TWO NEW CONDITIONS SUPPORTING
THE FIRST-ORDER APPROACH TO
MULTI-SIGNAL PRINCIPAL-AGENT PROBLEMS

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ABSTRACT

This paper presents simple new multi-signal generalizations of the two classic methods used to justify the first-order approach to moral hazard principal-agent problems. The paper first discusses some problems with previous generalizations. It then uses a state-space formulation to obtain a new multi-signal generalization of the Jewitt (1988) conditions. Next, using the Mirrlees formulation, it obtains new multi-signal generalizations of the Rogerson (1985)/Sinclair-Desgagné (1994) conditions. Vector calculus methods are used to translate the generalized Rogerson/Sinclair-Desgagné conditions into easy-to-check local conditions on the CDF’s of the underlying distributions.

KEYWORDS

Principal-Agent Model, Moral Hazard, First-Order Approach, Multiple Signals

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1. INTRODUCTION

Moral hazard principal-agent problems have always presented surprising technical challenges. In particular, a great deal of attention has focused on determining conditions under which the principal can predict the agent’s behavior using the agent’s first order conditions alone. This question was first raised by Mirrlees (1999, originally circulated in 1975), and Guesnerie and Laffont (1978), and was further discussed in Grossman and Hart (1983). Rogerson (1985) gave a rigorous set of sufficient conditions for the first-order approach in the one-signal case, based on the “Convexity of the Distribution Function Condition” (CDFC). Jewitt (1988) gave a different set of sufficient conditions, not using the CDFC, also for the one-signal case.

In addition, Jewitt (1988) gave two sets of sufficient conditions for the multi-signal case, one of which also avoided the CDFC. However, both sets of Jewitt’s multi-signal conditions assumed that signals were independent of one another.

Six years later, Sinclair-Desgagné (1994) developed a generalization of the CDFC, the GCDFC, for the multi-signal case. This allowed him to avoid Jewitt’s independence assumption. However, the GCDFC requires the marginal distribution of all but one of the principal’s signals to be linear in the agent’s action (see Section 3 below).

In addition, the CDFC and GCDFC do not capture the standard diminishing marginal returns intuition. As Rogerson (1985) explains, “if output is determined by a stochastic production function with diminishing returns to scale in each state of nature, the implied distribution function over output will not, in general, exhibit the CDFC.” Thus, neither Rogerson’s one-signal results nor Sinclair-Desgagné’s multi-signal results build on the usual economic notion of diminishing marginal returns.

Jewitt’s (1988) conditions, by contrast, do build on the standard economic sense of diminishing returns, so his conditions are closer to ordinary economic intuition. They
are also interesting because they suggest that reasonable conditions supporting the
first-order approach may naturally tend to use the rather strong requirement that the
endogenously contracted \textit{payment schedule} be concave in output.

However, the profession seems to have treated the Jewitt conditions as overly
technical. Thus, while most principal-agent textbooks present the Rogerson CDFC,
none present the Jewitt conditions. Also, little theoretical work has built on these
conditions. In particular, no multi-signal generalizations of Jewitt’s conditions are
available, beyond those in Jewitt’s original paper, which assume independent signals.

The goal of this paper is to clarify and extend both of the above lines of attack on
the first-order approach. First we suggest that the Rogerson conditions arise most nat-
urally from the Mirrlees formulation of the agent’s problem, while the Jewitt conditions
arise very naturally from the older state-space perspective of Spence and Zeckhauser
(1971) and Ross (1973). In addition we discuss limitations of previous multi-signal
generalizations of the Jewitt and Rogerson one-signal approaches.

We next present new multi-signal extensions of each set of conditions, which are
more general than previous extensions. The multi-signal generalization of Jewitt’s
approach yields essentially Jewitt’s main set of multi-signal conditions, but without
his restrictive independence assumption. The multi-signal extension of the CDFC
approach involves a new condition, the Concave Increasing-Set Probability (CISP)
condition, which generalizes the CDFC, and is more flexible than previous extensions.
In addition, if we want to allow for a risk averse principal, we need an additional
condition, the Nondecreasing Increasing-Set Probability (NISP) condition.

The multi-signal case is of practical interest since, in most actual environments,
the principal sees a wide range of signals of an agent’s effort. The multi-signal case
is also especially interesting because it is often used to underscore the central role of
information in principal-agent problems (Holmström, 1979, Shavell, 1979).

The next section introduces the basic framework. Section 3 discusses previous results, and Section 4 presents the new multi-signal generalization of Jewitt's conditions. Sections 5 and 6 derive simple new multi-signal generalizations of the CDFC approach, for risk neutral and risk averse principals, respectively. Section 7 reduces the key NISP and CISP conditions to the existence of probability density flows having certain local properties. Section 8 shows how to construct these density flows, and so, check NISP and CISP. Section 9 briefly discusses other extensions and Section 10 concludes.

2. BASIC FRAMEWORK

This section presents the general framework for multi-signal principal-agent problems, describing both the state-space and the Mirrlees formulations of the problem. It turns out that generalizing the multi-signal Jewitt approach is easiest using a state-space formulation, while generalizing the Rogerson/Sinclair-Desgagné approach is most convenient using the Mirrlees formulation.

Suppose there is one principal and one agent. The agent chooses an effort level $a \geq 0$. This is combined with a random vector of state variables $\tilde{\vartheta}$, with PDF $g(\tilde{\vartheta})$, to generate a random vector of signals,

\[(1_{SS}) \quad \tilde{x} = x(a, \tilde{\vartheta}),\]

with $x(\cdot, \cdot)$ taking values in $\mathbb{R}^n$. Equation $(1_{SS})$ is a state-space representation of the technology.

Assume, until Section 6, that the principal is risk neutral. The agent has a von Neumann-Morgenstern utility function $u(s) - a$, with $u(\cdot)$ strictly increasing. Let $s = s(x)$ be the function, chosen by the principal, specifying her payment to the agent as
a function of the signal $x$. Let the value of output be given by the function $\pi(x)$, as in Sinclair-Desgagné (1994). Then the principal’s expected payoff is

$$V(s(\cdot), a) = \int [\pi(x(a, \vartheta)) - s(x(a, \vartheta))] g(\vartheta) d\vartheta,$$

and the agent’s expected payoff is

$$U(s(\cdot), a) = \int u(s(x(a, \vartheta))) g(\vartheta) d\vartheta - a.$$

The principal’s problem is then to choose a payment schedule, $s^*(\cdot)$, and target action, $a^*$, by the agent, to maximize ($2_{SS}$), given two constraints:

$$(IC) \quad a^* \text{ maximizes the agent’s expected payoff, } U(s^*(\cdot), a),$$

and

$$(P) \quad \text{the resulting expected payoff to the agent, } U(s^*(\cdot), a^*) \geq U_0,$$

where $U_0$ is the agent’s reservation utility. Here (IC) and (P) are the usual incentive compatibility and participation (individual rationality) constraints.

To rephrase this in terms of a Mirrlees representation, let $x \leq y$, for $x, y \in \mathbb{R}^n$ mean $x_i \leq y_i$, $i = 1, 2, ..., n$, let

$$F(x|a) = \text{Prob}(x(a, \tilde{\vartheta}) \leq x)$$

be the CDF of $\tilde{x}$ implied by ($1_{SS}$), and let $f(x|a)$ be the corresponding PDF. Assume as usual that the support of $f(x|a)$ is compact and independent of $a$, and that $f(x|a)$ is bounded between two positive constants on its support. Using the Mirrlees notation, the principal’s expected payoff becomes

$$V(s(\cdot), a) = \int [\pi(x) - s(x)] f(x|a) dx,$$
and the agent’s expected payoff becomes

\[(3_M) \quad U(s(\cdot), a) = \int u(s(x)) f(x|a) dx - a.\]

The principal’s problem is still to maximize \(V(s(\cdot), a)\) subject to (IC) and (P).

Our primary focus is (3_S) and (3_M). These are equivalent ways of representing the agent’s expected payoff in terms of the payment function \(s(\cdot)\) and the agent’s action \(a\). However, (3_S) explicitly separates out the production technology \(x(a, \vartheta)\) from the PDF \(g(\vartheta)\), while (3_M) folds them together in \(f(x|a)\). On the other hand, the state space representation contains extraneous information in the sense that many different state space representations correspond to any given Mirrlees representation (see Section 4).

The first-order approach assumes that one can replace the constraint (IC) by a “relaxed” constraint – i.e., the agent’s first order condition

\[(IC_R) \quad U_a(s^*(\cdot), a^*) = 0,\]

where subscripts denote partial derivatives. To ensure that (IC_R) implies (IC), it is sufficient for the agent’s utility, \(U(s^*(\cdot), a)\), to be a concave function of her effort \(a\). The goal of the first-order approach is to find conditions which yield this concavity.

If, in fact, we replace (IC) with (IC_R) in the principal’s problem, then the principal’s optimal contract, \(s^*(x)\), for inducing the agent to choose some action \(a^*\) at minimum cost, satisfies the principal’s usual first order condition

\[(4) \quad \frac{1}{u'(s^*(x))} = \lambda + \mu \frac{f_a(x|a^*)}{f(x|a^*)},\]

where \(\lambda\) and \(\mu\) are the Lagrange multipliers for the constraints (P) and (IC_R), respectively (see Kadan and Swinkels, 2005, for an elegant treatment). As shown by Jewitt (1988, p. 1180), if (IC_R) and (4) hold, then \(\mu \geq 0\). Jewitt’s argument also applies to the multi-signal case (Jewitt, p. 1184).
3. PREVIOUS RESULTS

As indicated above, the first-order approach looks for conditions to ensure that the agent’s utility, \( U(s^*(\cdot), a) \), is a concave function of her effort \( a \). Rogerson (1985) and Jewitt (1988) obtain conditions which ensure this concavity in the one-signal case. Jewitt also presents some multi-signal generalizations of a slightly more restrictive version of his one-signal conditions. Sinclair-Desgagné (1994) obtains a multi-signal generalization of the Rogerson conditions.

Whether we obtain the Rogerson/Sinclair-Desgagné conditions or the Jewitt conditions turns out to depend on whether we represent \( U(s^*(\cdot), a) \) using (3\( _M \)) or (3\( _{SS} \)). If we represent \( U(s^*(\cdot), a) \) using (3\( _M \)), (i.e., \( \int u(s^*(x))f(x|a)dxd-a \)), we are naturally led to impose conditions on \( f(x|a) \), since that is where \( a \) appears. This yields the approach of Rogerson (1985) and Sinclair-Desgagné (1994), as is well known.

However, if we represent \( U(s^*(\cdot), a) \) using the state-space formulation, (3\( _{SS} \)) (i.e., 
\[
\int u(s^*(x(a, \vartheta)))g(\vartheta)d\vartheta-a,
\]
then it becomes obvious that \( U(s^*(\cdot), a) \) is concave in \( a \) if \( x(a, \vartheta) \) is concave in \( a \), \( s^*(x) \) is nondecreasing concave in \( x \), and \( u(s) \) is nondecreasing concave in \( s \). This turns out to lead very naturally to the main set of Jewitt’s (1988) multi-signal conditions. In fact, Jewitt himself used both the state space and Mirrlees formulations in his derivations.

Unfortunately, both the Jewitt and the Sinclair-Desgagné multi-signal generalizations remain more restrictive than necessary. First, Jewitt’s multi-signal results assume that the signals are independent from one another. It turns out, however, that Jewitt’s proof of his main result does not actually require the signals to be independent.

Nevertheless, the Mirrlees formulation, which Jewitt uses to present his results, makes it difficult to even state his multi-signal results without assuming independence. Section 4 below therefore initially uses the state-space formulation to state and prove
our generalization of Jewitt’s main multi-signal results. We then show how to restate these results using the Mirrlees formulation. This restatement turns out to be somewhat more difficult than the generalization itself.

Sinclair-Desgagné’s generalization of the Rogerson conditions is also more restrictive than it might initially seem to be. One of Sinclair-Desgagné’s conditions, his Assumption (A8), requires the marginal distribution of all but one of the signals to be independent of the agent’s action (see below). That is, it requires all but one of the signals to be “ancillary” for the agent’s action (Lehmann, 1983, pp. 45-46). It turns out that this assumption can be avoided if the principal is risk neutral (details available upon request). However, a second key assumption, his Assumption (A9), cannot be so easily avoided. Also, Assumption (A9) turns out to imply that the marginal distribution of all but one of the signals is linear in the agent’s actions.

To obtain these results, translate Sinclair-Desgagné’s key assumptions to the continuous signal-space case of the current paper. For any index \( h = 1, 2, \ldots, n \), let \( \mathbf{x}_{-h} = (x_1, x_2, \ldots, x_{h-1}, x_{h+1}, \ldots, x_n) \), and let

\[
Q(x_0^h, \mathbf{x}_{-h}|a) = \int_{x_h^0}^{\infty} f(x_h, \mathbf{x}_{-h}|a) dx_h.
\]

Intuitively, \( Q(x_0^h, \mathbf{x}_{-h}|a) \) is the probability density of being on the upper ray, starting from \((x_h^0, \mathbf{x}_{-h})\), pointing in the direction of increasing \( x_h \). Thus, \( Q(x_0^h, \mathbf{x}_{-h}|a) \) is a generalization of the upper cumulative distribution function to the multi-signal case, where this upper cumulative focuses on a one-dimensional ray in the \( n \)-dimensional space of signals. With this notation, Assumptions (A8) and (A9) are:

**Assumption (A8) (Generalized Stochastic Dominance):** For at least one index \( h \), \( Q(x_0^h, \mathbf{x}_{-h}|a) \) is nondecreasing in \( a \) for all \( x_0^h \) and \( \mathbf{x}_{-h} \).

**Assumption (A9) (Generalized Concave (Upper) Distribution Function Condition):** For at least one \( h \), \( Q(x_0^h, \mathbf{x}_{-h}|a) \) is concave in \( a \) for all \( x_0^h \) and \( \mathbf{x}_{-h} \).
PROPOSITION 1: If Assumption (A8) holds, then $f_{x-h}(x_{-h}|a)$, the marginal density of $\tilde{x}$ on $x_{-h}$, is independent of $a$ for almost all $x_{-h}$. If instead Assumption (A9) holds, then $f_{x-h}(x_{-h}|a)$ is linear in $a$ for almost all $x_{-h}$.

PROOF: Assume (A8). Then $Q(x_{h}^{0}, x_{-h}|a)$ is nondecreasing in $a$ for all $x_{h}^{0}$ and $x_{-h}$. Thus its limit as $x_{h}^{0} \to -\infty$ is also nondecreasing in $a$ for all $x_{-h}$, so

\[ f_{x-h}(x_{-h}|a) = Q(-\infty, x_{-h}|a) \]

is nondecreasing in $a$ for all $x_{-h}$. However, $\int f_{x-h}(x_{-h}|a)dx_{-h} = 1$. Thus, $f_{x-h}(x_{-h}|a)$ is constant in $a$ for almost all $x_{-h}$. In the same way, if Assumption (A9) holds, then (6) is concave in $a$ for all $x_{-h}$ so, using $\int f_{x-h}(x_{-h}|a)dx_{-h} = 1$ again, $f_{x-h}(x_{-h}|a)$ is linear in $a$ for almost all $x_{-h}$. QED

Note that these conditions are necessary but not sufficient. In addition, even though $f_{x-h}(x_{-h}|a)$ constant in $a$ implies $f_{x-h}(x_{-h}|a)$ linear in $a$, it does not follow that (A8) implies (A9). Thus, suppose $\tilde{x}_{-h}$ is ancillary, while the conditional CDF, $F_{h}(x_{h}|a, x_{-h})$, is decreasing, but not convex, in $a$. Then $\tilde{x}$ satisfies (A8) but not (A9).

Also, ancillarity of $\tilde{x}_{-h}$ does not mean that these signals cannot be very useful. For example, when a principal uses the output of other agents as a benchmark in “yardstick competition” (Holmström, 1982, Mookherjee, 1984), this involves signals which are ancillary relative to the agent’s action, but may nevertheless be very important.

As a second example, suppose the principal uses a scale to weigh output. If the principal does not know how accurate the scale is, she may first put an object of known weight on the scale to calibrate it, before weighing the agent’s output. In this case, the calibrating measurement is ancillary, but is nevertheless clearly relevant to the optimal payment schedule. It may therefore figure prominently in an optimal contract.

Multiple calibrating signals are also possible. One may then ask, e.g., how the optimal contract changes as the calibration becomes more and more accurate.
Thus, although Assumption (A8) is restrictive, it does allow the $\tilde{x}_h$ signals to play a very important role in helping the principal to interpret $\tilde{x}_h$. Alternatively, once the principal sees $\tilde{x}_h$, the $\tilde{x}_h$ signals may then contain a great deal of additional information. Similarly, if Assumption (A9) holds, it may nevertheless be the case that the distribution of $\tilde{x}_h$ given $\tilde{x}_h$ depends on $a$ in a highly nonlinear manner.

A related way to see this is that the contract ultimately depends on the likelihood ratio, and, as Sinclair-Desgagné (2007) shows, this likelihood ratio can depend, in interesting ways, even on ancillary signals. Thus, Assumptions (A8) and (A9) do permit a wide range of interesting contracts. Nevertheless, Proposition 1 shows that these assumptions are fairly restrictive. The following sections derive less restrictive versions of both the Jewitt and the Sinclair-Desgagné results.

4. A MULTI-SIGNAL EXTENSION OF JEWITT’S CONDITIONS

This section uses the state-space approach to give a simple generalization of Jewitt’s (1988) conditions. First, (3SS) above makes it clear that concavity in $a$ of $U(s^*(\cdot), a)$ follows from two standard conditions plus one nonstandard condition:

Condition (i): the coordinates of $x(a, \vartheta)$ are concave in $a$,

Condition (ii): $s^*(x)$ is nondecreasing concave in $x$, and

Condition (iii): $u(s)$ is nondecreasing concave in $s$.

Conditions (i) and (iii) are completely standard: (i) is diminishing marginal productivity – at least in the case where the signal is output – and (iii) is diminishing marginal utility, i.e., risk aversion. Condition (ii), however, is very problematic, since it depends on the shape of the endogenous payment schedule, $s^*(\cdot)$. The condition therefore depends on the solution.

Grossman and Hart (1983), however, give conditions – in the one-signal case –
which make the payment function progressive, so \( s^*(x) \) is indeed nondecreasing concave in \( x \), and (ii) is met. Moreover, these conditions generalize easily to the multi-signal case. Using the Mirrlees formulation, the principal’s cost minimizing schedule, \( s^*(\cdot) \), for inducing the agent to choose \( a^* \), given \( (\text{IC}_R) \) and \( (P) \), solves equation (4) above. Also, if \( (\text{IC}_R) \) and (4) hold, then \( \mu \geq 0 \), as mentioned above. In addition, even though the principal’s relaxed constraint, \( (\text{IC}_R) \), does not include the agent’s complementary slackness condition at \( a = 0 \), (4) nevertheless holds at that point as well, with \( \mu = 0 \).

Given this setup, the Grossman-Hart conditions for \( s^*(\cdot) \) to be nondecreasing concave are simply:

**Condition (iia):** \( f_a(x|a)/f(x|a) \) is nondecreasing concave in \( x \) for each \( a \), and

**Condition (iib):** \( 1/u'(s) \) is increasing convex in \( s \), so \( \nu(z) = (u')^{-1}(1/z) \) is increasing concave in \( z \).

Condition (iia) is a strengthening of the Monotone Likelihood Ratio property. It says, roughly, that the “good news” in the signal \( \tilde{x} \) grows more and more slowly as \( \tilde{x} \) rises, which tends to make \( s^*(x) \) concave. Condition (iib) says that the marginal utility of income falls quickly so, at large values of \( s \), further increases in \( s \) have less effect in terms of incentives. This also tends to make \( s^*(x) \) concave. Thus, assumptions like these are clearly relevant for the nonstandard condition that \( s^*(\cdot) \) be concave.

The Grossman-Hart conditions, plus \( \mu \geq 0 \), ensure that the function

\[
(7) \quad s^*(x) = \nu\left(\lambda + \mu \left[ f_a(x|a^*)/f(x|a^*) \right]\right)
\]

is nondecreasing concave in \( x \). This, combined with Conditions (i) and (iii), ensure that the first-order approach is valid. That is, one obtains the following chain of effects:

\[
(8) \quad a \xrightarrow{(i)} x \xrightarrow{\gamma_{\text{(iia)}}} f_a/f \xrightarrow{\gamma_{\text{(iib)}}} s \xrightarrow{\gamma_{\text{(iii)}}} u.
\]
Here Condition (i) ensures that the first mapping is concave, (iia) ensures that the second mapping is nondecreasing concave, and so on. These combine to ensure that \((3_{SS})\) is concave in \(a\).

In addition, Conditions (iib) and (iii) can be combined and weakened, because what matters is for the composition of the (iib) and (iii) mappings to be concave, not for each to be concave separately. But this composition is simply Jewitt’s function

\[
(9) \quad \omega(z) = u(\nu(z)) = u((u')^{-1}(1/z)),
\]

where \(\nu(\cdot)\) was defined in (iib). Thus, if \(\omega(\cdot)\) is concave, then the composition of the (iib) and (iii) mappings is concave, so we can replace (iib) and (iii) with:

**Condition (iib-iii_W):** The function \(\omega(\cdot)\) in (9) is nondecreasing concave.

Note that Jewitt’s function (9) can also be obtained very naturally from the dual representation of the principal-agent problem. That is, if we let \(\phi(z) = \max_x zu(x) - x\), then Condition (iib-iii_W) can be expressed as \(\phi'''(z) \leq 0\) (Jewitt, 1997, and personal communication; see also Fagart and Sinclair-Desgagné, 2007, who explore further economic implications of \(\phi'''(z)\) positive or negative).

**PROPOSITION 2:** Assume Conditions (i), (iia) and (iib-iii_W). Then any solution to the relaxed problem, maximizing \(V(s(\cdot), a)\) subject to (IC\(_R\)) and (P), is also a solution to the fully constrained problem of maximizing \(V(s(\cdot), a)\) subject to (IC) and (P).

**PROOF:** Conditions (i), (iia) and (iib-iii_W) imply that any solution to the relaxed problem yields an agent objective function \(U(s^*(\cdot), a)\), from \((3_{SS})\), which is concave in \(a\). Since \(U(s^*(\cdot), a)\) is concave, and \(a^*\) satisfies (IC\(_R\)), \(a^*\) also satisfies the full incentive compatibility constraint (IC), so it is a solution to the fully constrained problem. QED

Proposition 2 is similar to Jewitt’s (1988) Theorem 3, but without his independence assumption. In fact, the proof is also very similar. Jewitt (1988), footnote 6,
mentions that “[i]t is possible to derive some conditions without [independence], but we do not pursue the issue here.” Thus, Jewitt may have been aware of something like our Proposition 2. However, it is not clear how he would have expressed Condition (i) using the Mirrlees notation.

To translate Condition (i) into the Mirrlees notation, we must begin with a Mirrlees representation, $F(x|a)$, and find a state-space model that yields this CDF. We then impose Condition (i) on the state-space model, and translate back to the Mirrlees notation. Thus, for the random vector $\tilde{x}$ with CDF $F(x|a)$, we need a representation $x_i(a, \tilde{\vartheta})$, $i = 1, 2, ..., n$, where $\tilde{\vartheta}$ has some joint PDF $g(\vartheta)$ independent of $a$, such that the random variables $x_i(a, \tilde{\vartheta})$ have joint CDF $F(x|a)$.

It turns out that there are many such representations, and each provides different sufficient conditions. This is somewhat inelegant, since there is no one canonical representation. It may, however, supply some flexibility. If one representation does not work, the researcher can try another. The representation in Proposition 3 builds on what is known as the “standard construction” (see Müller and Stoyan, 2002, p. 88).

Consider, for specificity, the two signal case (the approach will clearly generalize). We construct one representation, and then indicate how others might be obtained.

**PROPOSITION 3:** Let $F^1(x_1|a)$ be the marginal CDF of $\tilde{x}_1$ as a function of $a$, and let $F^2(x_2|x_1, a)$ be the conditional CDF of $\tilde{x}_2$, given $\tilde{x}_1 = x_1$, as a function of $a$. Suppose that $F^1$ and $F^2$ are continuous, and that the support of the distribution of $\tilde{x}$ is compact and convex. Then there exist continuous functions $x_1(a, \theta_1, \theta_2)$ and $x_2(a, \theta_1, \theta_2)$, for $(\theta_1, \theta_2) \in [0, 1] \times [0, 1]$, that solve the system of equations

\[
F^1(x_1|a) = \theta_1, \quad F^2(x_2|x_1, a) = \theta_2.
\]

Also, if $\tilde{\vartheta} = (\tilde{\theta}_1, \tilde{\theta}_2)$ is uniformly distributed on the square $[0, 1] \times [0, 1]$, then $x(a, \tilde{\vartheta}) = (x_1(a, \tilde{\vartheta}), x_2(a, \tilde{\vartheta}))$ has the same joint distribution as $\tilde{x}$. 
PROOF: First, \( F^1(x_1|a) \) is continuous, and also increasing in \( x_1 \) on its (convex) support. Thus the function \( x_1(a, \theta_1, \theta_2) = (F^1)^{-1}(\theta_1|a) \), solving the first half of (10), exists, is continuous, and in fact depends only on \( \theta_1 \) and \( a \). Next, plug \( x_1 = (F^1)^{-1}(\theta_1|a) \) into the second half of (10). Then one can similarly solve \( F^2(x_2|(F^1)^{-1}(\theta_1|a), a) = \theta_2 \) for \( (F^2)^{-1}(\theta_2|(F^1)^{-1}(\theta_1|a), a) = x_2(a, \theta_1, \theta_2) \).

Next, \( \text{Prob}(x_1(a, \tilde{\theta}_1, \tilde{\theta}_2) < x_1) = \text{Prob}((F^1)^{-1}(\tilde{\theta}_1|a) < x_1) = \text{Prob}(\tilde{\theta}_1 \leq F^1(x_1|a)) = F^1(x_1|a) \), since \( \tilde{\theta}_1 \) is \( U[0, 1] \). A similar logic shows that \( \text{Prob}(x_2(a, \tilde{\theta}_1, \tilde{\theta}_2) < x_2|x_1(a, \tilde{\theta}_1, \tilde{\theta}_2) = x_1) = F^2(x_2|x_1, a) \). QED

Of course, many other representations are also possible. For example, the roles of \( x_1 \) and \( x_2 \) could be switched above. More generally, if one starts with \( \tilde{\theta} \) uniform on the square \([0, 1] \times [0, 1]\), and considers a density-preserving smooth deformation, \( \Phi_a(\vartheta) = \vartheta' \), of this square, then \( \tilde{x} = x(a, \Phi_a(\tilde{\vartheta})) \) gives a new state-space representation of the same Mirrlees model \( f(x|a) \).

Jewitt (1988), in his Theorem 3, assumes independence, so the functions in Proposition 3 reduce to \( x_i(a, \theta_i) = (F^i)^{-1}(\theta_i|a) \). He also assumes that the \( F^i(x_i|a) \) are quasiconvex in \( x_i \) and \( a \). This is how he imposes Condition (i) (i.e., concavity of the \( x_i(a, \theta_i) \) in \( a \)). His Theorem 3 then assumes this plus Conditions (iia) and (iib-iii_W).

To check Condition (i) more generally, one can implicitly differentiate (10) twice to obtain \( \partial^2 x_i/\partial a^2 \), and check concavity through the analogue of bordered Hessians (details available upon request). However, this may be computationally messy. It might therefore be easier to use a shortcut to check (i). For example, one might begin with a state-space representation, or solve (10) explicitly.

Thus, to find conditions for the concavity of \( x(a, \vartheta) \) in \( a \), consider for example \( x_2(a, \theta_1, \theta_2) \) from Proposition 3. For simplicity represent this as

\[
(11) \quad x_2(a, \theta_1, \theta_2) = (F^2)^{-1}(\theta_2|(F^1)^{-1}(\theta_1|a), a) = g(x_1(a, \theta_1), a) = g(h(a), a)
\]
where

\[(12) \quad g(x_1, a) = (F^2)^{-1}(\theta_2 | x_1, a) \quad \text{and} \quad h(a) = (F^1)^{-1}(\theta_1 | a) = x_1(a, \theta_1).\]

Here the $\theta_i$’s have been suppressed for brevity. Concavity now requires:

\[
\partial^2 x_2 / \partial a^2 = g_1(h(a), a)h''(a) + g_{11}(h(a), a)h'(a)^2 \\
+ g_{22}(h(a), a) + 2g_{12}(h(a), a)h'(a) \leq 0.
\]

Suppose $\tilde{x}_1$ and $\tilde{x}_2$ are positively related, so an increase in $\tilde{x}_1$ increases $\tilde{x}_2$ in the sense of first order stochastic dominance. Then $g_1(h(a), a)$ will be nonnegative. Thus if $h$ is concave (so $x_1$ depends concavely on $a$), then the first term will be nonpositive. Next, if $g$ is concave in each of its arguments (so $x_2$ depends concavely on $x_1$ and $a$), then the next two terms will be nonpositive. This leaves the last term. If the marginal distribution of $\tilde{x}_1$ is increasing in $a$ in the sense of first order stochastic dominance, then $h'(a)$ is nonnegative. The key issue therefore becomes whether $g_{12}(h(a), a)$ is nonpositive. This requires roughly that the correlation between $\tilde{x}_1$ and $\tilde{x}_2$ be nonincreasing in $a$.

A similar condition will be important in the CISP approach below (see Corollary 2, Section 8). Note also that if $g$ is concave jointly in $x_1$ and $a$, then its Hessian is negative semi-definite, so the sum of the last three terms is again nonpositive.

5. A MULTI-SIGNAL EXTENSION OF THE CDFC

As suggested in Section 3 above, if we express the agent’s payoff function using the Mirrlees formalism, (3M), (i.e., $U(s^*(\cdot), a) = \int u(s^*(x))f(x|a)dx - a$), then we are naturally led to impose concavity through the distribution function, $f(x|a)$, since that is where the agent’s choice variable, $a$, appears. Consider, for example, the one-signal
case, so $\bar{x}$ reduces to the univariate signal $\tilde{x}$, and let the support of $f(x|a)$ be $[\underline{x}, \bar{x}]$. Then integrating by parts gives

$$U(s^*(\cdot), a) = u(s^*(\underline{x})) + \int_{\underline{x}}^{\bar{x}} \left( \partial u(s^*(x)) / \partial x \right) \left[ 1 - F(x|a) \right] dx - a. \tag{14}$$

Thus, suppose $u(s^*(x))$ is nondecreasing in $x$, so $\partial u(s^*(x)) / \partial x \geq 0$. Then $U(s^*(\cdot), a)$ is clearly concave in $a$ if the upper CDF, $1 - F(x|a)$, is. This leads to the Rogerson CDFC, as is well known. Incidentally, a new derivation of Jewitt’s (1988) full one-signal conditions can be obtained similarly, by simply performing a second integration by parts (details available upon request).

In any case, the natural multi-signal generalization of Rogerson’s procedure would be to choose one of the $n$ signals, say $x_h$, and integrate (3M) by parts with respect to $x_h$. This yields essentially the GCDFC of Sinclair-Desgagné (1994).

To obtain a more flexible generalization of the CDFC, begin by returning to the one-signal case, and note that, if $u(s^*(x))$ is nondecreasing in $x$, then it can clearly be approximated by a sum of nondecreasing step functions. Specifically, it can be approximated as a sum, $u(s^*(x)) \approx \sum_j \alpha_j h(x, b_j)$, where the $\alpha_j$’s are positive and $h(x, b)$ is the step function which is zero for $x < b$ and one for $x \geq b$. The integral $\int u(s^*(x))f(x|a)dx$ in $U(s^*(\cdot), a)$ can then be approximated as

$$\int u(s^*(x))f(x|a)dx \approx \sum_j \alpha_j \int h(x, b_j)f(x|a)dx = \sum_j \alpha_j [1 - F(b_j|a)].$$

By the CDFC this sum is concave in $a$, so $U(s^*(\cdot), a)$ is concave in $a$, as desired.

To obtain a new multi-signal generalization of the CDFC approach, we must therefore simply construct a family of multi-variable generalizations of the functions $h(x, b)$ above, capable of generating all multi-variable nondecreasing functions.

Thus, consider the $n$ signal case, and say that the set $\mathbf{E} \subseteq \mathbb{R}^n$ is an increasing set (Nachbin, 1965, Milgrom and Weber, 1982) if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ (i.e., $y_i \geq x_i$, $i = 16$
1, 2, ..., n), implies $y \in E$. Note that if a function is nondecreasing, then its upper level sets are increasing sets. Given an increasing set $E$, let $h(x; E)$ be the characteristic function of $E$, so

$$h(x; E) = \begin{cases} 
0 & \text{if } x \not\in E, \\
1 & \text{if } x \in E.
\end{cases}$$

(15)

Then every nondecreasing function can be approximated using positive linear combinations of these $h(x; E)$ functions, yielding a multi-signal extension of the above approach. Thus, consider the following generalization of the CDFC:

**The Concave Increasing-Set Probability (CISP) Condition:** For every increasing set $E$, the probability $\text{Prob}(x \in E | a)$ is concave in $a$.

The CISP condition, plus the Monotone Likelihood Ratio (MLR) property and strict concavity of $u(\cdot)$, are sufficient to justify the first-order approach. To see this, we first prove a simple lemma.

**LEMMA:** The CDF, $F(\cdot | a)$, satisfies CISP if and only if the transformation

$$\varphi^T(a) = \int \varphi(x) dF(x | a) = E[\varphi(x) | a]$$

(16)

satisfies:

$$\text{for any nondecreasing function } \varphi(x), \varphi^T(a) \text{ is concave in } a.$$  

(17)

**PROOF:** The proof builds on a standard approximation (see, e.g., Müller and Stoyan, 2002, Theorem 3.3.4), and is contained in the appendix.

With this lemma in hand, it is easy to give conditions justifying the first-order approach in the multi-signal case. This is done in the following proposition.

**PROPOSITION 4:** Suppose (a) the MLR property holds, so $f_a(x | a)/f(x | a)$ is nondecreasing in $x$, (b) $u(\cdot)$ is strictly concave, and (c) CISP holds. Then any solution to the relaxed principal-agent problem also solves the fully constrained problem.
PROOF: First, the solution to the relaxed problem has the payment schedule \(s^*(x)\) solving (4). Also, since \(u(\cdot)\) is strictly concave, \(1/u'(s)\) is increasing in \(s\). This, plus \(\mu \geq 0\) (by the Jewitt, 1988, argument), and \(f_a(x|a)/f(x|a)\) nondecreasing in \(x\), implies that the payment schedule \(s^*(x)\) is also nondecreasing in \(x\). Thus \(u(s^*(x))\) is nondecreasing in \(x\). Using \(\varphi(x) \equiv u(s^*(x))\) and CISP in the Lemma then implies that \(U(s^*(\cdot), a) = \varphi^T(a) - a\), from (3_M) and (16), is concave in \(a\), so any solution to the relaxed problem is also a solution to the fully constrained problem. QED

6. A RISK AVERSE PRINCIPAL

Up to now, the principal has been assumed risk neutral. This was primarily because the Jewitt conditions go through more smoothly in this case. However, suppose now that the principal is risk averse, with von Neumann-Morgenstern utility function \(v(\cdot)\). Then the principal’s first order condition for \(s^*(\cdot), (4)\), changes to

\[
(4_{RA}) \quad \frac{u'\left(\pi(x) - s^*(x)\right)}{u'(s^*(x))} = \lambda + \mu \frac{f_a(x|a^*)}{f(x|a^*)}.
\]

Thus, even if we assume that the likelihood ratio \(f_a(x|a^*)/f(x|a^*)\) is concave in \(x\), it is difficult to impose the nonstandard condition \(s^*(x)\) concave in \(x\), because Condition (iib) is no longer adequate. Therefore Jewitt’s Condition (iib-iii_W) is also inadequate.

On the other hand, the CDFC approach, and its generalization in Section 5, are easily adapted to a risk averse principal. This requires two new assumptions:

**Nondecreasing** \(\pi(\cdot)\): The function \(\pi(x)\) is nondecreasing in \(x\).

**The Nondecreasing Increasing-Set Probability (NISP) Condition:** For any increasing set \(E\), the probability \(\text{Prob}(\tilde{x} \in E|a)\) is nondecreasing in \(a\).

Now, Jewitt’s (1988) proof of \(\mu \geq 0\) does not work if the principal is risk averse. We therefore replace (IC_R) with Rogerson’s (1985) *doubly* relaxed constraint,

\[
(IC_{DR}) \quad U_a(s^*(\cdot), a^*) \geq 0.
\]
Thus, the principal’s “doubly relaxed” problem is to maximize $V(s(\cdot), a)$ given (IC$_{DR}$) and (P). We then obtain the following proposition:

**PROPOSITION 5:** Assume $\pi(\cdot)$ nondecreasing, $u(\cdot)$ and $v(\cdot)$ strictly concave, and the MLR, CISP, and NISP conditions hold. Then any solution to the principal’s doubly relaxed problem is also a solution to her unrelaxed problem.

**PROOF:** Let $s^*(\cdot)$ and $a^*$ be solutions to the doubly relaxed problem. As usual, (4$_{RA}$) still holds. Also, since (IC$_{DR}$) is an inequality, $\mu \geq 0$. This, plus the MLR property and $\pi(x)$ nondecreasing, imply that $s^*(x)$ is nondecreasing in $x$.

If $U_a(s^*(\cdot), a^*) > 0$, then $a^*$ is not incentive compatible. Thus, we must show that this contradicts the assumption that $s^*(\cdot)$ and $a^*$ solve the doubly relaxed problem.

Suppose therefore that $U_a(s^*(\cdot), a^*) > 0$. Then $\mu = 0$, so (4$_{RA}$) shows that $u'(\pi(x) - s^*(x))/u'(s^*(x))$ is constant. Thus, since $s^*(x)$ is nondecreasing in $x$, $\pi(x) - s^*(x)$ is nondecreasing in $x$ as well, so $\varphi(x) \equiv v(\pi(x) - s^*(x))$ is also nondecreasing in $x$. Thus, by an argument like that in the proof of the lemma in Section 5, NISP implies that the principal’s payoff, $V(s^*(\cdot), a) = \varphi^T(a) = \int v(\pi(x) - s^*(x))f(x|a)d\mathbf{x}$, is nondecreasing in $a$. This, plus $U_a(s^*(\cdot), a^*) > 0$, implies that $s^*(\cdot)$ and $a^*$ are not solutions to the doubly relaxed problem, since increasing $a$ and shifting the $s^*(\cdot)$ schedule down can increase the principal’s payoff, while still satisfying the constraints. Thus $U_a(s^*(\cdot), a^*) > 0$ leads to a contradiction, so $U_a(s^*(\cdot), a^*) = 0$.

Next, since $s^*(\cdot)$ is nondecreasing, $u(s^*(\cdot))$ is nondecreasing, so CISP, plus the lemma from Section 5, imply that the agent’s objective function is concave. Thus $s^*(\cdot)$ and $a^*$ solve the unrelaxed problem and the first-order approach is valid. **QED**

As pointed out by a referee, NISP can be checked by adapting a version of Milingrom and Weber’s (1982) notion of affiliated random variables (equivalently MPT$_2$; see Müller and Stoyan, 2002, pp. 126-7).
Thus, assume that the PDF $f(x|a)$ is smooth jointly in $x$ and $a$, and that $a \in [0, 1]$. Though $a$ is not actually random, imagine that it is a random variable, with marginal distribution uniform on $[0, 1]$. Then the random vector $(\tilde{x}, \tilde{a})$, with joint PDF $f^\#(x, a) = f(x|a)$, is affiliated if the cross partials $\partial^2 \ln f^\#/\partial x_i \partial a$ and $\partial^2 \ln f^\#/\partial x_i \partial x_j$ are all nonnegative. NISP then follows immediately from Milgrom and Weber’s Theorem 5. Note, incidentally, that $\partial^2 \ln f^\#/\partial x_i \partial a \geq 0$ is the MLR property.

Unfortunately, it is not clear how this approach could be used to check CISP. In the next two sections we therefore use a different approach to derive conditions ensuring CISP. While affiliation is sufficient to ensure NISP, the approach in the following sections also gives less restrictive conditions ensuring NISP, with little extra effort.

7. REDUCING NISP AND CISP TO DENSITY FLOWS

The previous sections showed that CISP plays a role similar to CDFC in justifying the first-order approach. Also, CISP is less restrictive than the GCDFC. As a referee has pointed out, GCDFC implies that there is an index $h$ such that $\text{Prob}(\tilde{x} \in E|a)$ is concave in $a$ for all sets $E$ satisfying the condition: $(x_h, x_{-h}) \in E$ and $x'_h \geq x_h$ imply $(x'_h, x_{-h}) \in E$. Since this class of sets is larger than the class of increasing sets, GCDFC is more restrictive than CISP. However, it remains to see more clearly how flexible CISP actually is.

It turns out to be easy to construct distributions satisfying CISP. To do so, however, requires us to think of $f(x|a)$ in terms of probability density flows as $a$ increases. Analyzing these flows, in turn, requires tools from vector calculus (Taylor and Mann, 1983, Chapter 15; Lang, 1987, Chapters 10 and 12). For expository purposes focus on the two-signal case, though the approach generalizes easily to $n$ signals.

Thus, consider the random vector $\tilde{x} = (\bar{x}, \bar{y})$ on the square $S = [0, 1] \times [0, 1]$, with
PDF $f(x, y|a) = f(x|a)$. Also, consider a vector field $\mathbf{v}(x, a) = (u(x, a), v(x, a))$ on $S$. If the mass of $f(x|a)$ moves along the vector field $\mathbf{v}(x, a)$ as $a$ grows, then

$$f_a(x, y|a) = -[u_x(x, y, a) + v_y(x, y, a)] = -\text{div } \mathbf{v}(x, a).$$

This is the usual “divergence” formula for the flow of a compressible fluid, where $\mathbf{v}(x, a)$ is the “flux” (density $\times$ velocity of flow). Thus if, e.g., $u_x(x_0, y_0, a) > 0$, then $u(x_0 - \Delta x, y_0, a) < u(x_0 + \Delta x, y_0, a)$ so, in the $x$-direction, less mass is moving towards $(x_0, y_0)$ than away from it. That is, the mass really does diverge. This should make $f(x_0, y_0|a)$ fall in $a$, as in (18). A similar argument applies for the $y$-direction. This therefore does, in fact, represent $f(x, y|a)$ in terms of a density flow.

Next let $A$ be a simply connected two-dimensional subset of $S$ with boundary $\partial A$, and let $\partial A$ be traced out counterclockwise by the continuous piecewise differentiable simple closed curve $\mathbf{x}(t)$ as $t$ rises from zero to one. Let $\mathbf{n}(t) = (y'(t), -x'(t))$, which is an outward pointing “normal” (perpendicular) vector to $\partial A$ at $\mathbf{x}(t)$. Then

$$\frac{\partial}{\partial a} \int_A f(x|a) \, dx = -\int_A [u_x(x, y, a) + v_y(x, y, a)] \, dxdy$$

$$= -\int_0^1 [u(\mathbf{x}(t), a)y'(t) - v(\mathbf{x}(t), a)x'(t)] \, dt = -\int_0^1 \mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t) \, dt.$$

Here the first step follows from (18) and the second follows from the two dimensional “Divergence Theorem” (Taylor and Mann, 1983, p. 488, Lang, 1987, p. 345). Also, $\mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t) = u(\mathbf{x}(t), a)y'(t) - v(\mathbf{x}(t), a)x'(t)$ is the “dot product” of the vectors $\mathbf{v}(\mathbf{x}(t), a)$ and $\mathbf{n}(t)$. This dot product is positive if $\mathbf{v}(\mathbf{x}(t), a)$ points out from the boundary of $A$, and negative if it points into $A$. Thus the last integral in (19) is the flow of the vector field $\mathbf{v}(\mathbf{x}(t), a)$ across the boundary $\partial A$, so (19) gives the density flow of $\mathbf{v}(x, a)$ into $A$. The equation thus makes intuitive sense.
To illustrate (19) we show that \( f(x|a) \) from (18) is a PDF on the square \( S \) for all \( a \) if it is a PDF on \( S \) for \( a = 0 \) and if in addition

\[
(20) \quad u(0, y, a) = u(1, y, a) = v(x, 0, a) = v(x, 1, a) = 0.
\]

This last condition ensures that no mass is flowing across the boundaries of the square \( S \). To see that this works, translate into ordinary multiple integrals:

\[
(21) \quad \frac{\partial}{\partial a} \int_0^1 \int_0^1 f(x, y|a) dxdy = -\int_0^1 \int_0^1 [u_x(x, y, a) + v_y(x, y, a)] dxdy = \int_0^1 [u(1, y, a) - u(0, y, a)] dy - \int_0^1 [v(x, 1, a) - v(x, 0, a)] dx = 0.
\]

Here the first step uses (18), the second step uses the Fundamental Theorem of Calculus and the last step uses (20). Thus \( \int_S f(x|a) dx \) is constant in \( a \), so if \( \int_S f(x|0) dx = 1 \), then \( \int_S f(x|a) dx = 1 \) for all \( a \), and, assuming \( f(x|a) \geq 0 \), \( f(x|a) \) really is a PDF on \( S \). The third expression in (21), incidentally, is the analogue for the square, \( S \), of the second line of (19). Thus (21) actually gives a proof of a special case of (19). Note that this calculation suggests that the Divergence Theorem is a kind of generalization of the Fundamental Theorem of Calculus. Note also that integrating (18) gives:

\[
(22) \quad f(x, y|a) = f(x, y|0) - \int_0^a [u_x(x, y, \alpha) + v_y(x, y, \alpha)] d\alpha.
\]

With this machinery, it is easy to construct PDFs which satisfy NISP and CISP. We show how to do this in the following proposition.

**PROPOSITION 6:** Let the vector field \( v(x, a) \) satisfy (20). Then if its coordinates are nonnegative, the density function \( f(x, y|a) \) in (18) satisfies NISP. Similarly, if its coordinates are nonincreasing in \( a \), then \( f(x, y|a) \) satisfies CISP.

**PROOF:** Let \( E \) be an increasing set. Using (19) gives

\[
(23) \quad \frac{\partial}{\partial a} \text{Prob}(\tilde{x} \in E|a) = - \int_0^1 v(x(t), a) \cdot n(t) dt,
\]
with \( x(t) \) tracing \( \partial E \). Since \( E \) is an increasing set in \( S \), its boundary, \( \partial E \), consists of a downward sloping curve on its Southwest, whose normal vector \( n(t) \) has nonpositive coordinates, and other lines, from the boundary of \( S \). Now, if the vector field \( v(x, a) \) has nonnegative coordinates, it flows into \( E \) through its Southwest boundary, but does not flow out through the other parts of its boundary, by (20). Thus the derivative in (23) is nonnegative, so \( \text{Prob}(\bar{x} \in E | a) \) is nondecreasing in \( a \). Similarly, if the coordinates of \( v(x(t), a) \) are nonincreasing in \( a \) then the rate of flow is nonincreasing in \( a \). Thus \( \text{Prob}(\bar{x} \in E | a) \) is concave in \( a \) and \( f(x | a) \) satisfies CISP. \textbf{QED}

It is now easy to construct PDFs \( f(x | a) \) which satisfy NISP, CISP and also the MLR property. To do this choose \( v(x, a) \) as in Proposition 6 so that we also have

\[
(24) \quad f_a(x, y | a) / f(x, y | a) = -[u_x(x, y, a) + v_y(x, y, a)] / f(x, y | a)
\]

nondecreasing in \( x \) and \( y \). As an example, let \( f(x, y | 0) \) be uniform on \( S \), let \( u(x, y, a) = (x - x^2)(1 - a)\epsilon \), and let \( v(x, y, a) = (y - y^2)(1 - a)\epsilon \). Then (22) gives

\[
(25) \quad f(x, y | a) = 1 + (x + y - 1)(2a - a^2)\epsilon \text{ for } (x, y) \in S.
\]

Now, if \( \epsilon < 1 \), then this is a strictly positive PDF on \( S \) for all \( a \in [0, 1] \), and it satisfies CISP, NISP and the MLR property. Note also that this PDF satisfies neither Jewitt’s (1988) independence assumption nor Sinclair-Desgagné’s (1994) GCDFC. Clearly it would be easy to generate an enormous range of examples like this, for example, by building on ideas from LiCalzi and Spaeter (2003). Of course, if one uses affiliation to get NISP, as in the end of Section 6, then MLR follows automatically.

On the other hand, to use this approach to confirm NISP and CISP for a \textit{pre-specified} \( f(x | a) \), one must construct an appropriate probability density flow \( v(x, a) \) corresponding to \( f(x | a) \). We show how to do this in the next section.
8. LOCAL CONDITIONS SUFFICIENT FOR NISP AND CISP

The previous section reduced NISP and CISP to the existence of certain probability density flows. This allowed us to construct PDF’s satisfying NISP and CISP. However, to check NISP and CISP for a given PDF, \( f(x, y|a) \), we must construct vector flows satisfying equations (18) and (20), and then check the conditions in Proposition 6. This turns out to be relatively straightforward, and reduces the hard-to-check global NISP and CISP conditions to convenient local conditions.

Let \( g(x|a) \) and \( G(x|a) \) be the marginal PDF and CDF of \( \tilde{x} \), and let \( h(y|x, a) \) and \( H(y|x, a) \) be the conditional PDF and CDF of \( \tilde{y} \) given \( \tilde{x} = x \). Thus the joint PDF \( f(x, y|a) = g(x|a)h(y|x, a) \). The following then characterizes NISP and CISP.

**Proposition 7:** Suppose there are functions \( \phi(x, y, a) \) and \( \psi(x, y, a) \) with

\[
\phi_x(x, y, a) = -\psi_y(x, y, a)
\]

and

\[
\phi(0, y, a) = \phi(1, y, a) = \psi(x, 0, a) = \psi(x, 1, a) = 0,
\]

and such that the vector field

\[
v(x, a) = -(G_a(x|a)h(y|x, a) + \phi(x, y, a),
\]

\[
g(x|a)H_a(y|x, a) - G_a(x|a)H_x(y|x, a) + \psi(x, y, a))
\]

has nonnegative coordinates for all \((x, y) \in [0, 1] \times [0, 1]\). Then the NISP condition holds. Similarly, if there are functions \( \phi(x, y, a) \) and \( \psi(x, y, a) \) satisfying (26) and (27) and such that the coordinates of (28) are nonincreasing, then CISP holds.
**PROOF:** Using the product rule to calculate \( f_a(x, y|a) \) shows that \( v(x, a) \) satisfies (18). The coordinates also satisfy (20) since \( G_a(0|a) = G_a(1|a) = H_a(0|x, a) = H_a(1|x, a) = H_x(0|x, a) = H_x(1|x, a) = 0 \). Now apply Proposition 6. QED

The following corollaries give some illustrative applications of this result.

**COROLLARY 1:** Suppose \( G(x|a) \) is nonincreasing in \( a \), and \( H(y|x, a) \) is nonincreasing in \( x \) and \( a \). Then NISP holds.

**PROOF:** Let \( \phi \equiv \psi \equiv 0 \). Then since \( G(x|a) \) is nonincreasing in \( a \), the first coordinate of \( v(x, a) \) is nonnegative. Similarly, the second coordinate is nonnegative since \( g(x|a)H_a(y|x, a) \) is nonpositive and \( G_a(x|a)H_x(y|x, a) \) is nonnegative. QED

Note that \( H(y|x, a) \) nonincreasing in \( x \) says that, when \( \tilde{x} \) increases, the conditional distribution of \( \tilde{y} \) increases in the sense of first order stochastic dominance. This says that \( \tilde{x} \) and \( \tilde{y} \) are positively related, but is weaker than \( \tilde{x} \) and \( \tilde{y} \) affiliated (use Milgrom and Weber’s, 1982, Theorem 5). Similarly, \( G(x|a) \) and \( H(y|x, a) \) nonincreasing in \( a \) says that the marginal distribution of \( \tilde{x} \), and the conditional distribution of \( \tilde{y} \), respectively, are nondecreasing in \( a \) in the sense of first order stochastic dominance. These are weaker assumptions than the corresponding MLR properties. Thus, the conditions in Corollary 1 generalize the affiliation conditions from the end of Section 6.

**COROLLARY 2:** Suppose \( G_a(x|a) \) and \( H_a(y|x, a) \) are negative (this follows from strict versions of the corresponding MLR properties for \( \tilde{x} \) and \( \tilde{y} \)). Assume also that \( g(x|a) \) and \( h(y|x, a) \) are strictly positive on \([0, 1]\). Finally assume \( H_x(y|x, a) < 0 \) (so \( \tilde{x} \) and \( \tilde{y} \) are positively related). Then the conditions

\[
(29) \quad h_a(y|x, a)/h(y|x, a) \leq -G_{aa}(x|a)/G_a(x|a)
\]

\[
(30) \quad g_a(x|a)/g(x|a) \leq -H_{aa}(y|x, a)/H_a(y|x, a)
\]

\[
(31) \quad H_{ax}(y|x, a)/H_x(y|x, a) \leq -G_{aa}(x|a)/G_a(x|a)
\]
are sufficient to ensure CISP.

**PROOF:** Let $\phi \equiv \psi \equiv 0$ in (28) and consider the first coordinate of $v(x, a)$. This is nonincreasing in $a$ if

$$\partial[G_a(x|a)h(y|x, a)]/\partial a = G_a(x|a)h_a(y|x, a) + G_{aa}(x|a)h(y|x, a) \geq 0.$$  

Using $G_a(x|a) < 0$, $h(y|x, a) > 0$ and (29) shows (32) holds. Similarly, (30) and (31) show that the second coordinate of (28) is nonincreasing in $a$. **QED**

Note that, for all $x$, the left hand side of (29) is nonnegative for some values of $y$. Thus, since $G_a(x|a) < 0$, we get $G_{aa}(x|a) \geq 0$ for all $x$, so (29) implies the CDCC for $\tilde{x}$. Similarly (30) implies the CDCC for $\tilde{y}$, conditional on $\tilde{x} = x$. Of course, (29) and (30) put additional restrictions on the $\tilde{x}$ and $\tilde{y}$ distributions, over and above the CDCC. This is not surprising however since, as Jewitt (1988), p. 1184, points out, the CDCC for the random variables $\tilde{x}$ and $\tilde{y}$ is not enough to justify the first-order approach, even if the signals are independent.

Condition (31), finally, says that an increase in $a$ should not make $H_x(y|x, a)$ fall too quickly, i.e., should not, roughly speaking, increase the correlation between $\tilde{x}$ and $\tilde{y}$ too much. This is similar to the condition discussed at the end of Section 4.

Interestingly, the conditions in Corollary 2 do not imply, and are not implied by, the GCDFC. The GCDFC requires either that $g(x|a)[1 - H(y|x, a)]$ be concave in $a$ for all $(x, y) \in [0, 1] \times [0, 1]$, or that $h(y|a)[1 - G(x|y, a)]$ be concave in $a$ for all $(x, y) \in [0, 1] \times [0, 1]$ (with $h(y|a)$ and $G(x|y, a)$ the obvious marginal PDF and conditional CDF). While these conditions cannot be obtained from Corollary 2, they can be derived from Proposition 7, as shown in the following corollary.

**COROLLARY 3:** The GCDFC implies the CISP condition.

**PROOF:** For specificity, assume $g(x|a)[1 - H(y|x, a)]$ concave in $a$ for all $(x, y) \in [0, 1] \times [0, 1]$. Apply Proposition 7 with $\phi(x, y, a) = [1 - h(y|x, a)]G_a(x|a)$ and
\[ \psi(x, y, a) = G_a(x|a)H_x(y|x, a) + [H(y|x, a) - y]g_a(x|a). \] These functions satisfy (26) and (27). In addition, with these functions, the vector field in (28) becomes

\[ (33) \quad \mathbf{v}(x, a) = -\left( G_a(x|a), g(x|a)H_a(y|x, a) + [H(y|x, a) - y]g_a(x|a) \right). \]

Now, \( g(x|a)[1 - H(y|x, a)] \) is concave in \( a \) for all \((x, y) \in [0, 1] \times [0, 1] \), by GCDFC. Plugging \( y = 0 \) into this shows that \( g(x|a) \) is concave in \( a \), so \( g_{aa}(x|a) \leq 0 \) for all \( x \in [0, 1] \). Integrating gives \( 0 \geq \int_0^1 g_{aa}(x|a)dx \geq (\partial^2/\partial a^2) \int_0^1 g(x|a)dx = \partial^2 1/\partial a^2 = 0 \), so \( g_{aa}(x|a) = 0 \) almost everywhere (this also follows from Proposition 1 above). Thus, since \( G_{aa}(0|a) = 0 \), it follows that \( G_{aa}(x|a) = 0 \) for all \( x \), so the first coordinate of \( \mathbf{v}(x, a) \) in (33) is constant in \( a \), and so trivially nonincreasing.

Next, concavity of \( g(x|a)[1 - H(y|x, a)] \) in \( a \), plus \( g_{aa}(x|a) = 0 \), yields

\[ (34) \quad g(x|a)H_{aa}(y|x, a) + 2g_a(x|a)H_a(y|x, a) \geq 0. \]

Also, the second coordinate in (33) is nonincreasing if

\[ (35) \quad g(x|a)H_{aa}(y|x, a) + 2g_a(x|a)H_a(y|x, a) + [H(y|x, a) - y]g_{aa}(x|a) \geq 0. \]

Using \( g_{aa}(x|a) = 0 \), (34) implies (35). Both coordinates of (33) are therefore nonincreasing, so Proposition 7 yields CISP. Thus, GCDFC implies CISP. \textbf{QED}

Of course, it may be more natural to derive the GCDFC directly, as in Sinclair-Desgagné (1994). However, Corollary 3 illustrates the flexibility inherent in Proposition 7. Essentially, (18) and (20) hold for a range of possible vector fields. If any one of these vector fields has coordinates which are nonincreasing in \( a \), then CISP holds.

This can all be extended to the \( n \)-signal case. For example, consider the three signal case, with density \( f(x, y, z|a) = g(x|a) h(y|x, a) k(z|x, y, a) \). Then, leaving off the arguments for brevity, (28) should be replaced by the vector field

\[ \text{27} \]
\[ \mathbf{v} = -(G_a h k + \phi^1, \ g H_a k - G_a H_x k + \phi^2, \]
\[ ghK_a - G_a h K_x - g H_a K_y + G_a H_x K_y + \phi^3), \]

where \( G, H, \) and \( K \) are the obvious CDF’s, and the \( \phi^i(x, y, z, a) \)’s satisfy \( \phi^1_x + \phi^2_y + \phi^3_z = 0 \) and the analogue of (27). One can then obtain results like Corollaries 1 and 2 above. The approach is therefore quite flexible. Of course, it would also be interesting to find additional conditions, as well as other approaches, to check NISP and CISP.

9. OTHER EXTENSIONS

The above method allows other extensions. The basic approach is to find conditions which imply that the function \( u(s^*(x)) \) is in some restricted class, and then find conditions on the CDF, \( F(x|a) \) such that the mapping, (16), from \( \varphi(x) \) to \( \varphi^T(a) \), maps this restricted class into concave functions (see also Jewitt, 1988, p. 1189-90).

For example, consider a two-signal case, with signals \( \tilde{x} \) and \( \tilde{y} \). Suppose the likelihood ratio, \( f_a(x, y|a)/f(x, y|a) \), is nondecreasing and submodular in \((x, y)\) (so its cross partial is nonpositive), and that Condition (iib-iii\(_W\)) above holds. Then \( \varphi(x) \equiv u(s^*(x)) \) also turns out to be nondecreasing and submodular. Next suppose \( \text{Prob}( \tilde{x} \geq x_0 \text{ or } \tilde{y} \geq y_0|a) \) is concave in \( a \) for all \((x_0, y_0)\). Then the agent’s payoff \( \varphi^T(a) - a = U(s^*(\cdot), a) \) is concave, and the first-order approach is valid (this result was suggested by a referee; details available upon request). This generalizes Jewitt’s other multi-signal result, i.e. his Theorem 2. However, it is not clear how to extend this beyond the two-signal case.

Finally (as this same referee points out), if one is willing to assume that the \( n - 1 \) signals in \( \tilde{x}_{-h} \) are ancillary, then one can extend the Sinclair-Desgagné (1994) results by imposing a range of different conditions on the remaining dimension \( h \). For example,
using the Rogerson (1985) approach leads to the original Sinclair-Desgagné conditions, while the Jewitt (1988) approach would yield an alternative set of conditions. It would also be interesting to extend the approach in Brown et al. (1986), building on increasing marginal cost of effort and nonseparabilities between effort and income.

10. CONCLUSION

This paper has derived new multi-signal extensions of the two major sets of conditions justifying the first-order approach to moral hazard principal-agent problems. One set of conditions, i.e., that generalizing Jewitt’s (1988) Theorem 3, follows naturally from the state-space formulation of the agent’s problem. Unfortunately, these conditions, unlike those generalizing Rogerson’s (1985) conditions, depend upon the concavity, not only of the technology, but also of the payment schedule $s^*(\cdot)$.

One might consider this to be a drawback. However, there are interesting cases where the first-order approach should fail, since the payment schedule $s^*(\cdot)$ is often not concave. For example, managers are often rewarded with stock options, which introduce nonconcavities into their objective functions. Such managers may then face multiple local optima. This suggests, for example, that small changes in the environment might cause large changes in managers’ behavior (see Guesnerie and Laffont, 1978). Such possibilities should not be ruled out for mathematical convenience alone.

Nevertheless, if researchers want to ignore such problems, at least initially, and use the first-order approach even when the payment schedule may not be concave, then the generalizations of the CDGC approach, from Sections 5 and 6 above, can be used. However, the CDGC and its extensions must be very strong if they are able to yield concave agent objective functions, even when $s^*(\cdot)$ is not concave.
First, (17) clearly implies CISP, since $h(x; E)$ is nondecreasing in $x$ for $E$ an increasing set, and $\Pr(\tilde{x} \in E | a) = h^T(a; E)$. Thus, assume CISP and let $\varphi(\cdot)$ be nondecreasing. For $\alpha < \beta$ let

\[ (A1) \quad \varphi_{[\alpha, \beta]}(x) = \max(\alpha, \min(\beta, \varphi(x))). \]

This equals $\varphi(x)$ when $\alpha \leq \varphi(x) \leq \beta$, but equals $\alpha$ when $\varphi(x) < \alpha$ and equals $\beta$ when $\varphi(x) > \beta$. Now $\varphi_{[\alpha, \beta]}(\cdot)$ can be approximated uniformly by the sum

\[ (A2) \quad \varphi_{[\alpha, \beta]}^N(x) = \alpha + \sum_{i=1}^{N} \left[ (\beta - \alpha)/N \right] h(x; E_i), \]

where the sets

\[ (A3) \quad E_i = \{ x : \varphi(x) \geq \alpha + (\beta - \alpha)(i/N) \} \]

are increasing sets (note that (A2) essentially expresses the convex cone of increasing functions in terms of its extreme rays). Also, the transformation in (16) is continuous under the uniform norm. Thus, applying this transformation to (A2) indicates that $\varphi_{[\alpha, \beta]}^T(a)$ is approximated uniformly by

\[ (A4) \quad (\varphi_{[\alpha, \beta]}^N)^T(a) = \alpha + \sum_{i=1}^{N} \left[ (\beta - \alpha)/N \right] h^T(a; E_i). \]

Here $h^T(a; E_i) = \int h(x; E_i) dF(x | a) = \Pr(\tilde{x} \in E_i | a)$ is concave in $a$ by CISP, so $(\varphi_{[\alpha, \beta]}^N)^T(a)$ is concave in $a$ for all $N$. Thus $\varphi_{[\alpha, \beta]}^T(a)$ must also be concave in $a$. Next, letting $\alpha \to -\infty$, and using the monotone convergence theorem shows that $\varphi_{(-\infty, \beta]}^T(a)$ is concave in $a$. Similarly, $\varphi_{(-\infty, \infty)}^T(a) = \varphi_T(a)$ is concave in $a$. QED
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