# BOUNDARY CROSSING IDENTITIES FOR DIFFUSIONS HAVING THE TIME INVERSION PROPERTY

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Abstract. We review and study a one-parameter functional transformation, denoted by $(S^{(\beta)})_{\beta\in\mathbb{R}}$, which allow, in the case $\beta<0$, to provide a path realization of bridges of the family of diffusion processes which enjoy the time inversion property. This family includes the Brownian motion, the family of Bessel processes with a positive dimension and their conservative $h$-transforms. By means of these transformations, we derive an explicit and simple expression which relates the law of the boundary crossing times for these diffusions over a given function $f$ to those over the image of $f$ by the mapping $S^{(\beta)}$, for some fixed $\beta\in\mathbb{R}$. We give some new examples of the boundary crossing problems for the Brownian motion and the family of Bessel processes. In the case of a Brownian motion, we also provide an interpretation of the results obtained by the standard method of images and establish connections between the exact asymptotics for large time of the densities corresponding to various curves of each family.

## 1. Introduction

Let $X:=(X_t,t\geq 0)$ be a $E$-valued, where $E=\mathbb{R}$ or $[0,\infty)$, 2-self-similar conservative homogenous diffusion enjoying the time inversion property in the sense of Shiga and Watanabe [37]. Denoting by $(\mathbb{P}_x)_{x\in E}$ the family of probability measures of X which act on $\mathcal{C}(\mathbb{R}_+,E)$, the space of continuous functions from $\mathbb{R}_+$ into $E$, such that $\mathbb{P}_x(X_0=x)=1$, it means that, for any $x\in E$ and $c>0$, the law of the process

$$(c^{-1}X_{c^2t},t\geq 0;\mathbb{P}_{cx}) \text{ is } \mathbb{P}_x \qquad \text{(2-self-similarity)}$$

and the process

$$i(X):=(tX_{1/t},t>0;\mathbb{P}_x) \qquad \text{(time inversion)} \tag{1}$$

is a conservative diffusion. The standard notation $\mathcal{F}_t$ is used for the $\sigma$-algebra generated by the process $X$ up to time $t$ and we write simply $\mathcal{F}=\mathcal{F}_\infty$. It is well-known that the class of processes considered therein consists of Brownian motion and Bessel processes of positive dimensions, see Watanabe [39]. Next, let $f\in\mathcal{C}(\mathbb{R}_+,E)$, such that $f(0)\neq X_0$ and set

$$T^f=\inf\{s>0;\ X_s=f(s)\}$$

with the usual convention $\inf\{\emptyset\}=+\infty$. The study of the distribution of the stopping time $T^f$ is known as the boundary crossing or first passage problem. Unfortunately, the

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explicit determination of these distributions is only attainable for a few specific functions. In this paper, we suggest a new method which allows to relate, in a simple and explicit manner, the law of $T^f$ with the ones of the family of stopping times $(T^{f^{(\beta)}})_{\beta\in\mathbb{R}}$ with $f^{(\beta)} := S^{(\beta)}(f)$ and

$$(2)\qquad \begin{array}{rrcl} S^{(\beta)}: & \mathcal{C}(\mathbb{R}_+, E) & \to & \mathcal{C}([0,\zeta^{(\beta)}), E) \\ & f & \to & (1+\beta.)\, f\left(\dfrac{.}{1+\beta.}\right) \end{array}$$

where we set $\zeta^{(\beta)} = 1/|\beta|$ if $\beta < 0$ and equals $+\infty$ otherwise. The results extend to $h$-transforms of the prescribed processes leading to conservative diffusions.

The motivations for such investigations are of both practical and theoretical importance. Indeed, on the one hand, such studies were originally motivated by their connections to sequential analysis, see e.g. Robbins and Siegmund [35] and Smirnov-Kolmogorov test, see e.g. Lerche [25]. On the other hand, this problem has found many applications in several fields of sciences, such as mathematical physics, see Zambrini and Lescot [26], neurology, see Lanski and Sacerdote [22], epidemiology, see Martin-Löf [27] and mathematical finance, see Bielecki et al. [5].

Amongst general results, in the Brownian setting, Strassen [38] proved that if $f$ is continuously differentiable then the distribution of $T^f$ is absolutely continuous with a continuous probability density function. Moreover, for elementary curves, the density is known explicitly for linear, square root and parabolic functions. For these curves, specific technics proved to be efficient and we refer to Table 1 in Section 2 for a description of these cases. Besides, the distribution is known for some concave curves solving implicit equations obtained by the celebrated standard method of images, see Lerche [25]. For some recent investigations, we refer to Pötzelberger and Wang [33] and Borovkov and Novikov [7] for numerical approximations of the density, Peskir [30], [31] for the study of the small time behavior of the density and Kendall et al. [20] for statistical applications.

The remaining part of the paper is organized as follows. In the next Section we recall some recent results regarding diffusions which enjoy the time inversion property. Section 3 is devoted to the statement of our main results which are proved in Section 4. Section 5 is concerned with a detailed study of the Brownian case. In particular, we show how our results translate and agree with the standard method of images. Finally, in Section 6, we treat the example of Bessel processes and characterize the distribution of hitting times of straight lines, which completes the results of [32]. We also mention that, in the Brownian motion case, the boundary crossing identity (5), stated below, has been published without proofs in the note [1].

## 2. Preliminaries

Let us recall that $E$ is either $[0,\infty)$ or $\mathbb{R}$. We recall that $X := (X_t, t \geq 0)$ is a 2-self-similar conservative homogenous diffusion enjoying the time inversion property. In recent papers, Gallardo and Yor [17] and Lawi [23] characterized the class of self-similar Markov processes (which might have càdlàg paths) satisfying the time inversion property in terms of their semi-groups when the latter are assumed to be absolutely continuous and twice differentiable. More precisely, they showed that if we write $\mathbb{P}_x(X_t \in dy) = p_t(x,y)dy$, for some $t>0$ and all $x$ and $y\in E$, the transitions densities have the following form

$$p_t(x,y)=\frac{c}{\sqrt{t}}\Phi\left(\frac{xy}{t}\right)\left(\frac{y}{\sqrt{t}}\right)^{2\nu+1}e^{-\frac{x^2+y^2}{2t}}$$

where the reals $\nu$ and $c$ are related to the function $\Phi:E\to\mathbb{R}_+$ as follows.

(1) If $E=\mathbb{R}$, then $X$ is a Brownian motion, $\Phi(y)=e^y$, $y\in\mathbb{R}$, and necessarily $\nu=-1/2$ and $c=1/\sqrt{2\pi}$.

(2) If $E=[0,\infty)$ then $X$ is Bessel process of dimension $\delta>0$, $c=1$ and $\Phi(y)=y^{-\nu}I_\nu(y)$, $y\in\mathbb{R}_+$, for some $\nu=\frac{\delta}{2}-1$, where $I_\nu$ is the modified Bessel function of the first kind of index $\nu$ which admits the following power series representation, see e.g. [24, Section 5.2],

$$I_\nu(z)=\left(\frac{y}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)},\ |z|<\infty,\ |\arg(z)|<\pi.$$

Moreover, it is well-known that, in this case, the point 0 is
- an entrance boundary if $\delta\geq 2$,
- a reflecting boundary if $0<\delta<2$.

Next, recalling that, for every fixed $y\in E$, the mapping $x\mapsto\Phi(yx)$ is $\frac{y^2}{2}$-excessive, we can define a new family of probability measures as follows

$$\mathrm{d}\mathbb{P}_x^{(y)}{}_{|\mathcal{F}_t}=\frac{\Phi(yX_t)}{\Phi(yx)}e^{-\frac{1}{2}y^2t}\mathrm{d}\mathbb{P}_{x|\mathcal{F}_t}\quad t>0. \tag{3}$$

Note from the definition of $\Phi$ that the laws $\mathbb{P}_x^{(0)}$ and $\mathbb{P}_x$ coincide on $\mathcal{C}(\mathbb{R}^+,E)$. Also, observing that, for any $y\in E$, $\Phi(y)>0$, then, under $\mathbb{P}_x^{(y)}$, $X$ is conservative. Moreover, in [17], the authors showed that under $\mathbb{P}_x^{(y)}$, $X$ satisfies as-well the time inversion property (1) and $(\mathbb{P}_x^{(y)})_{x\in E}$ is the family of probability measures induced by $i$, under $\mathbb{P}_y$ for some $y\in E$. It is then plain that $i(X)$ has an absolutely continuous semi-group with densities given by

$$\begin{aligned}p_t^{(y)}(x,a) &= \frac{1}{\sqrt{t}}\frac{\Phi(ya)}{\Phi(yx)}\Phi\left(\frac{ax}{t}\right)\left(\frac{a}{\sqrt{t}}\right)^{2\nu+1}\\ &\times\ \exp-\frac{1}{2}\left(ty^2+(x^2+a^2)/t\right),\ x,a\in E.\end{aligned}$$

Note that

(1) if $E=\mathbb{R}$, then, under $\mathbb{P}_x^{(y)}$, $X$ is a Brownian motion with drift $y$,

(2) if $E=[0,\infty)$ then, under $\mathbb{P}_x^{(y)}$, $X$ is the so-called Bessel process in the wide sense introduced by Watanabe [39].

Next, the family of mappings $(S^{(\beta)})_{\beta<0}$, defined in (2), naturally appears in the process of construction of bridges associated to $X$. To explain the connection, for any fixed $T>0$ and $x\in E$, let us denote by $(\mathbb{P}_{x,z}^T, z\in E)$ a regular version of the family of conditional probability distributions $(\mathbb{P}_x(.|X_T=z), z\in E)$. That is, for some $z\in E$, the law of the bridge associated to $X$, under $\mathbb{P}_x$, between $x$ and $z$ over the time interval $[0,T]$. Fitzsimmons [16] observed that these laws remain invariant under changes of probability of type (3). From Pitman and Yor [32, Theorem 5.8], see also [17, Theorem 1], we know that, for any $x$, $z \in E$, the processes

$$\{X_u, 0 \le u < T; \mathbb{P}_x^{(y)} | X_T = Tz\} \text{ and } \{S^{(-1/T)}(X)_u, 0 \le u < T; \mathbb{P}_x^{(z)}\} \tag{4}$$

have the same law. This fact, when $X$ is a Brownian motion, can also be seen from the unique decomposition, as a semi-martingale in its own filtration, of the Brownian bridge of length $T = -\frac{1}{\beta} > 0$, between 0 and 0. Indeed, for any $0 \le t < T$, we can write

$$S^{(-1/T)}(X)_t = (T-t)\int_0^t \frac{d\tilde{X}_s}{T-s}, \quad X_t = T\int_0^{\frac{Tt}{T-t}} \frac{d\tilde{X}_s}{T-s}, \quad t < T,$$

where $\tilde{X}$ is a Brownian motion on $[0, \zeta^{(-\beta)}]$ with respect to the filtration generated by $(S^{(-\frac{1}{T})}(X)_t, 0 \le t < T)$. Thus, we have

$$S^{(-1/T)}(X)_t = \tilde{X}_t - \int_0^t \frac{S^{(-1/T)}(X)_s}{T-s} ds, \quad t < T.$$

which coincides with the canonical decomposition of the standard Brownian bridge. The previous two relationships extend to the case $\beta > 0$ and the analogue of the Brownian bridges is then a family of Gauss-Markov processes of Ornstein-Uhlenbeck type. We shed light on this extension in Lemma 4.2 of Section 4 below.

## 3. Main results

We keep the notations and setting of the previous section. We proceed by pointing out that the mapping $S^{(\beta)}$ can be defined similarly on the space of probability measures. For instance, in the absolutely continuous case, we associate to $\mu(dt) = h(t)dt$ the image $S^{(\beta)}(\mu)(dt) = h^{(\beta)}(t)dt$ where we recall that $h^{(\beta)}(t) := S^{(\beta)}(h)(t)$. We are now ready to state our main result.

**Theorem 3.1.** *Let $f \in \mathcal{C}(\mathbb{R}_+, E)$. Then, for any $x$, $y \in E$ such that $f(0) \neq x$, and $t < \zeta^{(\beta)}$, we have*

$$\begin{aligned} \mathbb{P}_x^{(y)}(T^{f^{(\beta)}} \in dt) &= (1+\beta t)^{\nu-2} \frac{\Phi(yf^{(\beta)}(t))}{\Phi(yf^{(\beta)}(t)/(1+\beta t))} e^{-\frac{\beta}{2}\frac{y^2t^2}{1+\beta t}} e^{-\frac{\beta}{2}\frac{f^{(\beta)2}(t)}{1+\beta t} + \frac{\beta}{2}x^2} \\ &\times \; S^{(\beta)}\left(\mathbb{P}_x^{(y)}(T^f \in dt)\right). \end{aligned}$$

*The particular case $y = 0$ yields*

$$\mathbb{P}_x\left(T^{f^{(\beta)}} \in dt\right) = (1+\beta t)^{\nu-2} e^{-\frac{\beta}{2}\frac{f^{(\beta)2}(t)}{1+\beta t} + \frac{\beta}{2}x^2} \times S^{(\beta)}\left(\mathbb{P}_x\left(T^f \in dt\right)\right). \tag{5}$$

**Remark 3.2.** Assuming that $X$ is a 2-self-similar strong Markov process with càdlàg paths (i.e. with possible jumps) enjoying the time inversion property, then Theorem 5.8 in [32] or Theorem 1 in [17] allow to extend Lemma 4.1 and Lemma 4.2 of Section 4. An analogue of Theorem 3.1 of Section 3 can be stated. Although, in [29], the second author characterizes, through its Mellin transform, the law of the first passage time above the square root boundary for spectrally negative positive 2-self-similar Markov processes, we did not succeed to construct examples of $\mathbb{R}_+$-valued processes enjoying the time inversion property other than Bessel processes in the usual or wide sense. This is the reason why the setting is restricted to the continuous one.

Theorem 3.1 considerably simplifies when the focus is on straight lines. Indeed, if we consider a constant function $f \equiv a$ where $a \neq 0$, then with $\beta = b/a$, for some $b \in \mathbb{R}$, we have $f^{(b/a)}(t) = a + bt$, $t < \zeta^{(b/a)}$. Note that if $b < 0$ then the support of $T^{a+b\cdot}$ is $(0, -b/a)$. The previous result reads as follows.

**Corollary 3.3.** *Let $a, b, x \in E$ such that $a \neq x$. Then, we have, for any $t < \zeta^{(b/a)}$,*

$$\begin{aligned} \mathbb{P}_x^{(y)}\left(T^{a+b\cdot} \in dt\right) &= (1+bt/a)^{\nu-2}\frac{\Phi(y(a+bt))}{\Phi(ya)} \exp -\frac{b}{2}\left(a+bt+\frac{t^2y^2}{a+bt}-\frac{x^2}{a}\right) \\ &\times \ S^{(\beta)}\left(\mathbb{P}_x^{(y)}(T^a \in dt)\right). \end{aligned}$$

*In particular, when $y = 0$, we obtain*

$$\begin{aligned} \mathbb{P}_x\left(T^{a+b\cdot} \in dt\right) &= (1+bt/a)^{\nu-2} \exp -\frac{b}{2}(a+bt-x^2/a) \\ &\times \ S^{(\beta)}\left(\mathbb{P}_x(T^a \in dt)\right). \end{aligned}$$

Corollary 3.3 follows readily from Theorem 3.1 and its proof is omitted. It seems natural to examine the images of some curves by the mapping $S^{(\beta)}$. We gathered, in Table 1, the images of the most studied curves in the Brownian setting. If we set $b = 0$ in row (1) then

| | $f$ | $f^{(\beta)}$ | References |
|---|---|---|---|
| (1) | $a + bt$ | $a + (b + a\beta)t$ | [4] |
| (2) | $\sqrt{1+2bt}$ | $\sqrt{(1+\beta t)(1+(\beta+2b)t)}$ | [8], [28] |
| (3) | $a + bt^2$, $ab > 0$ | $a(1+\beta t) + bt^2/(1+\beta t)$ | [18], [36] |
| (4) | $\frac{a}{2} - \frac{t}{a}\ln\left(\frac{b+\sqrt{b^2+4b_1e^{-\frac{a^2}{t}}}}{2}\right)$ | $\frac{a(1+\beta t)}{2} - \frac{t}{a}\ln\left(\frac{b+\sqrt{b^2+4\hat{b}_1e^{-\frac{a^2}{t}}}}{2}\right)$ | [9] |

TABLE 1. Image by the mapping $S^{(\beta)}$, where $a, b \in \mathbb{R}$ ( $b_1 > -b^2/4$ and $\hat{b} = b_1e^{-a^2\beta}$ in (4)), of well-known curves $f$ and references to the papers where an expression of the law of $T^f$ can be found.

our calculations lead to the Bachelier-Lévy formula recalled in the beginning of Section 5. The fact that the transformation $S^{(\beta)}$, for some $\beta > 0$, preserves straight lines is well-known, see for instance [32]. The family $(S^{(\beta)})_{\beta\in\mathbb{R}}$ has also some interesting algebraic and analytic properties which we list below.

**Proposition 3.4.** *Let $f \in \mathcal{C}([0,\infty),\mathbb{R})$. Then, we have the following items.*

*(1) For any fixed $\beta \in \mathbb{R}$, $S^{(\beta)}$ is a linear mapping on $\mathcal{C}([0,\infty),\mathbb{R})$. The unique invariant subspace of $\mathcal{C}([0,\infty),\mathbb{R})$ by the family of mapping $(S^{(\beta)})_{\beta\in\mathbb{R}}$ is given by the space of linear functions, i.e. the identity $f^{(\beta)} = f$ holds for any $\beta \in \mathbb{R}$ if and only if $f = l_\mu$ where $l_\mu(t) = \mu t$, for some $\mu \in \mathbb{R}$.*

*(2) Let $a > 0$ and denote by $d_a$ the dilatation operator, i.e. $d_a(f)(t) = f(at)$, then for any $\beta \in \mathbb{R}$, we have $d_a \circ f^{(\beta)} = f^{(a\beta)} \circ d_a$.*

*(3) For any $\alpha, \beta \in \mathbb{R}$, we have $S^{(\alpha)} \circ S^{(\beta)} = S^{(\alpha+\beta)}$. Thus, as $S^{(0)}(f) = f$, $(S^{(\beta)})_{\beta\geq 0}$ is a semi-group on $\mathcal{C}([0,\infty),\mathbb{R})$.*

*(4) Let $\beta \in \mathbb{R}$ and set $p_{\beta,\alpha}(t) = (1+\beta t)^{\alpha}$. Then, we have $p^{(-\beta)}_{\beta,\alpha} = p_{-\beta,1-\alpha}$. In particular, if $\beta > 0$ and $\alpha$ is odd and negative, then $p^{(-\beta)}_{\beta,\alpha} \in \mathcal{C}([0,\infty),\mathbb{R})$.*
*(5) For any $\beta \in \mathbb{R}$, $S^{(\beta)}$ preserves the concave and convex properties.*
*(6) Let $f$ be concave such that $f(0) > 0$ and $f(\frac{1}{\beta_0}) = 0$ for some $\beta_0 > 0$. If $f$ is affine then $f^{(\beta_0)}$ is constant on $\mathbb{R}^+$. Otherwise, for any $\beta \geq \beta_0$, $f^{(\beta)}$ is increasing on $\mathbb{R}^+$.*

**Remark 3.5.** Note that from item (2) and the 2-self-similarity of $X$, we have, for any $\beta \in \mathbb{R}$ and $a > 0$, the identity

$$T^{af^{(\beta)}} \overset{(d)}{=} a^2 T^{f\circ d_a^{(a^2\beta)}}$$

where $T^{af^{(\beta)}}$ stands for the first crossing time of $X$ over the curve $af^{(\beta)}$.

## 4. Proofs

4.1. **Proof of Theorem 3.1.** We start by proving two intermediate results. To this end, we fix $\beta \in \mathbb{R}$ and denote by $H^{(\beta,f)}$ the first time when $S^{(\beta)}(X)$ crosses $f$ i.e.

$$H^{(\beta,f)} = \inf\left\{0 < s < \zeta^{(\beta)};\ S^{(\beta)}(X)_s = f(s)\right\}.$$

The aim of the next result is to relate the stopping time $H^{(\beta,f)}$ to $T^{f^{(\beta)}}$.

**Lemma 4.1.** *The identities*

$$H^{(-\beta,f)} = \frac{T^{f^{(\beta)}}}{1+\beta T^{f^{(\beta)}}} \quad \text{and} \quad T^{f^{(\beta)}} = \frac{H^{(-\beta,f)}}{1-\beta H^{(-\beta,f)}}$$

*hold almost surely true. In particular, we have* $\left\{H^{(-\beta,f)} < \zeta^{(-\beta)}\right\} = \left\{T^{f^{(\beta)}} < \zeta^{(\beta)}\right\}$.

**Proof.** From the definition of $S^{(\beta)}(X)$ and by using a deterministic time-change, we get

$$\begin{aligned} H^{(-\beta)} &= \inf\left\{0 \leq s < \zeta^{(-\beta)};\ X_{\frac{s}{1-\beta s}} = \frac{f(s)}{1-\beta s}\right\} \\ &= \inf\left\{0 \leq s < \zeta^{(-\beta)};\ X_{\frac{s}{1-\beta s}} = S^{(\beta)}(f)\left(\frac{s}{1-\beta s}\right)\right\} \\ &= \frac{T^{f^{(\beta)}}}{1+\beta T^{f^{(\beta)}}}. \end{aligned}$$

The second identity is obtained in a similar way. The last statement follows then by observing that, for any $\beta \in \mathbb{R}$, we have $\zeta^{(\beta)} = \frac{\zeta^{(-\beta)}}{1-\beta\zeta^{(-\beta)}}$. □

For a fixed $\beta \in \mathbb{R}$, the image by the mapping $S^{(\beta)}$ of any homogeneous Markov process is clearly a non-homogeneous Markov process. However, the time inversion property allows to connect the law of $X$ and that of $S^{(\beta)}(X)$ via a simple time-space $h$-transform which is the content of the next result.

**Lemma 4.2.** *Let $x, y \in E$ and $\beta \in \mathbb{R}$. Under $\mathbb{P}^{(y)}_x$, the process $X^{(\beta)} = S^{(\beta)}(X)$, defined on $[0, \zeta^{(\beta)})$, is a non-homogeneous strong Markov process. Its law, denoted by $\mathbb{Q}^{y,\beta}_x$, is* *absolutely continuous with respect to* $\mathbb{P}_x^{(y)}$ *with Radon-Nikodym derivative* $M_x^{y,\beta}(t, X_t)$, $t < \zeta^{(\beta)}$, *given by*

$$M_x^{y,\beta}(t, X_t) = (1+\beta t)^{-\nu-1} \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yX_t)} e^{\frac{\beta}{2}\frac{y^2t^2}{1+\beta t} + \frac{\beta}{2}\frac{X_t^2}{1+\beta t} - \frac{\beta}{2}x^2}.$$

*This also holds for* $y = 0$ *obtained by letting* $y \to 0$.

**Proof.** Observing that, for any $c, d \in E$, $\frac{\Phi(cy)}{\Phi(dy)} \to 1$ as $y \to 0$, we first consider the case $y = 0$. Next, we assume that $\beta < 0$. We deduce readily that, for any $t < \zeta^{(\beta)}$,

$$M_x^{0,\beta}(t, X_t) = \lim_{\substack{z\to 0\\ z\in E}} \frac{p_{1+\beta t}(X_t, z)}{p_{1+\beta t}(x, z)}.$$

Thus, the martingale property of $(M_x^{0,\beta}(t, X_t), 0 \leq t < \zeta^{(\beta)})$ follows from the Markov and the self-similarity properties of $X$. Then, the claim, in this case, is obtained by combining the path realization (4) of bridges associated to $X$ and the construction, elaborated by Fitzsimmons et al. [15], of the law of bridges as a time-space $h$-transform of the law of $X$. Next, we investigate the case $\beta > 0$. We assume that $X$ is a Bessel process of dimension $\delta > 0$. Using Itô formula we see that the process $(M_x^{0,\beta}(t, X_t), t \geq 0)$ is a $\mathbb{P}_x$-local martingale. We shall now show that it is a true martingale. Recall that the squared Bessel process $X^2$ is the unique solution to the stochastic differential equation (for short sde)

$$dX_t^2 = 2X_t dB_t + \delta dt, \quad X_0^2 = x^2, \tag{6}$$

where $B$ is a Brownian motion, see [34]. Furthermore, we introduce the processes $Y$ and $\tilde{Y}$ which are defined, for a fixed $t \geq 0$, by setting $Y_t = (1+\beta t)\tilde{Y}_t = S^{(\beta)}(X)_t$. It follows from (6) and by performing a deterministic time change, that $\tilde{Y}$ solves the sde

$$d\tilde{Y}_t^2 = \frac{2\tilde{Y}_t d\gamma_t}{1+\beta t} + \frac{\delta dt}{(1+\beta t)^2}$$

where $(\gamma_t, t \geq 0)$ is a Brownian motion with respect to the filtration generated by the process $(\tilde{Y}_t, t \geq 0)$. Similarly, $Y^2$ satisfies

$$\begin{aligned} dY_t^2 &= d\left((1+\beta t)^2 \tilde{Y}_t^2\right) \\ &= 2Y_t \left(d\gamma_t + \frac{\beta}{1+\beta t} Y_t dt\right) + \delta dt. \end{aligned} \tag{7}$$

Now, from (6) and by Girsanov theorem, the law of the unique solution to (7) is obtained from the law of $X^2$ by a change of probability measure using the local martingale $(M_x^{0,\beta}(t, X_t), t \geq 0)$. The proof of the claim, in the case $y = 0$, is completed by invoking the conservativeness property of $Y^2$, which implies that $(M_x^{0,\beta}(t, X_t), t \geq 0)$ is a martingale.

Finally, to recover the case $y \neq 0$, we use (3) which gives

$$\begin{aligned} \mathrm{d}\mathbb{Q}_x^{y,\beta}|_{\mathcal{F}_t} &= \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yx)} e^{-\frac{1}{2}\frac{ty^2}{1+\beta t}} \mathrm{d}\mathbb{Q}_x^{0,\beta}|_{\mathcal{F}_t} \\ &= \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yx)} e^{-\frac{1}{2}\frac{ty^2}{1+\beta t}} M_x^{0,\beta}(t,X_t)\mathrm{d}\mathbb{P}_x|_{\mathcal{F}_t} \\ &= \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yX_t)} e^{\frac{\beta}{2}\frac{y^2t^2}{1+\beta t}} M_x^{0,\beta}(t,X_t)\mathrm{d}\mathbb{P}_x^{(y)}|_{\mathcal{F}_t} \\ &= M_x^{y,\beta}(t,X_t)\mathrm{d}\mathbb{P}_x^{(y)}|_{\mathcal{F}_t}. \end{aligned}$$

This completes the proof in the Bessel case. The Brownian case, for $\beta > 0$, follows by using the same arguments than for Bessel processes. □

We shall now turn back to the proof of Theorem 3.1. Lemmae 4.1 and 4.2, when combined with the optional stopping theorem, allow us to write, for any $\lambda > 0$,

$$\begin{aligned} \mathbb{E}_x^{(y)}\left[e^{-\lambda T^{f^{(\beta)}}}\mathbb{I}_{\{T^{f^{(\beta)}}<\zeta^{(\beta)}\}}\right] &= \mathbb{E}_x^{(y)}\left[e^{-\lambda\frac{H^{(-\beta,f)}}{1-\beta H^{(-\beta,f)}}}\mathbb{I}_{\{H^{(-\beta,f)}<\zeta^{(-\beta)}\}}\right] \\ &= \mathbb{E}_x^{(y)}\left[e^{-\lambda\frac{T^f}{1-\beta T^f}} M_x^{y,-\beta}(T^f,X_{T^f})\mathbb{I}_{\{T^f<\zeta^{(-\beta)}\}}\right] \\ &= \int_0^{\zeta^{(-\beta)}} e^{-\lambda\frac{t}{1-\beta t}} M_x^{y,-\beta}(t,f(t))\mathbb{P}_x^{(y)}\left(T^f\in dt\right) \\ &= \int_0^{\zeta^{(\beta)}} \frac{e^{-\lambda u}}{(1+\beta u)^3} M_x^{y,-\beta}\left(u/(1+\beta u), f(u/(1+\beta u))\right) \\ &\times\ S^{(\beta)}\left(\mathbb{P}_x^{(y)}(T^f\in du)\right) \end{aligned}$$

where we have performed the change of variables $t/(1-\beta t) = u$. From the injectivity of the Laplace transform, we get our first assertion by simplifying the expression above. The second statement follows by letting $y \to 0$.

**Remark 4.3.** It is plain that the results of Theorem 3.1 can be readily extended to any $h$-transform of the process $X$, but we need to take care of the life-times of the involved processes in the Bessel case. In the Brownian setting, one gets similar results for the process $X_t^\epsilon = X_t + y\epsilon t$, $t \geq 0$, where $\epsilon$ is an independent symmetric Bernoulli random variable taking values in $\{-1,1\}$. Observe that $i(X^\epsilon)$ is a strong Markov process. However, because the latter starts at the random point $y\epsilon$, $X^{(\epsilon)}$ does not satisfy the time inversion property (1).

4.2. **Proof of Proposition 3.4.** Assume that there exists a function $f \in \mathcal{C}(\mathbb{R}^+,\mathbb{R})$ such that $S^{(\beta)}(f) = f$ on $\mathbb{R}$ and for all $\beta \in \mathbb{R}$. On the one hand, we easily check that the space of linear functions satisfies this property. On the other hand, for all $n \in \mathbb{N}$ we have $S^{(n\beta)}(f) = f$ on $\mathbb{R}$. Moreover, we observe that $f$ is invariant by $S^{(\beta)}$ if and only if $\hat{f}(t) = f(t)/t$ is invariant by $(1+\beta.)^{-1}S^{(\beta)}$. Iterating this procedure, we obtain that $\hat{f}$ is invariant by the transformation $(1+n\beta.)^{-1}S^{(n\beta)}$ for any $n \in \mathbb{N}^*$. Thus, letting $n \to \infty$ and using the right continuity of $\hat{f}$ we see that $\hat{f}(t) = \lim_{s\to 0}\hat{f}(s)$ for $t \in \mathbb{R}^+$. In other words, $f$ has to be linear.

Statements (2), (3), (4) follows readily from the definition of the transformation $S^{(\beta)}$.
Let us now prove item (5). Assume, for instance, that $f$ is convex and let $\mu(dt)$ be the positive Radon measure corresponding to the second derivative of $f$ in the sense of distributions, see e.g. [34]. Then, the second derivative of $S^{(\beta)}(f)$ in the sense of distribution is given by the Radon measure $(1+\beta t)^{-4}S^{(\beta)}(\mu)(dt)$, $t < \zeta^{(\beta)}$, which is clearly positive. Item (5) follows.
The first claim of item (6) is obvious. Finally, we note that for $\beta \geq \beta_0$, $S^{(\beta)}(f)$ is positive on $\mathbb{R}^+$ which completes the proof since $S^{(\beta)}(f)$ is concave.

## 5. Brownian motion

In this paragraph, we take $E = \mathbb{R}$, thus $X$ is a Brownian motion. Thanks to the homogeneity property, in this case, it is clearly enough to consider the case $X_0 = 0$ and we simply write $\mathbb{P}$ for $\mathbb{P}_0$. Also, to simplify the discussion, we assume that $f \in \mathcal{C}^1(\mathbb{R}_+, E)$, with $f(0) \neq 0$, which implies that the studied distributions are absolutely continuous with respect to the Lebesgue measure with continuous densities. We write then $\mathbb{P}\left(T^f \in dt\right) = p^f(t)\,dt$ and read from Theorem 3.1 that, when $y = 0$, for any fixed $t < \zeta^{(\beta)}$, the identity

$$p^{f^{(\beta)}}(t) = \frac{1}{(1+\beta t)^{3/2}} e^{-\frac{1}{2}\frac{\beta}{1+\beta t} f^{(\beta)}(t)^2} p^f\left(\frac{t}{1+\beta t}\right).$$

A similar expression can be written down when working with a Brownian motion with a constant drift $y \neq 0$. Next, setting $f \equiv a$ and $\beta = b/a$, for some reals $a$ and $b$, we get $f^{(b/a)}(t) = a + bt$, $t < \zeta^{(b/a)}$. An immediate application of Corollary 3.3, combined with

$$\mathbb{P}(T^a \in dt) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}\,dt,\ t > 0,$$

yields the the well known Bachelier-Lévy formula

$$\mathbb{P}\left(T^{a+b\cdot} \in dt\right) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-ba-\frac{b^2}{2}t-\frac{a^2}{2t}}\,dt.$$

The example of the first crossing problem associated to new curves will be given in Section 5.3 below.

5.1. **Interpretation of the mapping $S^{(\beta)}$ via the method of images.** We aim to describe the impact of our methodology to the so-called standard method of images. To this end, we assume that $X_0 = 0$ and let $f \in \mathcal{C}([0,\infty], \mathbb{R}^+)$. Set

$$h(x,t)dx = \mathbb{P}\left(T^f > t, X_t \in dx\right)$$

and assume that the boundary $f$ has the following properties

- $f$ is infinitely often continuously differentiable.
- $f(t)/t$ is monotone decreasing.
- $f$ is concave.

Note that these properties are also satisfied by $f^{(\beta)}$ for any $\beta > 0$. Then, the function $h$ is the unique solution to the heat equation, see Lerche [25, Chap. I.1],

$$\frac{\partial h}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x,t) \quad \text{on } \{(x,t) \in \mathbb{R}\times\mathbb{R}^+;\ x \leq f(t)\},$$

with boundary conditions

$$h\left(f(t),t\right)=0,\quad h(.,0)=\delta_0(.)\quad \text{on}\quad ]-\infty,f(0^+)]$$

where $\delta_0$ stands for the Dirac function at 0 and $f(0^+)=\lim_{t\searrow 0} f(t)$. The standard method of images assumes that $h$ is known, whilst $f$ is unknown, and is given by

$$h(x,t)=\frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{x^2}{2t}}-\int_0^\infty e^{-\frac{(x-s)^2}{2t}}F(ds)\right) \tag{8}$$

where $F(ds)$ is some positive $\sigma$-finite measure satisfying $\int_0^\infty e^{-\frac{\epsilon s^2}{2}}F(ds)<\infty$, for all $\epsilon>0$. In [25], it is shown that if $f$ is the unique root of the equation

$$h(x,t)=0,$$

in the unknown $x$ for a fixed $t\geq 0$, then we have

$$\mathbb{P}\left(T^f\in dt\right)=\frac{dt}{\sqrt{2\pi t^3}}\int_0^\infty s e^{-\frac{(s-f(t))^2}{2t}}F(ds),\quad t>0. \tag{9}$$

Note that the implicit equation $h(f(.),.)=0$ may be written as

$$\int_0^\infty e^{-\frac{s^2}{2.}+s\frac{f(.)}{.}}F(ds)=1. \tag{10}$$

With $F(ds)$ replaced by $F(ds)e^{-\mu s}$, $\mu>0$, the unique solution to (10) is $f(t)+\mu t$ for a fixed positive real $t$ and this is easily checked by the Cameron-Martin formula. In the same spirit, replacing $F(ds)$ by $F_\beta(ds)=e^{-\beta\frac{s^2}{2}}F(ds)$ in (10) yields

$$h_\beta\left(f^{(\beta)}(t),\frac{t}{1+\beta t}\right)=0$$

where $h_\beta$ is defined in (8) by substituting $F(ds)$ by $F_\beta(ds)$. We state this observation in the following.

**Proposition 5.1.** *For a fixed $\beta>0$ let $h_\beta$ be defined by (8) where $F(ds)$ is replaced by $F_\beta(ds)=F(ds)e^{-\beta s^2/2}$. Then, for a fixed $t>0$, $f^{(\beta)}(t)$ is the unique solution to*

$$h_\beta\left(x,\frac{t}{1+\beta t}\right)=0. \tag{11}$$

*In other words, equation (9) is in agreement with Theorem 3.1 when $F(ds)$ and $f$ are replaced by $F_\beta(ds)$ and $f^{(\beta)}$ respectively.*

**Remark 5.2.** In [11], Durbin considered the studied problem for continuous gaussian processes and showed that, in the absolute-continuous case, the problem reduces to the computation of a conditional expectation. That is for the Brownian motion if $f$ is continuously differentiable and $f(0)\neq 0$ then $\mathbb{P}\left(T^f\in dt\right)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{f^2(t)}{2t}}h(t)dt$ where

$$h(t)=\lim_{s\nearrow t}\frac{1}{t-s}\mathbb{E}\left[B_s-f(s);T^f>s\middle|B_t=f(t)\right],\quad t>0.$$

There seems to be no-known way to compute the function $h$. It is shown in [12] that this method is in agreement with the standard method of images. We learned from Kendall [19] an intuitive interpretation of Durbin's expression involving the family of local times of $X$ at $f$ denoted by $(L_t^{X=f},t\geq 0)$. Indeed, observe that Durbin's formula can be rewritten as $\mathbb{P}\left(T^f \in dt\right) = h(t)\mathbb{E}[dL_t^{X=f}]$, $t \geq 0$. The latter when integrated over $[0,a]$, yields $\mathbb{P}\left(T^f < a\right) = \mathbb{E}\left[\int_0^a h(s)dL_s^{X=f}\right]$. Such a decomposition is not unique and many can be constructed from the above one. A natural non trivial one to consider is motivated by the following observation. If $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ solves $\mathbb{E}\left[\int_t^a g(s,a)dL_s^{X=f} \big| X_t = f(t)\right] = 1$, for $0 < t < a$, then clearly $\mathbb{E}\left[\int_0^a g(s,a)dL_s^{X=f}\right] = \mathbb{P}\left(T^f < a\right)$. However, it is not clear how to express $g$ in terms of $h$.

**Remark 5.3.** It is interesting to analyze the impact of our identities on some integral equations satisfied by the studied densities. However, some of them lead to obvious facts explainable by change of variables. As an example, we observe that if $f$ is positive and does not vanish then $X$, when started at $x > f(0)$, must hit $f$ before reaching 0. The strong Markov property then gives

$$\frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}} = \int_0^t \mathbb{P}_x(T^f \in dr)\frac{f(r)}{\sqrt{2\pi(t-r)^3}}e^{-\frac{f(r)^2}{2(t-r)}}, \quad t > 0.$$

This is easily shown to be in accordance with the result stated in Theorem 3.1. For the above and other classical integral equations we refer to [11], [14], [25] and also to [30] for some more recent ones.

5.2. **Large asymptotic for the density of $T^{f^{(\beta)}}$.** Following Anderson and Pitt [3], we consider the asymptotic of the density distribution of $T^{f^{(\beta)}}$ for a large time. We shall see that a general formula holds in the case $\beta > 0$. Since for $\beta < 0$, we are forced to close the curve at $\zeta^{(\beta)}$, we shall be concerned with the asymptotic as $t$ tends to $-1/\beta$. For the latter case, we need to list some assumptions borrowed from [3]. Let $\tilde{T}^f = \sup\{s > 0, X_s = f(s)\}$ be the last entrance time below $f$. If we assume that $f \geq 0$ then it is known that the distribution of $\tilde{T}^f$ is defective, i.e. $r = \mathbb{P}(\tilde{T}^f < \infty) < 1$, if and only if

$$\int_1^\infty t^{-3/2} f(t) e^{-f^2(t)/2t}\, dt = \infty. \tag{12}$$

By the classical Kolmogorov-Erdös-Petrovski theorem, see [13], we know that if $t^{-1/2}f(t)$ is increasing for sufficiently large $t$ then $f$ is transient if and only (12) holds. For the statement of our next result we need to recall the following three conditions:

- $H_1$ $f$ is increasing, concave, twice differentiable on $]0,\infty)$ and of regular variations at $\infty$ with index $\alpha \in [1/2, 1)$.
- $H_2$ For sufficiently large $t$, $f(t)/\sqrt{t}$ is monotonic increasing at $\infty$, and $f(t)/t$ is convex and decreases to 0.
- $H_3$ There exist positive constants $c < 1$ and $c'$ such that for a sufficiently large enough $t$ we have the inequalities $tf'(t) < cf(t)$ and $|t^2 f''(t)| \leq c' f(t)$.

We read from Theorem 1 in [3] that whenever $\tilde{T}^{(f)}$ is defective and $f$ satisfies conditions $H1$, $H2$ and $H3$, then holds

$$p^f(t) \sim (1-r)\,\frac{f(t) - tf'(t)}{\sqrt{2\pi}t^{3/2}}e^{-f^2(t)/2t}$$

where $h \sim g$ means that $h(t)/g(t) \to 1$ as $t \to \infty$. The behavior at $\infty$ imposed in the above conditions is granted in the examples where $f$ behaves like

$$f(t) = Ct^a (\log t)^b (\log\log t)^c (\log\log\log t)^d$$

with $1/2 \le a < 1$, for large time. It is clear that hypotheses $H1$, $H2$ and $H3$ are not all preserved by the mapping $S^{(\beta)}$. For instance, if $\beta > 0$ then $f^{(\beta)}(t)/t \to \beta f(1/\beta)$ as $t \to \infty$, so $f^{(\beta)}$ always fails to satisfy the second condition of $H2$. In contrast, still with $\beta > 0$ we have $f^{(\beta)}(t)/\sqrt{t} \sim \beta f(1/\beta)\sqrt{t}$ so that the first condition of $H2$ is preserved. When $f$ is concave then so is $f^{(\beta)}$ and the second condition of $H2$ implies that $\lim_{t\to\zeta^{(\beta)}} f^{(\beta)}(t) = 0$ for any $\beta < 0$. Equipped with this, we are now ready to state the following result which proof is left to the reader.

**Proposition 5.4.** *let $f \in \mathcal{C}^1(]0,\infty), \mathbb{R}^+)$ and $\beta \in \mathbb{R}$. Then hold the following assertions.*
*1) If $\beta > 0$ then*

$$p^{f^{(\beta)}}(t) \sim (\beta t)^{-3/2} e^{-f^{(\beta)^2}(t)/2t} p^f\left(\beta^{-1}\right) \qquad \text{as } t \to \infty.$$

*2) If $f$ is transient then $f^{(\beta)}$ is transient. Moreover, in the case $\beta < 0$, $f$ is transient and satisfies conditions $H1$, $H2$ and $H3$. Finally, we have, as $t \to -1/\beta$,*

$$p^{f^{(\beta)}}(t) \sim (1-r)\,(-\beta)^{3/2}\,\tilde{f}(\beta,t)$$

*where*

$$\tilde{f}(\beta,t) = \left(f\left(-\frac{1}{\beta\,(1+\beta t)}\right) + \frac{1}{\beta\,(1+\beta t)} f'\left(-\frac{1}{\beta(1+\beta t)}\right)\right).$$

**Remark 5.5.** In the defective case, the density of the distribution of the last crossing time $\tilde{T}^f = \sup\{s > 0; X_s = f(s)\}$ is shown to have an atom at 0 and its asymptotic as $t \to 0$ is determined in [38]. Similar questions can be treated for $\tilde{T}^{f^{(\beta)}} = \sup\{s > 0; X_s = f^{(\beta)}(s)\}$ using this method.

5.3. **Two new examples.** We now consider the boundary crossing problem associated to two new curves: the square root of a second order polynomial and the reciprocal of an affine function.

We start with the distribution of the stopping time

$$T_a^{(\lambda_1,\lambda_2)} = \inf\left\{s > 0;\ X_s = a\sqrt{(1+\lambda_1 s)\,(1+\lambda_2 s)}\right\}$$

where $a$ and $\lambda_1 < \lambda_2$ are some fixed reals. We do not consider the case $\lambda_1 = \lambda_2$ which can be studied in a elementary way, with the extra cost of making use of the strong Markov property when $\lambda_1 < 0$.

First, consider the case $\lambda_2 = 0$ and, to simplify notation, set $\lambda_1 = \lambda$ and $T_a^{(\lambda,0)} = T_a^{(\lambda)}$. This is the setting of the classical example studied by Breiman in [8]. It is related to the hitting time of a constant level by an Ornstein-Uhlenbeck process and we refer to Alili et al. [2] for a recent survey on this topic. That is, with

$$U_t = e^{-\lambda t/2} \int_0^t e^{\lambda s/2} dX_s, \quad t \ge 0,$$

and $H_a = \inf\{s > 0;\ U_s = a\}$, we have

$$T_a^{(\lambda)} \overset{(d)}{=} \lambda^{-1}\left(e^{\lambda H_a} - 1\right) \tag{13}$$

where $\overset{(d)}{=}$ stands for the identity in distribution. By symmetry, it is enough to consider the case where $a$ is positive. We proceed by recalling the distribution of $H_a$ given, see for instance [2], by

$$\mathbb{P}\left(H_a \in dt\right) = -\frac{1}{2}\lambda e^{-\lambda a^2/4} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,-a\sqrt{\lambda}}}(0)}{D^{(\nu)}_{\nu_{n,-a\sqrt{\lambda}}}(-a\sqrt{\lambda})}\, e^{-\lambda \nu_{n,-a\sqrt{\lambda}} t/2}, \quad t>0,$$

where we used the notation $D^{(\nu)}_{\nu_{n,b}}(b) = \frac{\partial D_\nu(b)}{\partial \nu}|_{\nu=\nu_{n,b}}$ and $(\nu_{j,b})_{j\geq 0}$ stands for the ordered sequence of the positive zeros of the parabolic function $\nu \to D_\nu(b)$. By means of the identity (13), we get that

$$\mathbb{P}\left(T_a^{(\lambda)} \in dt\right) = \frac{1}{1+\lambda t}\mathbb{P}\left(H_a \in d\cdot\right)|_{\cdot=\frac{1}{\lambda}\log(1+\lambda t)} dt, \quad t>0.$$

Next we assume that $\lambda_1 < \lambda_2$. Then the support of $T_a^{(\lambda_1,\lambda_2)}$ is $[0, \zeta^{(\lambda_1)})$ if $\lambda_1$ is positive and is the positive real line otherwise. We have

$$S^{(\lambda_1)}\left(\sqrt{1+(\lambda_2-\lambda_1)\cdot}\right) = \sqrt{(1+\lambda_2\cdot)\,(1+\lambda_1\cdot)}.$$

We are now ready to use Theorem 3.1 and write

$$\mathbb{P}\left(T_a^{(\lambda_1,\lambda_2)} \in dt\right) = \frac{1}{(1+\lambda_1 t)^{5/2}} e^{-\frac{1}{2}\lambda_1(1+\lambda_2 t)} S^{(\lambda_1)}\left(\mathbb{P}\left(T_a^{(\lambda_2-\lambda_1)} \in dt\right)\right), \quad t<\zeta^{(\lambda_1)}.$$

We now turn to the second example. We are interested in computing the distribution of the stopping time, defined for any $\beta \in \mathbb{R}$, by

$$T^{h^{(\beta)}} = \inf\left\{0 < s < \zeta^{(\beta)};\ X_s = \frac{1}{1+\beta s}\right\}.$$

To this end, we recall that Groeneboom [18] computed the density of $T^{\tilde{h}}$ with $\tilde{h}(t) = 1+\beta^2 t$ as follows

$$\mathbb{P}(T^{\tilde{h}} \in dt) = 2(\beta^2 c)^2 e^{-\frac{2}{3}\beta^4 t^3} \sum_{k=0}^{\infty} \frac{Ai\left(z_k + 2c\beta^2\right)}{Ai'(z_k)} e^{-z_k t}\, dt,\ t>0,$$

where $(z_k)_{k\geq 0}$ is the decreasing sequence of negative zeros of the Airy function, see e.g. [24], and we have set $c = (2\beta^2)^{-\frac{1}{3}}$. Next, by means of the Cameron-Martin formula, we obtain, with $h(t) = (1-\beta t)^2$,

$$\mathbb{P}(T^h \in dt) = 2(\beta^2 c)^2 e^{2\beta - 2\beta^2 t\left(1+\frac{2}{3}\beta^2 t^2 - \beta t\right)} \sum_{k=0}^{\infty} \frac{Ai\left(z_k + 2c\beta^2\right)}{Ai'(z_k)} e^{-z_k t}\, dt,\ t>0,$$

Finally, observing that $h^{(\beta)} = S^{(\beta)}(h)$, we get from Theorem 3.1,

$$\mathbb{P}(T^{h^{(\beta)}} \in dt) = \frac{2(\beta^2 c)^2 e^{2\beta}}{(1+\beta t)^{3/2}} e^{-2\beta^2 t\left(1+\frac{1}{4\beta}+\beta t+\frac{2}{3}\beta^2 t^2\right)} \sum_{k=0}^{\infty} \frac{Ai\left(z_k + 2c\beta^2\right)}{Ai'(z_k)} e^{-z_k \frac{t}{1+\beta t}}\, dt,\ t<\zeta^{(\beta)}.$$

We complete the example by stating the following asymptotic results, for $\beta > 0$,

$$\mathbb{P}(T^{h^{(\beta)}} \in dt) \sim \left(2(\beta^2 c)^2 e^{2\beta} \sum_{k=0}^{\infty} \frac{Ai\left(z_k + 2c\beta^2\right)}{Ai'(z_k)} e^{-\frac{z_k}{\beta}}\right) e^{-\frac{4}{3}\beta^4 t^3}\, dt \ \text{ as } t \to \infty,$$

and

$$\lim_{\beta\to 0}\mathbb{P}(T^{h^{(\beta)}}\in dt)=\frac{1}{\sqrt{2\pi t^3}}e^{-\frac{1}{2t}}\,dt.$$

## 6. Bessel processes and straight lines

In this part, we investigate the case when $X$ is a Bessel process of dimension $\delta>0$ and we refer to Revuz and Yor [34, Chap. XI] for a concise treatment of these processes. For $\delta\geq 2$, and $x>0$, the Bessel process of dimension $\delta$ is the unique solution to the equation

$$X_t=B_t+x+\frac{\delta-1}{2}\int_0^t\frac{ds}{X_s},\quad t>0.$$

where $B$ is a standard Brownian motion. For $0<\delta<2$, $X$ is defined as the square root of the unique non-negative solution of (6). We denote by $\mathbb{P}_x^{\nu}$ (resp. $\mathbb{E}_x^{\nu}$) the law (resp. the expectation operator) of the Bessel process of index $\nu=\delta/2-1$, starting at $x>0$. The spatial Laplace transform of $X^2$ is given by

$$\mathbb{E}_x^{\nu}\left[e^{-\lambda X_t^2}\right]\;=\;(1+2\lambda t)^{-\delta/2}e^{-\frac{\lambda x^2}{1+2\lambda t}},\quad \lambda,t\geq 0. \tag{14}$$

The densities of the semi-group of $X$, with respect to the Lebesgue measure, are

$$p_t^{\nu}(x,y)\;=\;\frac{y}{t}\left(\frac{y}{x}\right)^{\nu}e^{-\frac{x^2+y^2}{2t}}I_{\nu}\left(\frac{xy}{t}\right),\quad t,x,y>0.$$

For $f\in\mathcal{C}(\mathbb{R}_+,E)$, we keep the notations used in the introduction and Section 3, for the various involved hitting times. In the case $y=0$ and for $t<\zeta^{(\beta)}$ , Theorem 3.1 then reads

$$\mathbb{P}_x^{\nu}\left(T^{f^{(\beta)}}\in dt\right)=(1+\beta t)^{\nu-2}e^{-\frac{\beta}{2}\frac{f^{(\beta)}(t)^2}{1+\beta t}+\frac{\beta}{2}x^2}S^{(\beta)}\left(\mathbb{P}_x^{\nu}(T^f\in dt)\right).$$

We shall now end up our discussion by computing the distribution of the hitting time by $X$ of a straight line $a+b\cdot$ where $a>0$ and $b$ is a real. We keep the notations used in Corollary 3.3 and the reader is reminded about observations preceding its statement. When $a=0$ and $b>0$, $T^{b\cdot}$ and $1/\sup\{s>0;X_s=b\}$ have the same distribution determined in [32], by making use of the time inversion property. For $b\neq 0$ and up to our knowledge, this question, which was raised in [32], remained as an open problem. We recall that the law of $T^a$, which corresponds to $b=0$, is characterized, for $\lambda\geq 0$, by

$$\mathbb{E}_x^{\nu}\left[e^{-\frac{\lambda^2}{2}T^a}\right]=\begin{cases}\frac{x^{-\nu}I_{\nu}(x\lambda)}{a^{-\nu}I_{\nu}(a\lambda)}, & x\leq a,\\[2ex] \frac{x^{-\nu}K_{\nu}(x\lambda)}{a^{-\nu}K_{\nu}(a\lambda)}, & x\geq a,\end{cases} \tag{15}$$

where $K_{\nu}$ is the modified Bessel functions of the second kind of index $\nu$, see for instance Borodin and Salminen [6]. Observe that when $x>a$, the distribution of $T^a$ is defective and $\mathbb{P}_x^{\nu}(T^a<\infty)=(a/x)^{2\nu}$. In the case $x<a$, we have

$$\mathbb{P}_x^{\nu}(T^a\in dt)\;=\;\sum_{k=1}^{\infty}\frac{x^{-\nu}j_{\nu,k}J_{\nu}(j_{\nu,k}\frac{x}{a})}{a^{2-\nu}J_{\nu+1}(j_{\nu,k})}e^{-j_{\nu,k}^2t/2a^2}\,dt,\quad t>0, \tag{16}$$

where $(j_{\nu,k})_{k\geq 1}$ is the ordered sequence of positive zeros of the Bessel function of the first kind $J_{\nu}(.)$, see [6].

**Theorem 6.1.** *For $0 \le x < a$ and $b \in \mathbb{R}$, we have for any $t < \zeta^{(b/a)}$*

$$\mathbb{P}_x^\nu(T^{a+b\cdot} \in dt) = \frac{e^{\frac{b}{2a}(a^2-x^2)+\frac{b^2}{2}t}}{(1+\frac{b}{a}t)^{\nu+2}} \sum_{k=1}^\infty \frac{x^{-\nu} j_{\nu,k} J_\nu(j_{\nu,k}\frac{x}{a})}{a^{2-\nu} J_{\nu+1}(j_{\nu,k})} e^{-j_{\nu,k}^2 \frac{t}{2a(a+bt)}}\, dt.$$

*For any $x \ge 0$ and $a, b > 0$, we have*

$$\mathbb{E}_x^\nu\left[e^{-\frac{\lambda^2}{2}T^{a-b\cdot}}\right] = \begin{cases} e^{\frac{b}{2a}(a^2-x^2)} \frac{x^{-\nu}}{a^{-\nu}} \int_0^\infty \frac{I_\nu(\sqrt{2}xu)}{I_\nu(\sqrt{2}au)} p_{b/2a}^\nu(\lambda_b, u)\, du, & x \le a, \\ e^{\frac{b}{2a}(a^2-x^2)} \frac{x^{-\nu}}{a^{-\nu}} \int_0^\infty \frac{K_\nu(\sqrt{2}xu)}{K_\nu(\sqrt{2}au)} p_{b/2a}^\nu(\lambda_b, u)\, du, & x \ge a, \end{cases}$$

*where $\lambda_b = \sqrt{(\lambda^2+b^2)/2}$, $\lambda \in \mathbb{R}$.*

**Proof.** The first statement results from a combination of Corollary 3.3 and relation (15). Next, using Lemmae 4.1 and 4.2, with $\beta = -b/a$, we can write

$$\begin{aligned} &\mathbb{E}_x^\nu\left[e^{-\frac{\lambda^2}{2}T^{a-b\cdot}}; T^{a-b\cdot} < \frac{a}{b}\right] \\ = \;& \mathbb{E}_x^\nu\left[e^{-\frac{\lambda^2}{2}\frac{H^{(b/a,a)}}{1+\frac{b}{a}H^{(b/a,a)}}}; H^{(b/a,a)} < \infty\right] \\ = \;& \mathbb{E}_x^\nu\left[e^{-\frac{\lambda^2}{2}\frac{T^a}{1+\frac{b}{a}T^a}}\left(1+\frac{b}{a}T^a\right)^{-\delta/2} e^{\frac{b}{2a}\left(\frac{a^2}{1+\frac{b}{a}T^a}-x^2\right)}\right] \\ = \;& e^{\frac{b}{2a}(a^2-x^2)}\mathbb{E}_x^\nu\left[e^{-\frac{\lambda_b^2 T^a}{1+\frac{b}{a}T^a}}\left(1+\frac{b}{a}T^a\right)^{-\delta/2}\right] \\ = \;& e^{\frac{b}{2a}(a^2-x^2)}\int_0^\infty p_{b/2a}^\nu(\lambda_b, u)\mathbb{E}_x^\nu\left[e^{-u^2 T^a}\right] du, \end{aligned}$$

where the last line follows from (14). It remains to use (15) to conclude. □

**Remark 6.2.** The process $(Y_t := X_t + bt, t \ge 0)$ is a non-homogeneous Markov process and solves the sde

$$Y_t = B_t + \frac{\delta-1}{2}\int_0^t \frac{ds}{Y_s - bs} + bt, \quad t \ge 0.$$

This is to distinguish from a Bessel process with a "naïf" drift $b$ introduced in [40] and defined as a solution to

$$Z_t = B_t + \frac{\delta-1}{2}\int_0^t \frac{ds}{Z_s} + bt, \quad t \ge 0.$$

**Remark 6.3.** The Bessel process of dimension $\delta = 1$ is, in fact, the reflected Brownian motion. It follows that the associated first hitting times can be interpreted as double barrier hitting times. That is, with $f$ as above, the time when a Brownian motion $B$ hits one of the curves $x \pm f(\cdot)$, i.e. $\inf\{s > 0;\ B_s = x \pm f(s)\}$.

**Remark 6.4.** The technics used in the example treated in Subsection 5.3, for Brownian motion, can be applied to Bessel processes and square root curves. The required results, for that end, can be found in Delong [10] or in Yor [40]. This law is connected via a deterministic time-change to the one of the first passage time to a fixed level by the radial norm of a $\delta$-dimensional (in the case $\delta$ is an integer) Ornstein-Uhlenbeck processes.

Due to the stationarity of this later class of processes, the law, for both first passage time above and below, can be expressed as infinite convolutions of mixture of exponential distributions, see Kent [21].

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