q_L} |\nabla \varphi_L|^2 \leq \int_{Q_L} |\nabla \varphi_L^\circ|^2$$

By linearity, $\varphi_L^\circ(x) = \sum_n U_L(x - x_n)$

where $-\Delta U_L + \nabla P_L = (\mathbb{1}_B - L^{-d}|B|) e \qquad Q_L$

is the (locally averaged) periodic Stokeslet.

Hence, $\lambda_L |e| \bar{V}_L^\circ = \mathbb{E}_L\left[|\nabla\varphi_L^\circ|^2\right] = \mathbb{E}_L\left[\left|\sum_n \nabla U_L(x_n)\right|^2\right] = \left|\mathrm{var}_L\left[\sum_n \nabla U_L(x_n)\right]\right|$

$(\sigma_L^\circ)^2 = |\mathrm{var}_L[\varphi_L^\circ]| = \left|\mathrm{var}_L\left[\sum_n U_L(x_n)\right]\right|$

(II) Linear analysis : control of variances

To have a hint on what to expect,

look at variance of number of points in a box Q_L:

→ Poisson point process $\mathcal{P} = \{x_n\}_n$, intensity λ

$$\mathrm{Var}\left[L^{-d}\#(\mathcal{P}\cap Q_L)\right] = \lambda^2 L^{-d}$$

↑ CLT scaling

→ Hyperuniform point process $\mathcal{P} = \{x_n\}_n$

$$\mathrm{Var}\left[L^{-d}\#(\mathcal{P}\cap Q_L)\right] \sim L^{-d-2}$$

↑ better than CLT

⟶ suppression of number fluctuations

In this section : alternative formulation in terms of $\boxed{\text{variance estimates}}$

(II) Linear analysis : Caflisch-Luke behavior

Assumption : $\mathbb{P}_L$ is <u>weakly correlated</u> in the sense that

$$\forall \text{ function } \zeta, \quad \mathrm{var}_L\Big[\sum_n \zeta(x_n)\Big] \lesssim \rho_L^2 \int_{Q_L} |\zeta|^2$$

CLT scaling

$\mathbb{E}[L^{-d} \#\{x_n\}]$ (intensity of point process)

Let $\varphi : Q \to \mathbb{R}_+$ and set $\varphi_L := L^{-d}\varphi(\cdot/L)$
with $\int_Q \varphi = 1$, $\int_Q \varphi^2 \lesssim 1$ so that $\int_{Q_L} \varphi_L = 1$, $\int_{Q_L} \varphi_L^2 \lesssim L^{-d}$

Then, since $|U_L(x)| \sim |e|(1+|x|)^{2-d}$, $|\nabla U_L(x)| \sim |e|(1+|x|)^{1-d}$

$$\lambda_L |e| \bar{V}_L^\circ \lesssim \rho_L^2 \int_{Q_L} |\nabla U_L|^2 \lesssim \rho_L^2 |e|^2 \begin{cases} L & d=1 \\ \log L & d=2 \\ 1 & d>2 \end{cases}$$

Batchelor

$$(\sigma_L^\circ)^2 \lesssim \rho_L^2 \int_{Q_L} |U_L|^2 \lesssim \rho_L^2 |e|^2 \begin{cases} L^3 & d=1 \\ L^2 & d=2 \\ L & d=3 \\ \log L & d=4 \\ 1 & d>4 \end{cases}$$

Caflisch-Luke

(II) Linear analysis: additional screening

Assumption: $\mathbb{P}_L$ is weakly correlated and hyperuniform in the sense that $\forall$ function ς, $\mathrm{var}_L\left[\sum_n \varsigma(x_n)\right] \lesssim \rho_L^2 \int_{Q_L} |\nabla \varsigma|^2$

$\mathbb{E}[L^{-d} \# \{x_n\}]$ (intensity of point process)

better than CLT scaling

$$\boxed{\begin{array}{l}\text{Let } \varsigma : Q \to \mathbb{R}_+ \quad \text{and set } \varsigma_L := L^{-d} \varsigma(\cdot / L) \\ \text{with } \int_Q \varsigma = 1,\ \int_Q \varsigma^2 \lesssim 1 \quad \text{so that } \int_{Q_L} \varsigma_L = 1,\ \int_{Q_L} |\nabla \varsigma_L|^2 \lesssim L^{-d-2}\end{array}}$$

Then, since $|\nabla U_L(x)| \sim |e|(1+|x|)^{1-d}$, $|\nabla^2 U_L(x)| \sim |e|(1+|x|)^{-d}$

$$\lambda_L |e| \bar{V}_L^0 \lesssim \rho_L^2 \int_{Q_L} |\nabla^2 U_L|^2 \lesssim \rho_L^2 |e|^2 \quad (d \geq 1)$$

$$(\sigma_L^0)^2 \lesssim \rho_L^2 \int_{Q_L} |\nabla U_L|^2 \lesssim \rho_L^2 |e|^2 \left| \begin{array}{ll} L & d=1 \\ \log L & d=2 \\ 1 & d>2 \end{array} \right.$$

screening

Koch & Shaqfeh

(II) Linear analysis : conclusion

(1) $\bar{V}_L$ well defined : $\left| \begin{array}{l} d>2 \\ d\geq 1 \end{array}\right.$ for "CLT scaling" / for "hyperuniform scaling"

P_L : $\int_{Q} |D\varphi_L|^2 \leq \int_{Q_L} |D\varphi_L^\circ|^2$

(2) σ_L : no rigorous bound yet (P_L in the way)

σ_L° gives a hint : $\left| \begin{array}{l} \text{Caflisch-Luke for "CLT scaling"} \\ \text{screening for "hyperuniform scaling"} \end{array}\right.$

(3) A matter of stochastic cancellations

→ when do we have $\mathrm{var}_L\left[\sum_n g(x_n)\right] \overset{?}{\lesssim} \rho_L^2 \begin{cases} \int_{Q_L} |g|^2 \\ \int_{Q_L} |\nabla g|^2 \end{cases}$

→ where does L_* enter the picture (transition)?

→ P_L is nonlinear : it would be convenient to have "nonlinear versions" of these inequalities

→ where would these stochastic cancellations come from physically ?

(III) Standard and hyperuniform functional calculus

(III) Standard functional calculus

Standard functional calculus :

$X = X(\mathcal{P})$ with $\mathcal{P} = \{x_n\}_{n\in\mathbb{N}}$ Poisson point process

then $$\operatorname{var}[X] \lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{osc}_{B(x)} X\right)^2\right] dx$$

with $$\partial^{osc}_{D} X = \underset{\substack{\mathcal{P}' = \mathcal{P} \\ \text{on } \mathbb{R}^d \setminus D}}{\operatorname{ess\,sup}} X(\mathcal{P}') - \underset{\substack{\mathcal{P}'' = \mathcal{P} \\ \text{on } \mathbb{R}^d \setminus D}}{\operatorname{ess\,inf}} X(\mathcal{P}'')$$

For a hardcore Poisson process, multiscale version:

$$\operatorname{var}[X] \lesssim \int_1^{+\infty} \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{osc}_{B_\ell(x)} X\right)^2\right] dx \, e^{-\ell} d\ell$$

Example: $X_L(\mathcal{P}) = \sum_n g_L(x_n)$

$$\operatorname{var}[X_L] \lesssim \int_1^\infty \int_{\mathbb{R}^d} \mathbb{E}\Big[\sup_{\substack{\mathcal{P}' = \mathcal{P} \\ \text{on } \mathbb{R}^d \setminus B_\ell(x)}} \underbrace{\Big(\sum_n g_L(x_n') \mathbb{1}_{x_n' \in B_\ell(x)}\Big)^2}_{\lesssim \int_{Q_L \cap B_\ell(x)} |g_L|}\Big] dx \, e^{-\ell} d\ell$$

$$\lesssim \int_1^\infty \int_{\mathbb{R}^d} |Q_L \cap B_\ell(x)| \int_{B_\ell(x) \cap Q_L} |g_L|^2 dx \, e^{-\ell} d\ell \qquad \lesssim L^{-d} \int_{Q_1} |g|^2$$

$\left[\ell = 1 : \int_{Q_L} |g_L|^2 , \quad \ell \geq 1 : \ell^{2d} e^{-\ell} \int_{Q_L} |g_L|^2\right]$

(III) Hyperuniform functional calculus ($L_\varphi = 1$)

Hyperuniform functional calculus :

Elementary example : $\mathcal{P} = \{x + u_x, x \in \mathbb{Z}^d\} \overset{\text{notation}}{=} \{x_n\}_n$

with $\{u_z\}_z$ iid, uniformly distributed in $(-1/3, 1/3)^d$

$\mathbb{Z}^d$ $\qquad$ $\mathcal{P}$ (random displacement model)

Efron-Stein inequality : $\mathrm{var}[X] \leq \sum_y \mathbb{E}\left[(X(P) - X(P_y))^2\right]$,

where $P_y = \{x + u_{x,y}, x \in \mathbb{Z}^d\}$ with $\begin{cases} u_{x,y} = u_x & \text{for } x \neq y \\ u_{y,y} & \text{independent of } \{u_x\}_x \end{cases}$

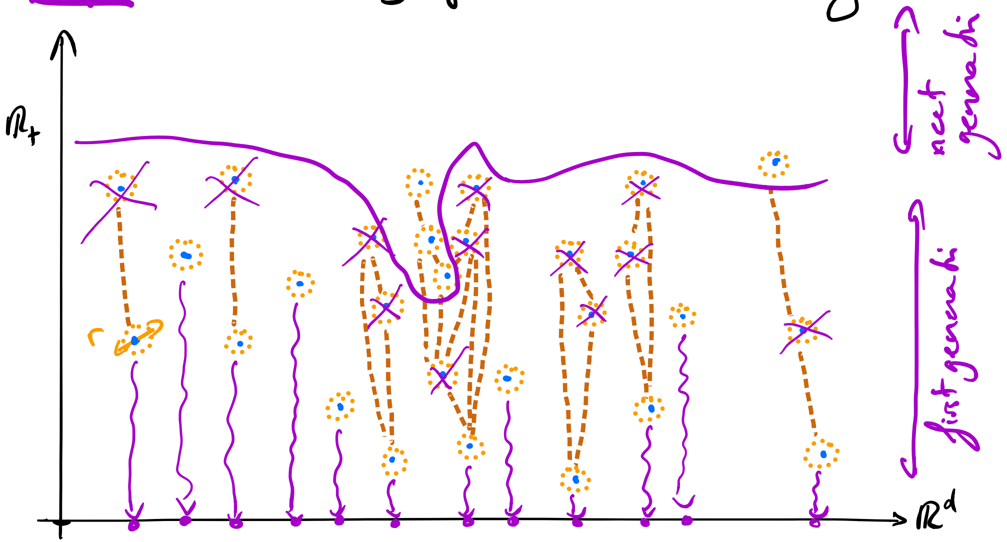
Example : $X_L(P) = \sum_n \varphi_L(x_n) = \sum_{x \in \mathbb{Z}^d} \varphi_L(x + u_x)$

$\mathrm{var}[X_L] \leq \sum_y \mathbb{E}\left[(\varphi_L(y + u_y) - \varphi_L(y + u_{yy}))^2\right]$

$\leq \sum_y \mathbb{E}\left[\int_{Q(y)} |\nabla \varphi_L|^2\right] = \int_{\mathbb{R}^d} |\nabla \varphi_L|^2 \leq L^{-d-2} \int_Q |\nabla \varphi|^2$

(III) A more sophisticated hyperuniform model ($L_\star \gg 1$)

Step 0: Penrose graphical construction $\mathcal{G}$



$\mathcal{P}_r^t(\mathbb{R}^d)$
r hardcore

$= \mathcal{G}(\text{Poisson}_{\mathbb{R}^d \times [0,t]})$

$\mathcal{P}_r^\infty(\mathbb{R}^d)$

$= \mathcal{G}(\text{Poisson}_{\mathbb{R}^d \times \mathbb{R}_+})$

Random parking measure (packing)

Step 1 : $\mathcal{P}_\star := \mathcal{P}^\infty_{L_\star}(\mathbb{R}^d)$ for hardcore radius $\boxed{L_\star \gg 1}$

$\mathcal{V}_\star$:= Voronoi tessellation of $\mathbb{R}^d = \{C_n, n \in \mathbb{N}\}$

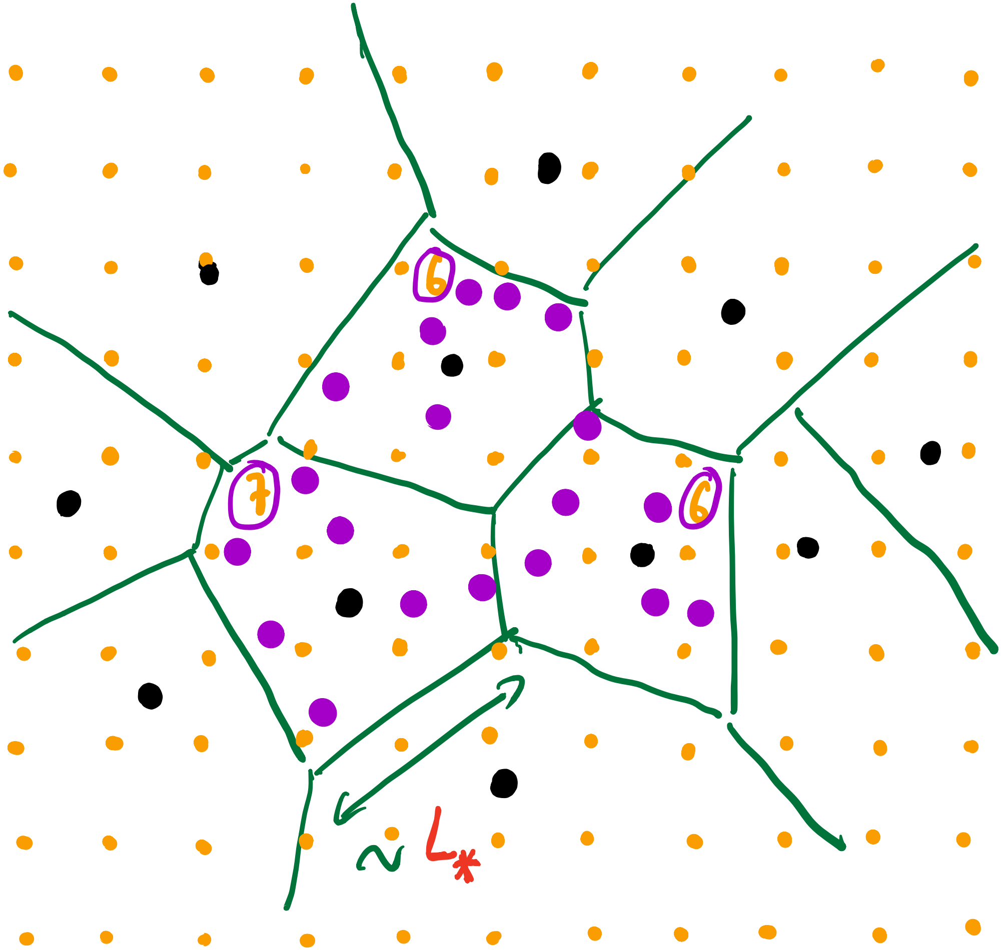
Property : $\exists\, \gamma$ s.t. $B(x_n, L_\star) \subset C_n$, $|C_n| \leq \gamma L_\star^d$

Step 2 : $\mathcal{P}_1 = \{(x, t_x) \mid x \in \mathcal{P}_1^\infty(\mathbb{R}^d),\ t_x \text{ its arrival time}\}$

Set $N_n := \mathrm{card}(C_n \cap \mathbb{Z}^d)$, and define $\mathcal{P}$ via

$\mathcal{P}|_{C_n} := \{N_n \text{ points of } \mathcal{P}_1 \cap (C_n \times \mathbb{R}_+) \text{ that arrived first}\}$.

(III) A more sophisticated hyperuniform model



$\mathbb{Z}^d$

Voronoi $\mathcal{P}_*$

$\mathcal{P}$

If X only depends on $\mathcal{P}_{|Q_L}$ for $L \le L_*$, then usual spectral gap:

$$\mathrm{Var}[X] \le \int_0^L \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{osc}_{B_\ell(x)} X\right)^2\right] dx\, e^{-\ell} d\ell$$

(III) A more sophisticated hyperuniform model

Multiscale hyperuniform variance inequality: $X = \sum_n \varphi(x_n; P)$

$$\mathrm{Var}[X] \lesssim \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\partial^{\mathrm{mov}}_{B_\ell(x)} X\right)^2\right] dx$$

where $$\partial^{\mathrm{mov}}_{B_\ell(x)} = \underset{\substack{P' = P \text{ on } \mathbb{R}^d \setminus B_\ell(x) \\ |P' \cap B_\ell(x)| = |P \cap B_\ell(x)|}}{\mathrm{esssup}} X(P') - \underset{\substack{P'' = P \text{ on } \mathbb{R}^d \setminus B_\ell(x) \\ |P'' \cap B_\ell(x)| = |P \cap B_\ell(x)|}}{\mathrm{essinf}} X(P'')$$

same number of points

Example: $X_L(P) = \sum_n \varphi_L(x_n)$, $L \gg L_\circ$

$$\mathrm{Var}[X_L] \lesssim L_\circ^{-1} \int_{L_\circ}^{\infty} \int_{\mathbb{R}^d} \mathbb{E}\Big[\sup_{\substack{x_i, x_i' \\ \in B_\ell(x)}} \Big| \sum_{i=1}^{N_\ell(x)} \underbrace{\varphi_L(x_i) - \varphi_L(x_i')}_{\leq |D\varphi_L(x_i^\circ)| \underbrace{|x_i - x_i'|}_{\leq \ell}} \Big|^2\Big] dx \, e^{-\ell/L_\circ} d\ell$$

$$\lesssim L_\circ^{-1} \int_{L_\circ}^{\infty} \int_{\mathbb{R}^d} \ell^2 \, \mathbb{1}_{B_\ell(x) \cap Q_L \neq \emptyset} \, \ell^{2d} \sup |D\varphi_L|^2 \, dx \, e^{-\ell/L_\circ} d\ell$$

$$\lesssim L_\circ^{1+2d} \int_1^{\infty} \mathbb{1}_{B_\ell(x) \cap Q_{L/L_\circ} \neq \emptyset} \sup |D\varphi_L|^2 \, dx \, e^{-\ell} d\ell$$

$$\lesssim L_*^{1+d} L^d \sup |D\varphi_L|^2 \lesssim L_*^{1+d} L^{-d-2}$$

(III) More about hyperuniformity

(1) Hyperuniformity yields improvement on CLT for (only?) linear functionals of the random field

Counter-example: Random displacement model $\mathcal{P} = \{x + u_x,\ x \in \mathbb{Z}^d\}$

$$X_L = L^{-d} \sum_{x \in \mathbb{Z}^d \cap Q_L} \underbrace{d(\mathcal{P}, x)}_{\text{nonlinear}} = L^{-d} \sum_{x \in \mathbb{Z}^d \cap Q_L} \underbrace{|u_x|}_{\text{iid rv}}$$

We have $\mathrm{Var}[X_L] \sim L^{-d}$ not L^{-d-2}

(2) this section is descriptive: hyperuniform systems exist

+ why would a system be hyperuniform?

→ Talk by S. Serfaty on Coulomb gases

+ for a systematic study:

→ Talk by S. Torquato

(IV) Nonlinear analysis: a matter of locality

(IV) nonlinear analysis : a matter of locality

Summary :

- long-range order (like hyperuniformity) combined with linear analysis of sedimentation allows to see both Caflisch–Luke and long-range screening
- sedimentation is nonlinear (via the projection P_2)
- hyperuniformity "only" improves linear functionals

Question : why would the linear analysis of the system be representative of the nonlinear system ?

Remark : In stochastic homogenization of linear elliptic equations in divergence and non divergence form, assuming the coefficients are hyperuniform does not improve the CLT scaling !

(IV) nonlinear analysis : an instructive example

Let a be random coefficients and consider

$$- \operatorname{div}(a \nabla \varphi) = (a - \mathbb{E}[a]) : E$$

This is a hybrid between the corrector equations in homogenization of linear elliptic operators :

- in divergence-form : $- \operatorname{div}(a \nabla \varphi) = \operatorname{div}(ae)$
- in non-divergence form : $-a : \nabla^2 \varphi = (a - \gamma) : E$

Functional calculus : replace a by $a + t\,\delta a$ and differentiate wrt t :

$$- \operatorname{div}(a \nabla \delta\varphi) = \underbrace{\delta a : E}_{\text{linear response}} + \underbrace{\operatorname{div}(\delta a \nabla \varphi)}_{\substack{\text{nonlinear response} \\ (\text{depends on } \varphi)}}$$

<u>Questions</u> : order of magnitude of the contributions ?

Effect of hyperuniformity ?

(IV) nonlinear analysis : an instructive example

(1) Locality : Set G : Green's function, sol of $-\mathrm{div}(a\nabla G(x,y)) = \delta(x-y)$

Then : $\delta\varphi(x) = \int_{\mathbb{R}^d} G(x,y)\,\delta a(y) : E\,dy - \int_{\mathbb{R}^d} \nabla_2 G(x,y)\cdot\delta a\,\nabla\varphi(y)\,dy$

Locality of each term : assume $\operatorname{supp}\delta a = B(z)$ (and nothing else)

$$|\delta\varphi(x)| \sim \underbrace{G(x,z)}_{|x-z|^{2-d}} + \underbrace{|\nabla_2 G(x,z)|\,|\nabla\varphi(z)|}_{|x-z|^{1-d}}$$

$$|x-z|^{2-d} \gg |x-z|^{1-d}$$

Caflisch-Luke

Screening

linear response is less local than nonlinear response

(2) Hyperuniformity : In this case, δa must have vanishing average :

$$\int_{\mathbb{R}^d} G(x,y)\,\delta a(y) : E\,dy = \int_{B(z)} \underbrace{\Big(G(x,y) - \fint_{B(z)} G(x,\cdot)\Big)}_{\sim \nabla_2 G(x,y)\,\sim\,|x-z|^{1-d}}\delta a(y) : E\,dy$$

linear response gets more local with hyperuniformity

For hyperuniform coefficients, the linear and the nonlinear responses have the same order of magnitude for this model

(V) Main results

(V) Main results: statement

Theorem (Duerinckx - Gloria '20)

Let $\mathbb{P}_L$ be the L-periodization in law of the hyperuniform model of parameter $L_\# \gg 1$. Then

(I) For all $\boxed{1 \leq L \ll L_\#}$, we have

$$\frac{\bar{V}_L}{\rho_L |e|} \lesssim \begin{cases} 1 & : d > 2 \\ \log L & : d = 2 \\ L & : d = 1 \end{cases}, \qquad \sigma_L \lesssim |e| \begin{cases} 1 & : d > 4 \\ (\log L)^{1/2} & : d = 4 \\ L^{1/2} & : d = 3 \\ L & : d = 2 \\ L^{3/2} & : d = 1 \end{cases}$$

Batchelor

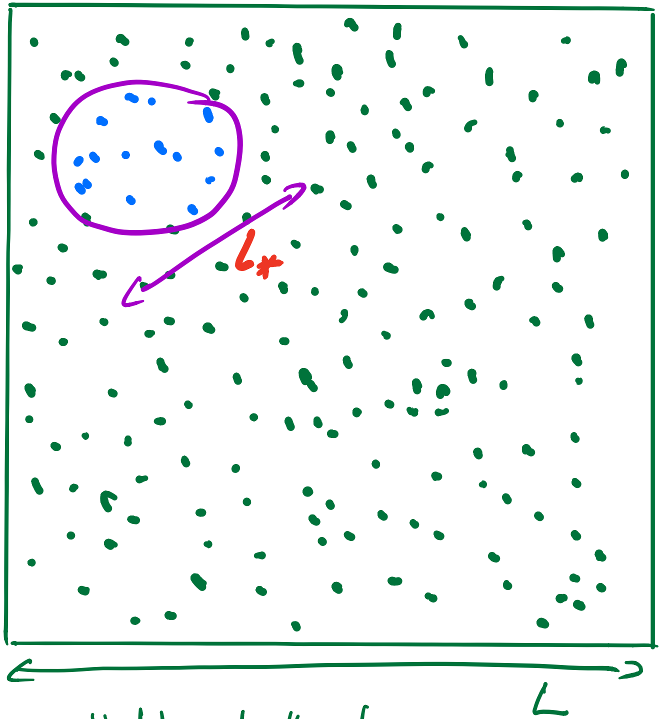
Caflisch-Luke

(II) For all $\boxed{L \gtrsim L_\#}$, we have

$$\frac{\bar{V}_L}{\rho_L |e|} \lesssim_{L_\#} 1 : d \geq 1, \qquad \sigma_L \lesssim_{L_\#} |e| \begin{cases} 1 & : d > 2 \\ (\log L)^{1/2} & : d = 2 \\ L^{1/2} & : d = 1 \end{cases}$$

Screening

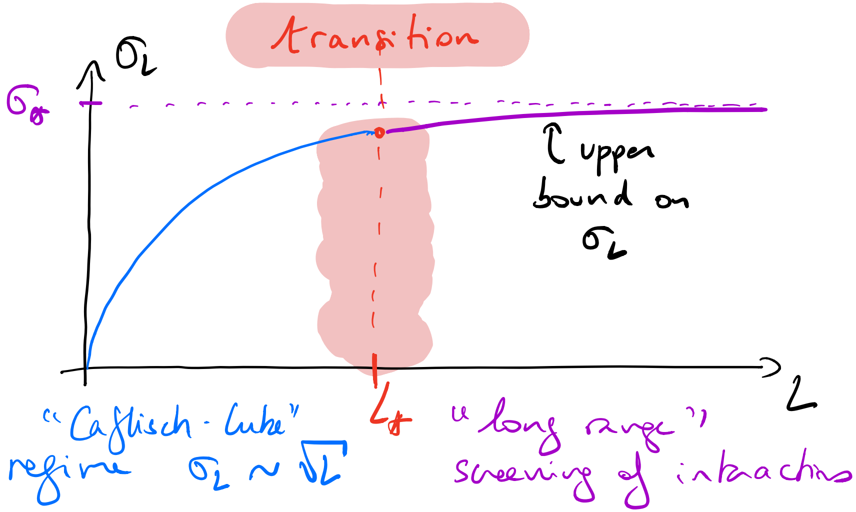
(V) Main results: interpretation $d = 3$



"Hinch"-type model

like hardcore Poisson at scale $\leq L_*$

suppression of number fluctuations at scale $\gtrsim L_*$



Typical graph $L \mapsto \sigma_L = \sqrt{\mathrm{Var}[V_L]}$ recovered as upper bound

(also lower bound for linear model)

(V) Main results : effective velocity

Starting point is identity

insert $\mathrm{Id} = -\Delta(-\Delta)^{-1}$

$$\int_{Q_L} |\nabla \varphi_L|^2 = (1+\alpha_L) \int_{Q_L} \varphi_L \cdot \Big(\sum_n e(\mathbb{1}_{I_n} - L^{-d}|B|)\Big)$$

$$\overset{(ipp)}{=} (1+\alpha_L) \int_{Q_L} \nabla\varphi_L : \Big(e \otimes \sum_n \nabla(-\Delta)^{-1}(\mathbb{1}_{I_n} - L^{-d}|B|)\Big)$$

$$\overset{(C.S.)}{\lesssim} (1+\alpha_L)^2 |e|^2 \int_{Q_L} \Big| \sum_n \underbrace{\nabla(-\Delta)^{-1}(\mathbb{1}_{I_n} - L^{-d}|B|)}_{\text{linear analysis !}} \Big|^2$$

$\rightsquigarrow$ yields the claimed scalings depending on $L \ll \ell_0$, $L \gtrsim \ell_0$

(even without functional-analytic version of ergodicity)

Subtle part : fluctuations

(V) Main results : bounds on the variance

Based on the combination of two propositions :

Proposition 1 For all $L \geq 1$, $g \in C^\infty_\#(Q_L)^{d\times d}$, all $1 \leq R \leq L$, and $q \geq 1$ and $1 \ll p < \infty$:

- If $1 \leq L \ll L_*$ $(d > 2)$ Caffisch-Luke

$$\left\| \int_{Q_L} g : D\varphi_L \right\|^2_{L^{2p}(\Omega)} \lesssim_p \|g\|^2_{L^{\frac{2d}{d+2}}(Q_L)} + \|\langle D\rangle^{1/2} g\|^2_{L^2(Q_L)} \left\| \left(1 + \int_{Q_R} |D\varphi_L|^{2q}\right)^{1/q} \right\|_{L^p(\Omega)}$$

Screening

stochastic integrability

- If $L \gtrsim L_*$,

$$\left\| \int_{Q_L} g : D\varphi_L \right\|^2_{L^{2p}(\Omega)} \lesssim_p \|\langle D\rangle^{1/2} g\|^2_{L^2(Q_L)} \left\| \left(1 + \int_{Q_R} |D\varphi_L|^{2q}\right)^{1/q} \right\|_{L^p(\Omega)}$$

(V) Main results : bounds on the variance

Proposition 2 Let $\chi \in C_c^\infty(B)$, $\int_B \chi = 1$, set $\chi_r : x \mapsto r^{-d}\chi(x/r)$.

There exists $\eta_0 > 0$ such that for all $1 \le r \ll_\chi R \le L$, $1 \le q \le 1+\eta_0$ and $p \ge 1$,

$$\left\| \left(1 + \int_{Q_R} |D\varphi_L|^{2q}\right)^{1/q} \right\|_{L^p(\Omega)} \lesssim_\chi R^2 + \left\| \int_{Q_L} \chi_r D\varphi_L \right\|^2_{L^p(\Omega)}$$

Proof of thm: (1) buckling on $\bullet_R$:

$$\bullet_R \lesssim R^2 + \bullet_r$$

$$\lesssim R^2 + \bullet_r + \bullet_r \bullet_1$$

reverse Jensen + stationarity

$$\lesssim R^2 + r^{2-d} + r^{-d} R^{d(1-1/p)} \bullet_R$$

Take $r = R/2$, choose $R \gg 1$, and get $\bullet_R \lesssim 1$.

(2) $\|\phi_L(x) - \phi_L(0)\|_{L^{2p}(\Omega)} \le \|D\varphi\|_{L^{2p}(\Omega)} + \|\fint_{B(x)}\varphi_L - \fint_{B(0)}\varphi_L\|_{L^{2p}(\Omega)}$

set $-\Delta h_L = |B|^{-1} 1_B - L^{-d}$ in Q_L

$$= \underbrace{\int_{Q_L} D\varphi_L \cdot Dh_L(x+\cdot)}_{\bullet_L}$$

(IV) Main results: bounds on the variance

Proposition 2 solely relies on Caccioppoli's inequality

Proposition 1 relies on two ingredients:

- sensitivity calculus and use of hyperuniformity
- annealed Calderón-Zygmund estimates for colloidal suspensions

Argument 1: Let $\mathcal{P}_L$ and $\mathcal{P}_L'$ differ on $B(x)$

then:
$$\int_{Q_L \setminus \mathcal{I}_L} g : (D\varphi_L - D\varphi_L') = -(\alpha_L - \alpha_L')\, e \cdot \int_{\mathcal{I}_L} v_L$$
$$+ (\alpha_L' + 1)\Big(\sum_{x_n \in B(x)} e \cdot \int_{I_n} v_L - \sum_{x_n' \in B(x)} e \cdot \int_{I_n'} v_L \Big)$$
$$- \sum_{x_n' \in B(x)} \int_{\partial I_n'} \Big(v_L - \fint_{I_n'} v_L\Big) \cdot \sigma(\phi_L', \Pi_L' - c_L')\,\nu$$

[duality argument]

where
$$\begin{cases} -\Delta v_L + \nabla P_L = \nabla \cdot g \quad \& \quad \operatorname{div} v_L = 0 \text{ in } Q_L \setminus \mathcal{I}_L \\ D(v_L) = 0 \text{ in } \mathcal{I}_L \\ \int_{\partial I_n} (g + \sigma(v_L, P_L))\nu = 0, \quad \int_{\partial I_n} \Theta \nu \cdot (g + \sigma(v_L, P_L))\nu = 0 \end{cases}$$

(V) Main results : bounds on the variance

$$\int_{Q_L\setminus I_L} g : (D\varphi_L - D\varphi_L') = -(\alpha_L-\alpha_L')\, e\cdot \int_{I_L} v_L$$
$$+ (\alpha_L'+1)\Big(\sum_{x_n\in B(x)} e\cdot \int_{I_n} v_L - \sum_{x_n'\in B(x)} e\cdot \int_{I_n'} v_L\Big)$$
$$- \sum_{x_n'\in B(x)} \int_{\partial I_n'} \big(v_L - \textstyle\int_{I_n'} v_L\big)\cdot \sigma(\phi_L', \Pi_L' - c_i')\,\nu$$

Without hyperuniformity : CL

$$\Big|\int_{Q_L\setminus I_L} g : D\phi_L - \int_{Q_L\setminus I_L} g : D\phi_L'\Big| \lesssim L^{-d}\Big|e\cdot\int_{I_L} v_L\Big| + \boxed{\int_{B(x)} |v_L|}$$
$$+ \Big(\int_{B_3(x)} |Dv_L|^2 + |\langle D\rangle^{1/2} g|^2\Big)^{1/2}\Big(1 + \int_{B_3(y)} |D\phi_L'|^2\Big)^{1/2}$$

With hyperuniformity : $\mathcal{P}$ and $\mathcal{P}'$ have same number of points

$\alpha_L = \alpha_L'$ and $\mathrm{Card}\{n \mid x_n \in B_\ell(x)\} = \mathrm{Card}\{n \mid x_n' \in B_\ell(x)\}$

$\longrightarrow$ gives a gradient

$$\Big|\int_{Q_L\setminus I_L} g : D\phi_L - \int_{Q_L\setminus I_L} g : D\phi_L'\Big|$$
$$\lesssim \underbrace{\boxed{\int_{B_2(x)} |Dv_L|}}_{\text{screening}} + \Big(\int_{B_3(y)} |Dv_L|^2 + |\langle D\rangle^{1/2} g|^2\Big)^{1/2}\Big(1+\int_{B_3(y)} |D\phi_L'|^2\Big)^{1/2}$$

(V) Main results : bounds on the variance

Argument 2 : related to two terms :

- $\| \int_{Q_L} (|\nabla v_L|^2 + |\langle \nabla \rangle^{1/2} g|^2) |\nabla \phi_L|^2 \|_{L^p(\Omega)}$
- $\| \int_{Q_L} |v_L|^2 \|_{L^p(\Omega)}$

stochastic improvement on reflection method : any q !

Annealed CZ theory for equation for v_L

$$\begin{cases} -\Delta v_L + \nabla P_L = \operatorname{div} g \quad \& \quad \operatorname{div} v_L = 0 \text{ in } Q_L \setminus I_L \\ D(v_L) = 0 \text{ in } I_L \\ \int_{\partial I_n} (g + \sigma(v_L, P_L))\nu = 0, \int_{\partial I_n} \Theta\nu \cdot (g + \sigma(v_L, P_L))\nu = 0 \end{cases}$$

screening

- perturbative : $|q-2|, |p-2| \leq \eta_0$ $\quad \| \nabla v_L \|_{L^q(Q_L, L^p(\Omega))} \lesssim \| g \|_{L^q(Q_L, L^p(\Omega))}$

(use also duality in probability)

CL

- non perturbative : $\forall\, p, q, \forall \eta > 0$ $\quad \| \nabla v_L \|_{L^q(Q_L, L^p(\Omega))} \lesssim \| g \|_{L^q(Q_L, L^{p+\eta}(\Omega))}$

$$\left\| \int_{Q_L} |v_L|^2 \right\|_{L^p(\Omega)} \overset{d>2}{\lesssim} \left\| \left(\int_{Q_L} |\nabla v_L|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{d}} \right\|_{L^p(\Omega)} = \| \nabla v_L \|^2_{L^{\frac{2d}{d+2}}(Q_L, L^{2p}(\Omega))}$$

$$\lesssim \| g \|^2_{L^{\frac{2d}{d+2}}(Q_L, L^{2p+\eta}(\Omega))} = \| g \|^2_{L^{2d/d+2}(Q_L)} .$$

A remark on annealed CZ estimates

$$\begin{cases} -\Delta v_L + \nabla P_L = \operatorname{div} g \quad \& \quad \operatorname{div} v_L = 0 \text{ in } Q_L \setminus I_L \\ \qquad\qquad\qquad D(v_L) = 0 \text{ in } I_L \qquad\qquad (\ast) \\ \int_{\partial I_n} (g + \sigma(v_L, P_L))\nu = 0, \ \int_{\partial I_n} \Theta\nu \cdot (g + \sigma(v_L, P_L))\nu = 0 \end{cases}$$

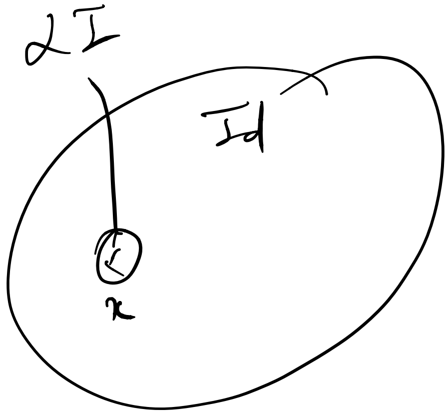
this is the colloidal case, that is: no divergence issue like in sedimentation [cf. talk tomorrow]

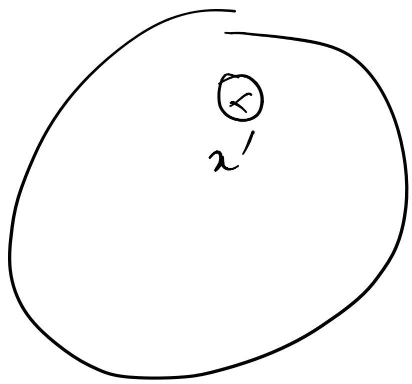
Indeed, energy estimate directly gives

$$\int_{Q_L} |\nabla v_L|^2 \lesssim \int_{Q_L} |g|^2$$

Approach to annealed CZ estimates as in stochastic homogenization of divergence-form elliptic equations:

1) large-scale $C^{1,\alpha}$ estimates [difficulty: pressure]
2) Lipschitz $\Longrightarrow$ large scale quenched (weighted) CZ
3) Quenched CZ $\Longrightarrow$ annealed CZ





$$A' = A + \delta A$$

$$\delta A = (1-\alpha)\, \mathrm{Id}\, \mathbb{1}_{B(x)} + (\alpha - 1)\, \mathrm{Id}\, \mathbb{1}_{B(x')}$$