Phenomenology at colliders (2)

P. Marage
Université Libre de Bruxelles
Egyptian School on High Energy Physics
BUE – Cairo – May 27 to June 4, 2009
Plan

I. INTRODUCTION AND MOTIVATION

II. STRUCTURE FUNCTIONS AND PARTON DISTRIBUTION FUNCTIONS
1. Deep inelastic scattering and structure functions
2. Quark parton model
3. Scaling violation
4. QCD evolution and DGLAP equations

III. FACTORIZATION THEOREMS; PDF PARAMETERISATIONS
1. Factorisation theorems
2. Drell-Yan production with CMS
3. Parton distribution function parameterisations
4. Parton distribution uncertainties
5. Some (of many) uncovered topics
II. Structure functions and parton distribution functions
II.1 Deep inelastic scattering and structure functions
1. **Rutherford formula**

scattering of spin 0 (α particle) by spin 0 nucleus (no recoil)

\[
\frac{d\sigma}{d\Omega} = Z^2 \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cdot F(q^2) = \int \rho(R) e^{iqR} d^3R
\]

form factor: extended target (Fourier transform of charge distribution)

2. **Mott formula**

scattering of spin 1/2 (electron) by spin 0 nucleus (neglecting electron mass)

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E} \cos^2(\theta/2)
\]

Rutherford electron spin recoil mass

forbidden

3. **Spin 1/2 – spin 1/2 point-like scattering (« e – µ »)**

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E} \left[ \cos^2(\theta/2) + \frac{Q^2}{2M^2} \sin^2(\theta/2) \right]
\]

Mott magnetic interaction (\(\sigma^{\mu\nu}\))

\[
= \frac{4\pi\alpha^2}{Q^4} \left( 1 - y + y^2 / 2 \right) \quad \text{with} \quad y = 1 - E'/E \cos^2(\theta/2)
\]

\[
Q^2 = -q^2 = 4EE' \sin^2(\theta/2) = p_T^2
\]

NB \(q^2 < 0\): scattering \(large \ p_T \leftrightarrow small \ transverse \ distance\)
4. Extended spin 1/2 target ; form factors

\[ M = \frac{4\pi\alpha^2}{Q^4} J_{\mu}^{(e)}(q) J_{\mu}^{(p)}(q) \]

with \[ J_{\mu}^{(e)} = \bar{u}(k') \gamma_\mu u(k) \]
\[ J_{\mu}^{(p)} = \bar{u}(p') \left[ F_1(q^2) \gamma_\mu + i \frac{\kappa}{2M} F_2(q^2) q_\nu \sigma^{\mu\nu} \right] u(p) \]

\[
\text{covariance : } \gamma_\mu \quad q_\mu \quad q_\nu \quad \sigma^{\mu\nu} \quad \oplus \quad \text{current conservation : } \partial_\mu J_{\mu}^{(p)}(x) = 0 \Rightarrow q_\mu J_{\mu}^{(p)}(q) = 0
\]

\[ F_1(q^2) \quad F_2(q^2) : 1 \text{ invariant variable } (+ \text{ trivial azimuthal angle}) \quad + \quad \text{CM energy } \sqrt{s} \]

In practice, combine \( F_1 \) and \( F_2 \) \( \rightarrow \text{ GE and GM "form factors"} \)

\( \rightarrow \text{Rosenbluth formula} \) for \( e \ p \) elastic scattering

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega} \text{(Mott)} \cdot \left[ \frac{G_E^2 + (Q^2/4M^2) G_M^2}{1 + Q^2/4M^2} + \frac{Q^2}{4M^2} G_M^2 g^2(\theta/2) \right]
\]

Experimentally : \( G_E = G_M \approx \frac{1}{(Q^2 + 0.71^2)^2} \) (dipole parameterisation)

i.e. \( 1/Q^8 \) compared to point-like target !

photon wave length decreases with \( Q^2 \) => probability to break proton increases
(the photon wave must embrace coherently the whole proton)
5. Deep inelastic scattering

\[ M \sim \left[ J_{\mu}^{(e)}(q) J_{\mu}^{(p)}(q) \right] + cc = L_{\mu\nu} W^{\mu\nu} \]

\[ L_{\mu\nu} = 2k_{\mu} k_{\nu} + 2k_{\mu} k_{\nu} - Q^2 g_{\mu\nu} \quad \text{ (for em interactions, \( \gamma \) exchange)} \]

\[ W^{\mu\nu} = -W_1 g^{\mu\nu} + W_2 p^{\mu} p^{\nu} + W_4 q^{\mu} q^{\nu} + W_5 (p^{\mu} q^{\nu} + q^{\mu} p^{\nu}) \]

Current conservation \( q_{\nu} W^{\mu\nu} = q_{\mu} W^{\mu\nu} = 0 \quad \Rightarrow \quad \text{only 2 of the 4 } W \text{ functions contribute: } W_1 \text{ and } W_2 \)

Proton dissociation \( \rightarrow \) one additional invariant \( W \) in addition to \( Q^2 \) (and \( s \))

\( \rightarrow \) \( W_{1,2}(Q^2, W) \) or any combination, in particular \( x = Q^2 / 2 p.q = Q^2 / 2M \) in proton rest frame

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E} \left[ W_2(v, Q^2) \cos^2(\theta/2) + 2W_1(v, Q^2) \sin^2(\theta/2) \right] \]

\( W_{1,2}(Q^2, W) : \text{physical observables} \) (measured quantities)
DIS cross section

\[ F_1(x, Q^2) = MW_1 \quad F_2(x, Q^2) = \nu W_2 \]

\[
\frac{d^2\sigma}{dx dy} = \frac{d^2\sigma}{dx dQ^2} \times s \frac{4\pi\alpha^2}{Q^4} \left[ (1 - y) F_2(x, Q^2) + \frac{y^2}{2} 2xF_1(x, Q^2) \right] \quad \text{em interaction : NC } \gamma \text{ exchange}
\]

\[
= \ldots \pm \frac{G_F^2}{8\pi^2} \frac{Q^4}{(1 + Q^2 / M^2)} \left[ y(1 - y / 2) xF_3(x, Q^2) \right] \quad \text{weak interaction : CC } W \text{ exchange}
\]

\[ F_1, F_2, F_3(x, Q^2) = \text{structure functions – physical observables (measured quantities)} \]

In the following, we shall concentrate on the structure function behaviour,

but don’t forget the \(1/Q^4\) factor in the cross section!
For completeness (bis)

"polarised cross sections" (with different $y$ dependences)

\[
\sigma_T = \frac{1}{2}(\sigma^+ + \sigma^-) = \frac{4\pi\alpha^2}{s} - 2 F_1 \\
\sigma_L = \sigma^0 = \frac{4\pi\alpha^2}{s x} (F_2 - 2 x F_1) \\
F_L = \frac{1}{2x} (F_2 - 2 x F_1)
\]

$\sigma_T$ and $\sigma_L$ can be separated by different dependences in $y$

Kinematical variables definitions and relations

\[
s = (p + k)^2 \\
Q^2 = -q^2 \\
\nu = \frac{p.q}{2M} \\
x = \frac{Q^2}{2M\nu} \\
y = \frac{p.q}{p.k}
\]

\[
W^2 = Q^2 (1/ x - 1) + M^2 = \frac{Q^2}{x} = y s \\
Q^2 = x y s \\
0 \leq x \leq 1 \\
0 \leq y \leq 1
\]

Laboratory frame \quad \nu = E - E' \quad y = \frac{\nu}{E} \\
Q^2 = 4EE' \sin^2(\theta/2) \quad y = 1 - E'/E \cos^2(\theta/2)

CM frame \quad 1 - y = \frac{1}{2}(1 + \cos \theta^*) \quad \rightarrow \text{controls helicity} : y = 1 \leftrightarrow \text{backward scattering, forbidden for long. photon}

NB At high $Q^2$ (HERA), also NC Z exchange + $\gamma-Z$ interference
II.2 Quark parton model
Scaling

ep scattering SLAC 1969, for sufficient energy and $Q^2$: observation of approximate «scaling» i.e. no strong $Q^2$ dependence of cross section (except for common $1/Q^4$)
**Point-like partons**

Compare

\[
\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4} \left[ \cos^2(\theta/2) + \frac{Q^2}{2M^2} \sin^2(\theta/2) \right] \delta(\nu - \frac{Q^2}{2M}) \quad (1) \text{ elastic scattering on point-like spin 1/2 target}
\]

\[
= \frac{4\alpha^2 E'^2}{Q^4} \left[ W_2(\nu, Q^2) \cos^2(\theta/2) + 2W_1(\nu, Q^2) \sin^2(\theta/2) \right] \quad (2) \text{ deep inelastic scattering}
\]

If DIS is in fact elastic scattering on spin 1/2 pointlike « partons » with charge e, momentum p, mass m,

then one has (using \(\delta(x/a) = a \delta(x)\))

\[
\nu W_2(\nu, Q^2) = e^2 \delta(1 - \frac{Q^2}{2m\nu}) \quad 2mW_1(\nu, Q^2) = e^2 \frac{Q^2}{2m\nu} \delta(1 - \frac{Q^2}{2m\nu}) \quad (3)
\]

→ scattering on partons explains scaling,

i.e. the fact that the structure functions \(W_{1,2}\) depend on one variable only:

\[x = \frac{Q^2}{2m\nu}\] – or equivalently \(W\) or \(\nu\) –

and not separately on \(Q^2\) and \(\nu\)
Interpretation of the $x$ variable

if masses and transverse momenta can be neglected

-> in the proton rest frame

quark : $(\xi M 0 0 0)$

photon : $(\nu 0 0 (Q^2+\nu^2)^{1/2})$ since $q^2 = -Q^2 = \nu^2 - p_g^2$

$\Rightarrow 2 \xi M \nu - Q^2 = m_q^2 = 0$

$x = \xi = \text{fraction of proton momentum carried by the quark}$

$x$ is the momentum fraction of the proton carried by the struck quark

Rigourously, in the Breit frame, i.e. where photon is purely space-like :

$$W^2 = 2M\nu \approx Q^2 / x$$
**Interpretation of the x variable (more rigourously)**

Let the hit quark carry be parallel to the proton and carry the fraction $\xi$ of the proton momentum $p$.

In the Breit frame, i.e. where photon is purely space-like:

\[
\begin{align*}
\text{initial state inv. mass} &= m_q^2 + (\xi p)^2 - (\xi p + q)^2 \\
&= m_q^2 + (\xi p)^2 - (\xi p)^2 - 2\xi p.q - q^2 = m_q^2 \quad \text{(final state quark)}
\end{align*}
\]

\[\Rightarrow \xi = \frac{Q^2}{2p.q} = x\]

$x$ is, in the Breit frame, the momentum fraction of the proton carried by the struck quark.

NB Breit frame is also called the « brick wall » frame:

\[Q^2 = 2\xi p.q \quad \Rightarrow \quad q = -2\xi p\]

More generally: $x = \text{fraction of proton momentum carried by the quark in IFM (infinite momentum frame)}$, where masses and transverse momenta can be neglected.

NB: $W^2 = Q^2 / x$
**Parton distribution functions**

*Incoherent* scattering on constituent partons, « frozen » in the proton by *time dilatation* (NB also longitudinal contraction):

parton-parton interaction time \( \sim \gamma / R_p \ll \) high energy \( \gamma p \) interaction time

hence:

\[
\sigma(e p) = \sum_i \int dx f_i(x) \sigma(e q_i) \quad (4)
\]

where \( f_i(x) = \) probability to find in the proton parton of species \( i \) carrying momentum fraction \( x \) (in IMF)

NB : \( f_i \) = valence + sea

\[
\begin{align*}
\nu W_2(\nu, Q^2) &= e^2 \delta(1 - \frac{Q^2}{2m\nu}) \\
MW_1(\nu, Q^2) &= e^2 \frac{Q^2}{2m\nu} \delta(1 - \frac{Q^2}{2m\nu})
\end{align*}
\]

\[
\begin{align*}
\rightarrow F_2(x) &= \sum_i e_i^2 x f_i(x) \\
\rightarrow F_1(x) &= \frac{1}{2x} F_2(x)
\end{align*}
\]

\( QPM \)

Structure functions depend only on \( x \)

cross section given by quark distributions \( f(x) \)

\[
\frac{d^2 \sigma}{dx dy} = \frac{2\pi\alpha^2}{Q^4} s \left[ 1 + (1 - y)^2 \right] \sum_i e_i^2 x f_i(x) \quad QPM
\]

Note that in QPM \( F_2 = 2xF_1 \) (Callan Gross relation)
**Parton distribution functions (bis)**

*Incoherent* scattering on constituent partons, « frozen » in the proton by time dilatation (NB also longitudinal contraction):

\[ \text{parton-parton interaction time } \sim \gamma / R_p \ll \text{high energy } \gamma \text{ p interaction time} \]

hence:

\[ \sigma(e \ p) = \sum_i \int dx \ f_i(x) \sigma(e \ q_i) \quad (4) \]

where \( f_i(x) \) = probability to find in the proton parton of species \( i \) carrying momentum fraction \( x \) (in IMF)

NB : \( f_i = \text{valence} + \text{sea} \)

Using \( p_i = x P_p \) and thus formally \( m = xM (= 0 \text{ in IMF !}) \), putting in \( (4) \) the \( W_{1,2} \) structure functions \( (3) \) and integrating over the \( \delta \) function, only an \( x \) dependence remains at high energy, high \( Q^2 \) (*DIS regime*)

\[ \nu W_2(\nu, Q^2) = e^2 \delta(1-\frac{Q^2}{2m\nu}) \quad \rightarrow \quad F_2(x) = \sum_i e_i^2 x f_i(x) \]

\[ MW_i(\nu, Q^2) = e^2 \frac{Q^2}{2m\nu} \delta(1-\frac{Q^2}{2m\nu}) \quad \rightarrow \quad F_i(x) = \frac{1}{2x} F_2(x) \]

\[ \frac{d^2\sigma}{dx\,dy} = \frac{2\pi\alpha^2}{Q^4} \left[ 1+(1-y)^2 \right] \sum_i e_i^2 x f_i(x) \quad \text{QPM} \]

Note that in QPM \( F_2 = 2xF_1 \) (*Callan Gross relation*)

→ measurement of \( F_L = 0 \) indicates that *partons* are massless spin 1/2 objects → identified with *quarks*

(Note also if quark spin were 0, \( \sigma_T = 0 – \text{cf. Mott formula} \)
**Sum rules, first pdf measurements**

\[ \sum_i \int_0^1 dx \times f_i(x) = 1 \quad \text{momentum conservation} \]

\[ \int_0^1 dx \left[ u(x) - \bar{u}(x) \right] = 2 \quad \int_0^1 dx \left[ d(x) - \bar{d}(x) \right] = 1 \quad \text{valence quarks} \]

\[ \int_0^1 dx \left[ s(x) - \bar{s}(x) \right] = 0 \quad \text{sea - idem for } c, b, t \]

\[ \frac{1}{x} F_2(x) = \sum_q e_q^2 \times f_q(x) \quad \text{proton } \leftrightarrow \text{ neutron: } u(x) \leftrightarrow d(x) \quad \text{(isospin)} \]

\[ \frac{1}{x} F_2^{ep}(x) = \frac{4}{9} u_v + \frac{1}{9} d_v + \left( \sum_{\text{SEA}} e_q^2 S(x) \right) \quad \frac{1}{x} F_2^{en}(x) = \frac{1}{9} u_v + \frac{4}{9} d_v + \left( \sum_{\text{SEA}} e_q^2 S(x) \right) \]

\[ \frac{F_2^{en}(x)}{F_2^{ep}(x)} \quad x \to 0 \to 1 \quad \frac{F_2^{en}(x)}{F_2^{ep}(x)} \quad x \to 1 \to \frac{u_v + 4d_v}{4u_v + d_v} \]

\[ \frac{1}{x} \left( F_2^{ep}(x) - F_2^{en}(x) \right) = \frac{1}{3} (u_v(x) - d_v(x)) \]
Fixed target electron and muon scattering on hydrogen and nuclei \( \rightarrow F_2 \) for \( p \) and \( n \)

Neutrino scattering \( \rightarrow F_2 \) and \( xF_3 \)

\[
F_2^{\nu+\bar{\nu}} = x \sum_q (q(x)+\bar{q}(x)) \\
F_2^{\nu-\bar{\nu}} = x \sum_d d^\nu(x) - x \sum_u u^\nu(x) \\
F_3^{\nu+\bar{\nu}} = \sum_q (q(x)\bar{q}(x)) = \sum_q q^\nu(x) \\
F_3^{\nu-\bar{\nu}} = \sum_d (d(x)+\bar{d}(x)) - \sum_u (u(x)+\bar{u}(x))
\]

Charm production by neutrinos \( \rightarrow \) strange sea

Using \( \int_0^1 dx F_2^{en}(x) \) and \( \int_0^1 dx F_2^{np}(x) \) \( \rightarrow \) gluons \( = 0.46 \) proton momentum

First determinations of pdf's

NB Remember that structure functions are observables, but pdf's are « theoretical » quantities

Measured quantities are cross sections / structure functions
http://durpdg.dur.ac.uk/hepdata/
II.3 Scaling violation
H1 and ZEUS Combined PDF Fit

\( \sigma_1(x, Q^2) \)

\( Q^2 = 0.05 \text{ GeV}^2 \)
\( Q^2 = 0.06 \text{ GeV}^2 \)
\( Q^2 = 0.09 \text{ GeV}^2 \)
\( Q^2 = 0.11 \text{ GeV}^2 \)

\( Q^2 = 0.15 \text{ GeV}^2 \)
\( Q^2 = 0.4 \text{ GeV}^2 \)
\( Q^2 = 1.2 \text{ GeV}^2 \)

\( Q^2 = 2 \text{ GeV}^2 \)
\( Q^2 = 2.7 \text{ GeV}^2 \)
\( Q^2 = 3.5 \text{ GeV}^2 \)
\( Q^2 = 4.5 \text{ GeV}^2 \)

\( Q^2 = 6.5 \text{ GeV}^2 \)
\( Q^2 = 8.5 \text{ GeV}^2 \)

\( Q^2 = 10 \text{ GeV}^2 \)
\( Q^2 = 18 \text{ GeV}^2 \)

\( Q^2 = 20 \text{ GeV}^2 \)
\( Q^2 = 25 \text{ GeV}^2 \)
\( Q^2 = 30 \text{ GeV}^2 \)

\( Q^2 = 40 \text{ GeV}^2 \)
\( Q^2 = 50 \text{ GeV}^2 \)
\( Q^2 = 65 \text{ GeV}^2 \)
\( Q^2 = 80 \text{ GeV}^2 \)

\( Q^2 = 100 \text{ GeV}^2 \)
\( Q^2 = 120 \text{ GeV}^2 \)

\( Q^2 = 300 \text{ GeV}^2 \)
\( Q^2 = 400 \text{ GeV}^2 \)
\( Q^2 = 500 \text{ GeV}^2 \)
\( Q^2 = 600 \text{ GeV}^2 \)
\( Q^2 = 800 \text{ GeV}^2 \)

\( Q^2 = 1000 \text{ GeV}^2 \)
\( Q^2 = 1200 \text{ GeV}^2 \)
\( Q^2 = 1500 \text{ GeV}^2 \)
\( Q^2 = 2000 \text{ GeV}^2 \)

\( Q^2 = 3000 \text{ GeV}^2 \)
\( Q^2 = 5000 \text{ GeV}^2 \)
\( Q^2 = 8000 \text{ GeV}^2 \)
\( Q^2 = 12000 \text{ GeV}^2 \)

\( Q^2 = 20000 \text{ GeV}^2 \)
\( Q^2 = 30000 \text{ GeV}^2 \)

\( Q^2 = 40000 \text{ GeV}^2 \)

\( Q^2 = 50000 \text{ GeV}^2 \)
\( Q^2 = 80000 \text{ GeV}^2 \)
\( Q^2 = 120000 \text{ GeV}^2 \)

\( Q^2 = 200000 \text{ GeV}^2 \)
\( Q^2 = 300000 \text{ GeV}^2 \)

\( Q^2 = 400000 \text{ GeV}^2 \)
\( Q^2 = 500000 \text{ GeV}^2 \)
\( Q^2 = 800000 \text{ GeV}^2 \)
\( Q^2 = 1200000 \text{ GeV}^2 \)

\( Q^2 = 2000000 \text{ GeV}^2 \)
\( Q^2 = 3000000 \text{ GeV}^2 \)

\( Q^2 = 4000000 \text{ GeV}^2 \)
\( Q^2 = 5000000 \text{ GeV}^2 \)
\( Q^2 = 8000000 \text{ GeV}^2 \)
\( Q^2 = 12000000 \text{ GeV}^2 \)

\( Q^2 = 20000000 \text{ GeV}^2 \)
\( Q^2 = 30000000 \text{ GeV}^2 \)

\( Q^2 = 40000000 \text{ GeV}^2 \)
\( Q^2 = 50000000 \text{ GeV}^2 \)
\( Q^2 = 80000000 \text{ GeV}^2 \)
\( Q^2 = 120000000 \text{ GeV}^2 \)

\( Q^2 = 200000000 \text{ GeV}^2 \)
\( Q^2 = 300000000 \text{ GeV}^2 \)

\( Q^2 = 400000000 \text{ GeV}^2 \)
\( Q^2 = 500000000 \text{ GeV}^2 \)
\( Q^2 = 800000000 \text{ GeV}^2 \)
\( Q^2 = 1200000000 \text{ GeV}^2 \)

\( Q^2 = 2000000000 \text{ GeV}^2 \)
\( Q^2 = 3000000000 \text{ GeV}^2 \)

\( Q^2 = 4000000000 \text{ GeV}^2 \)
\( Q^2 = 5000000000 \text{ GeV}^2 \)
\( Q^2 = 8000000000 \text{ GeV}^2 \)
\( Q^2 = 12000000000 \text{ GeV}^2 \)

\( Q^2 = 20000000000 \text{ GeV}^2 \)
\( Q^2 = 30000000000 \text{ GeV}^2 \)

\( Q^2 = 40000000000 \text{ GeV}^2 \)
\( Q^2 = 50000000000 \text{ GeV}^2 \)
\( Q^2 = 80000000000 \text{ GeV}^2 \)
\( Q^2 = 120000000000 \text{ GeV}^2 \)

\( Q^2 = 200000000000 \text{ GeV}^2 \)
\( Q^2 = 300000000000 \text{ GeV}^2 \)
Scaling violations

Q$^2$ evolution of structure functions

- Photon resolution improves with $Q^2$
  - $\rightarrow$ disentangles virtual gluon emission

As $Q^2$ increases,
- Quark content decreases at large $x$ (valence)
  - and increases at low $x$
- Also: at low $x$, the gluon content and the sea increase
  - (low $x$ since due to bremsstrahlung $\rightarrow$ soft)

**Parton distribution function evolutions**
Q$^2$ evolution of the cross section (structure functions)

quark evolution

H1 and ZEUS Combined PDF Fit

HERA Structure Functions Working Group

April 2009
gluon evolution
II.4 QCD evolution
DGLAP equations
**structure of the quark**

Gluon emission by the quark:

a quark « structure » shows up

Take over the SF formalism, with

\[ p \quad \rightarrow \quad p_i = \zeta p \]
\[ x = Q^2/2p.q \quad \rightarrow \quad z = Q^2/2p_i.q = x/\zeta \]

Hence

\[ 2F_1(x, Q^2) = \frac{\sigma_T(x, Q^2)}{\sigma_0} \bigg|_{\gamma^*p} = \sum \int_0^1 dz \int_0^1 d\xi f_i(\xi) \delta(x - \zeta z) \frac{\hat{\sigma}_T(z, Q^2)}{\hat{\sigma}_0} \bigg|_{\gamma^*\text{quark}} \]

where \( \sigma_0 = \frac{4\pi \alpha_s^2(Q^2)}{s} \) and similarly for \( \hat{\sigma}_0 \) with \( \hat{s} = \zeta s \)

\( f_i(\xi) \) is the probability to find in the proton a (« primary ») quark with momentum fraction \( \xi \),
\( \hat{\sigma}_T(z, Q^2) \) is the photon-quark transverse cross section,
for a (« secondary ») quark of momentum fraction \( z \);
\( \xi \) and \( z \) can vary from 0 to 1, but \( x = \zeta z \) is fixed (hence the \( \delta \) function)

After integration on \( z \):

\[ 2F_1(x, Q^2) = \sum \int_0^1 \frac{d\xi}{\zeta} f_i(\xi) \frac{\hat{\sigma}_T(x, \zeta, Q^2)}{\hat{\sigma}_0} \]
« structure of the quark » (bis)

Gluon emission by the quark:

a quark « structure » shows up

\[ z p_i = x p \]

\[ p_i = \xi p \]

NB 1. we consider hard gluon emission, over timescale comparable to interaction time \( \rightarrow \) large \( p_T \),

well separated jets \( \leftrightarrow \) soft gluon emission during hadronisation (see later)

2. « before » and « after » are frame dependent - the second diagram for gauge invariance

Take over the SF formalism, with proton \( \rightarrow \) quark

\[
p = p_i = \xi p
\]

\[
x = Q^2/2p.q
\]

\[
z = Q^2/2p_i.q = x/\xi
\]

Hence

\[
\frac{1}{x} F_2(x, Q^2) = 2F_1(x, Q^2) = \left. \frac{\sigma_T(x, Q^2)}{\sigma_0} \right|_{y^*p} = \sum_i \int_0^1 dz \int_0^1 d\xi f_i(\xi) \delta(x - \xi z) \frac{\hat{\sigma}_T(z, Q^2)}{\hat{\sigma}_0} y^*\text{quark}
\]

where \( \sigma_0 = \frac{4\pi \alpha_s^2(Q^2)}{\hat{s}} \) and similarly for \( \hat{\sigma}_0 \) with \( \hat{s} = \xi s \)

\( f_i(\xi) \) is the probability to find in the proton a (« primary ») quark with momentum fraction \( \xi \),

\( \hat{\sigma}_T(x, Q^2) \) is the photon-quark transverse cross section, for a (« secondary ») quark of momentum fraction \( z \),

\( \xi \) and \( z \) can vary from 0 to 1, but \( x = \xi z \) is fixed (hence the \( \delta \) function)

After integration on \( z \):

\[
2F_1(x, Q^2) = \sum_i \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \frac{\hat{\sigma}_T(x/\xi, Q^2)}{\hat{\sigma}_0}
\]
quark evolution equation

At first order: \( \gamma^* q \rightarrow q \) where \( z = x / \xi = 1 \)

Hence \( 2F_1(x,Q^2) = \sum_I \int_0^1 \frac{d\xi}{\xi} f_I(\xi) \frac{\hat{\sigma}_I(x/\xi,Q^2)}{\hat{\sigma}_0} \)

\[ \rightarrow \sum_I e_i^2 \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \delta(1 - \frac{x}{\xi}) = \sum_I e_i^2 f_i(x) \]

At next order, the photon quark cross section contains a \( \gamma^* q \rightarrow q g \) contribution

with for \[ \frac{d\hat{\sigma}}{dp_T^2} = e_q^2 \frac{1}{p_T} \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(z) \]

where \( P_{qq}(z) = \frac{4}{3} \left( \frac{1+z^2}{1-z} \right) \)

\( P_{qq}(z) \) is the probability of a quark emitting a gluon and reducing its momentum by the factor \( z \):

« splitting function »
DGLAP

Thus

$$\hat{\sigma}(\gamma^* q \rightarrow qg) = \int \frac{d^2}{\mu_F^2} \frac{d p_T^2}{d p_T^2} \frac{d \hat{\sigma}}{d p_T^2} = e_q^2 \hat{\sigma}_0 \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(z) \log \frac{Q^2}{\mu_F^2}$$

$$\mu_F = \text{cut off for } p_T \rightarrow 0$$

and

$$\frac{1}{x} F_2(x, Q^2) = \sum_q e_q^2 \int_x^1 \frac{d \xi}{\xi} q(\xi) \left( \delta(1 - \frac{x}{\xi}) + \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(\frac{x}{\xi}) \log \frac{Q^2}{\mu_F^2} \right)$$

logarithmic scaling violation

$$= \sum_q e_q^2 \left[ q(x) + \Delta q(x, Q^2) \right]$$

where log dependence is formally absorbed in quark density redefinition

Hence integro-differential **evolution equation** for quark distribution:

$$\frac{d q(x, Q^2)}{d \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{d \xi}{\xi} q(\xi, Q^2) P_{qq}(\frac{x}{\xi})$$
QCD evolution – DGLAP equations (bis)

Low Q$^2$ probes $x$, high Q$^2$ probes $x' \leq x$

$P_{qq}(z)$ is the probability of a quark emitting a gluon of quark momentum factor $z$ : « splitting function »

$2F_1(x,Q^2) = \frac{\sigma_T(x,Q^2)}{\sigma_0} \bigg|_{\gamma^* p} = \sum_T \int_0^1 d\xi \int_0^1 d\xi' f_i(\xi) \delta(x - \xi z) \frac{\hat{\sigma}_T(z,Q^2)}{\hat{\sigma}_0} \bigg|_{\gamma^* \text{quark}}$

One has for

$$\frac{d\hat{\sigma}}{dp_T^2} = e_q^2 \frac{\alpha_s(Q^2)}{2\pi} \frac{1}{p_T^2} P_{qq}(z)$$

where $P_{qq}(z) = \frac{4}{3} \left( \frac{1 + z^2}{1 - z} \right)$

$P_{qq}(z)$ is the probability of a quark emitting a gluon of quark momentum factor $z$ : « splitting function »

Hence

$$\hat{\sigma}(\gamma^* q \to qg) = \int_{\mu_F^2}^{S^2/4} dp_T^2 \frac{d\hat{\sigma}}{dp_T^2} = e_q^2 \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(z) \log \frac{Q^2}{\mu_F^2} \quad \mu_F = \text{cut off for } p_T \to 0 \text{ (see below)}$$

$$\frac{1}{x} F_2(x,Q^2) = \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi) \left( \delta(1 - \frac{x}{\xi}) + \frac{\alpha_s(Q^2)}{2\pi} P_{qq}(\frac{x}{\xi}) \log \frac{Q^2}{\mu_F^2} \right)$$

logarithmic scaling violation, formally absorbed in quark density redefinition

DGLAP integro-differential equations

$$\frac{dq(x,Q^2)}{d\log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} q(\xi,Q^2) P_{qq}(\frac{x}{\xi})$$
**DGLAP equations**

Similarly: quark in gluon $P_{qg}$

\[ \text{gluon in gluon } P_{qg} \]

**Notation**

\[ P_{ij} \otimes f_i(x, Q^2) = \int_x^1 \frac{d\xi}{\xi} P_{ij}(\frac{X}{\xi}) f_i(\xi, Q^2) \]

\[
\begin{align*}
\frac{dq(x, Q^2)}{d \log Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \left[ P_{qq} \otimes q(x, Q^2) + P_{qg} \otimes g(x, Q^2) \right] \\
\frac{dg(x, Q^2)}{d \log Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \left[ P_{gq} \otimes q(x, Q^2) + P_{gg} \otimes g(x, Q^2) \right]
\end{align*}
\]
Remarks

1. **DGLAP equations = Renormalisation group equations (RGE)**

\[ q(x, Q^2; \mu_F^2) = q(x) + \frac{\alpha_S(Q^2)}{2\pi} \log \frac{Q^2}{\mu_F^2} \int_x^1 \frac{d\xi}{\xi} P_{qq}(\frac{x}{\xi}) q(\xi) \]

Choice of factorisation scale \( \mu_F \) is arbitrary \( \Rightarrow q(x, Q^2) \) should not depend on \( \mu_F \):

\[ \frac{dq(x, Q^2; \mu_F^2)}{d\log \mu_F} = 0 \rightarrow \text{the DGLAP equations} \]

2. **Singularities in splitting functions**

Remember \( P_{qq} \) comes from

\[ + \]

\[ \] and is singular

\[ P_{qq}(z) = \frac{4}{3} \left( \frac{1 + z^2}{1 - z} \right) \]

But interference of virtual corrections with leading order diagram regularise the singularity in \( P_{qq} \)

3. **Higher orders**

NLO and NNLO splitting functions have been calculated. Very complicated!