A FEW REMARKS ON THE SUPREMUM OF STABLE PROCESSES

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Abstract. In [1], Bernyk et al. offer a power series and an integral representation for the density of S_1, the maximum up to time 1, of a regular spectrally positive α-stable Lévy process. They also state the asymptotic behavior for large values of the density. A fact which was proved by Doney [4], by investigating the integral representation. In this note, we provide the asymptotic expansion of the density and of its successive derivatives from the power series representation. We also show that the density of the positive random variable $S_1^{-\alpha}$ is the Laplace transform of a function which takes negative values on $\mathbb{R}^+$ and thus it is not completely monotone.

1. Introduction

Let X be a regular spectrally positive α-stable Lévy process starting from 0, which means that $\alpha \in (1,2)$ and the Lévy measure $\nu(dx) = x^{-\alpha-1}/\Gamma(-\alpha)dx, x > 0$, with Γ the gamma function. Let $S_t = \sup_{0<s\leq t} X_s$ be the maximum of X up to time $t > 0$.

In a recent paper [1], Bernyk et al. invert, in a non-obvious way, the double time and space Laplace transforms of S_t, which is obtained from the Wiener-Hopf factorization of X. More specifically, they show that the law of S_1 is absolutely continuous with a density, denoted by s, which admits the following power series representation

$$s(x) = x^{-2} \sum_{n=1}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n - 1)\Gamma(1 - n + \tilde{\alpha})}, \quad x > 0, \tag{1.1}$$

where we set $\tilde{\alpha} = \alpha^{-1}$. They also provide an integral representation for s. The law of S_t, for any $t > 0$, is then obtained by a scaling argument since $S_t \overset{(d)}{=} t^{\tilde{\alpha}} S_1$. Moreover, they state the following asymptotic behavior of the density

$$s(x) \sim \frac{x^{-1-\alpha}}{\Gamma(-\alpha)} \quad \text{as } x \to \infty, \tag{1.2}$$

where $f(x) \sim g(x)$ as $x \to \infty$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. A result which was proved by Doney in [4]. Therein, the author develops a detailed asymptotic analysis of the integral representation of s. Note that in [5], the asymptotic behavior of the successive derivatives of the density s is also provided. Although the following asymptotic of the distribution function of S_1

$$\mathbb{P}(S_1 > x) \sim -\frac{x^{-\alpha}}{\Gamma(1-\alpha)} \quad \text{as } x \to \infty,$$

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can be found in [2, Proposition 4], as mentioned in [4], one can not use classical results from asymptotic analysis to deduce (1.2). Indeed, we do not know whether s is ultimately monotone or not. The first aim of this note is to express the density s in (1.1) in terms of a Wright hypergeometric function in order to use general results on this class of functions to obtain the asymptotic expansion. Then, we show that the density of the positive random variable $S_1^{-\alpha}$ is the Laplace transform of a function which takes negative values on $\mathbb{R}^+$ and thus it is not completely monotone.

2. **Main results**

We start by introducing the so-called Wright hypergeometric functions which admit the following power series representation

$$ {}_p\Psi_q\left(\begin{array}{c} (A_1,a_1),\dots(A_p,a_p) \\ (B_1,b_1)\dots(B_q,b_q) \end{array} \middle| \, z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(A_i n + a_i)}{\prod_{i=1}^{q}\Gamma(B_i n + b_i)} \frac{z^n}{n!} $$

where p, q are nonnegative integers, $a_i \in \mathbb{C}$ $(i = 1\dots p), b_j \in \mathbb{C}$ $(j = 1\dots q)$, the coefficients $A_i \in \mathbb{R}^+$ $(i = 1\dots p)$ and $B_j \in \mathbb{R}^+$ $(j = 1\dots q)$ are such that

$$ A_i n + a_i \neq 0, -1, -2, \dots \qquad (i = 0, 1, \dots, p; n = 0, 1, \dots). $$

A very nice account of this class of functions is given in the (lengthy) paper of Braaksma [3]. In particular, if we assume that

$$ S = 1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i > 0 $$

the series is convergent for all values of $z \in \mathbb{C}$. We proceed by recalling a general result regarding the asymptotic expansion of this class of function which can be found in [3, Theorem 19, p. 330]. To this end, we introduce the following function

$$ P(z) \;=\; \sum_{s\in R_p} z^s \Gamma(-s) Res \left(\frac{\prod_{i=1}^{p}\Gamma(A_i s + a_i)}{\prod_{i=1}^{q}\Gamma(B_i s + b_i)} \right) $$

where Res stands for residuum and we have set $R_p = \{r_{i,n} = -\frac{a_i+n}{A_i}, i = 0, 1, \dots, p; n = 0, 1, \dots\}$. Next, suppose $S > 0$ and $p > 0$. Then, the following algebraic asymptotic expansion

$$ {}_p\Psi_q(z) = P(-z) \tag{2.1} $$

holds for $\mid z \mid \to \infty$ uniformly on every closed subsector of $\mid \arg(-z) \mid < \left(1 - \frac{S}{2}\right)\pi$. We also mention the following notation, for any $z \in \mathbb{C}$,

$$ {}_p\Psi_q(-z) = {}_p\Psi_q(e^{i\pi} z). $$

We are now ready to state the following which is simply a reformulation of the main results obtained in [1] and [4].

Proposition 2.1. *For any $x > 0$ and $1 < \alpha < 2$, the density s admits the following representation*

$$ s(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2} {}_2\Psi_1\left(\begin{array}{c} (1,1),(1,1-\tilde{\alpha}) \\ (\alpha,\alpha-1) \end{array} \middle| \, -x^{\alpha} \right). $$

We also have the following asymptotic expansion, as $x \to \infty$,

$$ s(x) \;=\; \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{-2} \left(\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(-n-\tilde{\alpha})}{\Gamma(-\alpha n - 1)} x^{-\alpha n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(n+1-\tilde{\alpha})\Gamma(-n+\tilde{\alpha})}{\Gamma(-\alpha n)} x^{-\alpha n + 1} \right). $$

In particular, (1.2) *holds.*

Moreover, denoting by $s^{(m)} = \frac{d^m}{dx^m}s$, $m = 1, 2\ldots$, *the successive derivatives of* s, *we have, for any* $x > 0$,

$$s^{(m)}(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2-m} {}_2\Psi_1 \left(\begin{array}{c} (1,1),(1,1-\tilde{\alpha}) \\ (\alpha,\alpha-1-m) \end{array} \middle| -x^{\alpha} \right),$$

and, as $x \to \infty$,

$$s^{(m)}(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{-2-m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(-n-\tilde{\alpha})}{\Gamma(-\alpha n - 1 - m)} x^{-\alpha n} + \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n+1-\tilde{\alpha})\Gamma(-n+\tilde{\alpha})}{n!\Gamma(-\alpha n - m)} x^{-\alpha n + 1} \right).$$

In particular,

$$s^{(m)}(x) \sim \frac{x^{-\alpha-1-m}}{\Gamma(-\alpha-m)} \quad \textit{as } x \to \infty.$$

Proof. From the expression (1.1) of s and using Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$, $\mathfrak{Re}(z) > 0$, see e.g.[6, 1.2.3], one gets readily, for any $x > 0$, that

$$\begin{aligned} s(x) &= \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1-\tilde{\alpha})}{\Gamma(\alpha n + \alpha - 1)} x^{\alpha n} \\ &= \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2} {}_2\Psi_1 \left(\begin{array}{c} (1,1),(1,1-\tilde{\alpha}) \\ (\alpha,\alpha-1) \end{array} \middle| -x^{\alpha} \right). \end{aligned}$$

Next, note that $S = \alpha - 1 > 0$, $p = 2$ and

$$R_2 = \{r_{1,n} = -(1+n),\ r_{2,n} = -(1+n-\tilde{\alpha}); n = 0, 1, \ldots\}.$$

Then, recalling that for any integer n, $Res_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}$, one obtains, from (2.1), the following asymptotic expansion

$$s(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2} \left(\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(-n-\tilde{\alpha})}{\Gamma(-\alpha n - 1)} x^{-\alpha(n+1)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(n+1-\tilde{\alpha})\Gamma(-n+\tilde{\alpha})}{\Gamma(-\alpha n)} x^{-\alpha(n+1)+1} \right)$$

where we have used the fact that $0, -1$ are poles of the Gamma function. In particular, as $x \to \infty$,

$$\begin{aligned} s(x) &\sim -\frac{\sin(\tilde{\alpha}\pi)}{\pi} \frac{\Gamma(2-\tilde{\alpha})\Gamma(\tilde{\alpha}-1)}{\Gamma(-\alpha)} x^{-\alpha-1} \\ &\sim \frac{x^{-\alpha-1}}{\Gamma(-\alpha)} \end{aligned}$$

where we have used Euler's reflection formula. The proof of the statements for the density s is then completed. For the successive derivatives, let us start with the case $m = 1$. Then, recalling that s is expressed in terms of an absolutely convergent series, one may differentiate its expression terms by terms to get, by means of the recurrence relation of the Gamma function,

$$s^{(1)}(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2-1} {}_2\Psi_1 \left(\begin{array}{c} (1,1),(1,1-\tilde{\alpha}) \\ (\alpha,\alpha-1-1) \end{array} \middle| -x^{\alpha} \right).$$

By an induction argument one easily deduces the expression of $s^{(m)}$ for any $m = 2, 3 \ldots$. The proof is completed by following the same pattern as for the density. □

A nice feature of some class of Wright functions is their stability with respect to the Laplace transform operator, i.e. the Laplace transform of some Wright functions gives rise to another Wright functions. Thus, a natural question when dealing with a density expressed in terms of such a function is to check whether it is completely monotone or not, i.e., by Bernstein's theorem, it is the Laplace transform of a non-negative valued function. Note that this problem is closely related to the existence of real zeros for this former function. Finally, we mention that if a density is completely monotone then it is log-convex and hence infinitely divisible, see e.g. [7, Theorem 51.6].

Now, let us denote the density of the positive random variable $S_1^{-\alpha}$ by $s_{-\alpha}$. Thus, one has $s_{-\alpha}(x) = \frac{1}{\alpha}x^{-\tilde{\alpha}-1}s(x^{-\tilde{\alpha}})$, $x > 0$, which means

$$\begin{aligned} s_{-\alpha}(x) &= \frac{\sin(\tilde{\alpha}\pi)}{\alpha\pi}x^{\tilde{\alpha}-2}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+1-\tilde{\alpha})}{\Gamma(\alpha n+\alpha-1)}x^{-n} \\ &= \frac{\sin(\tilde{\alpha}\pi)}{\alpha\pi}x^{\tilde{\alpha}-2}{}_2\Psi_1\left(\begin{array}{c}(1,1),(1,1-\tilde{\alpha})\\(\alpha,\alpha-1)\end{array}\middle|\,-x^{-1}\right). \end{aligned}$$

We also need the mapping $\hat{s}_{-\alpha}$ defined by

$$\widehat{s}_{-\alpha}(v) = \frac{\sin(\tilde{\alpha}\pi)}{\pi}v^{1-\tilde{\alpha}}{}_1\Psi_1\left(\begin{array}{c}(1,1)\\(\alpha,\alpha)\end{array}\middle|\,-v\right).$$

Note that the mapping $z \longrightarrow {}_1\Psi_1\left(\begin{array}{c}(1,1)\\(\alpha,\alpha)\end{array}\middle|\,z\right)$ is an entire function and is a specific instance of the generalized Mittag-Leffler functions. We are now ready to state the following fact.

Proposition 2.2. *The density $s_{-\alpha}$ is the Laplace transform of the function $\widehat{s}_{-\alpha}$ which takes negative values on $\mathbb{R}^+$. Hence, it is not completely monotone.*

Proof. First, note, from (2.1), that the function $\widehat{s}_{-\alpha}$, admits the following asymptotic behaviors, as $v \longrightarrow \infty$,

$$\begin{aligned} \widehat{s}_{-\alpha}(v) &= \frac{\sin(\tilde{\alpha}\pi)}{\pi}v^{1-\tilde{\alpha}}\sum_{n=1}^{\infty}(-1)^n\frac{1}{\Gamma(-\alpha n)}v^{-(n+1)} \\ &\sim -\frac{\sin(\tilde{\alpha}\pi)}{\pi}\frac{v^{-\tilde{\alpha}-1}}{\Gamma(-\alpha)}, \end{aligned} \tag{2.2}$$

and as $v \longrightarrow 0$

$$\widehat{s}_{-\alpha}(v) \sim \frac{\sin(\tilde{\alpha}\pi)}{\pi}\frac{v^{1-\tilde{\alpha}}}{\Gamma(\alpha)}. \tag{2.3}$$

Thus, since $\tilde{\alpha} \in (\frac{1}{2}, 1)$, for any $x > 0$, the mapping $v \mapsto e^{-vx}$ is integrable with respect to the measure $\widehat{s}_{-\alpha}(v)dv$. Next, recall the identity, for any $\mathfrak{Re}(\beta) > 0$ and $x > 0$,

$$x^{-\beta} = \frac{\int_0^{\infty} e^{-ux}u^{\beta-1}du}{\Gamma(\beta)}$$

and the recurrence relation $\Gamma(z+1) = z\Gamma(z)$. Thus, by means of the analycity of the mapping $z \longrightarrow {}_2\Psi_1\left(\begin{matrix} (1,1),(1,1-\tilde{\alpha}) \\ (\alpha,\alpha-1) \end{matrix} \middle| \, z \right)$ and a dominated convergence argument, one gets

$$\begin{aligned}
\frac{\pi}{\sin(\tilde{\alpha}\pi)} s_{-\alpha}(x) &= \frac{1}{\alpha}\sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(n+1-\tilde{\alpha})}{\Gamma(\alpha n+\alpha-1)} x^{\tilde{\alpha}-2-n} \\
&= \frac{1}{\alpha}\sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(n+1-\tilde{\alpha})\int_0^\infty e^{-xv} v^{n+1-\tilde{\alpha}} dv}{\Gamma(\alpha n+\alpha-1)\Gamma(n+2-\tilde{\alpha})} \\
&= \int_0^\infty e^{-xv} \sum_{n=0}^{\infty}(-1)^n \frac{v^{n+1-\tilde{\alpha}}}{(\alpha n+\alpha-1)\Gamma(\alpha n+\alpha-1)} dv \\
&= \int_0^\infty e^{-xv} \sum_{n=0}^{\infty}(-1)^n \frac{v^{n+1-\tilde{\alpha}}}{\Gamma(\alpha n+\alpha)} dv \\
&= \int_0^\infty e^{-xv} \widehat{s}_{-\alpha}(v) dv.
\end{aligned}$$

Thus, $\widehat{s}_{-\alpha}$ is the Laplace transform of $s_{-\alpha}$. However, after checking easily that for $\alpha \in (1,2)$, $\frac{\sin(\tilde{\alpha}\pi)}{\pi}\frac{1}{\Gamma(-\alpha)} > 0$ and $\frac{\sin(\tilde{\alpha}\pi)}{\pi}\frac{1}{\Gamma(\alpha)} > 0$ we deduce from the asymptotic behaviors (2.2) and (2.3) that $\widehat{s}_{-\alpha}$ takes negative values on $\mathbb{R}^+$. By Bernstein's theorem, $s_{-\alpha}$ is not completely monotone. $\square$

We end up this note with the following remark. In the limit case $\alpha = 2$, the function $\widehat{s}_{-2}$ can be expressed in terms of the modified Bessel function of the first kind, which is known to have positive zeros, see e.g. [6, 5.13]. Hence, the density of S_1^{-2}, which is the inverse-gaussian density, is not completely monotone either.

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