

# $q$ -INVARIANT FUNCTIONS FOR SOME GENERALIZATIONS OF THE ORNSTEIN-UHLENBECK SEMIGROUP

P. PATIE

ABSTRACT. We show that the multiplication operator associated to a fractional power of a Gamma random variable, with parameter  $q > 0$ , maps the convex cone of the 1-invariant functions for a self-similar semigroup into the convex cone of the  $q$ -invariant functions for the associated Ornstein-Uhlenbeck (for short OU) semigroup. We also describe the harmonic functions for some other generalizations of the OU semigroup. Among the various applications, we characterize, through their Laplace transforms, the laws of first passage times above and overshoot for certain two-sided  $\alpha$ -stable OU processes and also for spectrally negative semi-stable OU processes. These Laplace transforms are expressed in terms of a new family of power series which includes the generalized Mittag-Leffler functions and generalizes the family of functions introduced by Patie [20].

## 1. Introduction and main results

Let  $E = \mathbb{R}, \mathbb{R}^+$  or  $[0, \infty)$  and let  $X$  be the realization of  $(P_t)_{t \geq 0}$ , a Feller semigroup on  $E$  satisfying, for  $\alpha > 0$ , the  $\alpha$ -self-similarity property, i.e. for any  $c > 0$  and every  $f \in \mathfrak{B}(E)$ , the space of bounded Borelian functions on  $E$ , we have the following identity

$$(1.1) \quad P_{c^\alpha t} f(cx) = P_t (d_c f)(x), \quad x \in E,$$

where  $d_c$  is the dilatation operator, i.e.  $d_c f(x) = f(cx)$ . We denote by  $(\mathbb{P}_x)_{x \in E}$  the family of probability measures of  $X$  which act on  $D(E)$ , the Skorohod space of càdlàg functions from  $[0, \infty)$  to  $E$ , and by  $(F_t^X)_{t \geq 0}$  its natural filtration. We also mention that, throughout the paper,  $\mathbb{E}$  stands for a reference expectation operator. Moreover,  $\mathbf{A}$  (resp.  $\mathfrak{D}(\mathbf{A})$ ) stands for its infinitesimal generator (resp. its domain). We have in mind the following situations

- (1)  $E = \mathbb{R}$  and  $X$  is an  $\alpha$ -stable Lévy process.
- (2)  $E = \mathbb{R}^+$  or  $[0, \infty)$  and  $X$  is a  $\frac{1}{\alpha}$ -semi-stable processes in the terminology of Lamperti.

More precisely, let  $\xi$  be a Lévy process starting from  $x \in \mathbb{R}$ , Lamperti [16] showed that the time change process

$$(1.2) \quad X_t = e^{\xi_{A_t}}, \quad t \geq 0,$$

---

2000 *Mathematics Subject Classification.* Primary 31C05, 60G18; Secondary 33E12, 20C20.

*Key words and phrases.* Invariant functions, generalized Ornstein-Uhlenbeck processes, first exit times, special functions.

I wish to thank an anonymous referee for very helpful comments and careful reading that led to improving the presentation of the paper. This work was partially carried out while I was visiting the Project OMEGA of INRIA at Sophia-Antipolis and the Department of Mathematics of ETH Zürich. I would like to thank the members of both groups for their hospitality.

where

$$A_t = \inf\{u \geq 0; V_u := \int_0^u e^{\alpha \xi_s} ds > t\},$$

is an  $\frac{1}{\alpha}$ -semi-stable positive Markov process starting from  $e^x$ . Further, we assume that  $-\infty < b := \mathbb{E}[\xi_1] < \infty$  and denote the characteristic exponent of  $\xi$  by  $\Psi$ . We also suppose that  $\xi$  is not arithmetic (i.e. does not live on a discrete subgroup  $r\mathbb{Z}$  for some  $r > 0$ ). Then, on the one hand, for  $b > 0$  (resp.  $b \geq 0$  and  $\xi$  is spectrally negative), it is plain that  $X$  has infinite lifetime and we know from Bertoin and Yor [2, Theorem 1] (resp. Bertoin and Yor [3, Proposition 1]), that the family of probability measures  $(\mathbb{P}_x)_{x>0}$  converges in the sense of finite-dimensional distributions to a probability measure, denoted by  $\mathbb{P}_0$ , as  $x \rightarrow 0+$ . On the other hand, if  $b < 0$ , then it is plain that  $\xi$  drift towards  $-\infty$  and  $X$  has a finite lifetime which is  $T_0^X = \inf\{u \geq 0; X_{u-} = 0, X_u = 0\}$ . In this case, we assume that there exists a unique  $\theta > 0$  which yields the so-called Cramér condition

$$(1.3) \quad \mathbb{E}[e^{\theta \xi_1}] = 1.$$

Then, under the additional condition  $0 < \theta < \alpha$ , Rivero [24] showed that the minimal process  $(X, T_0^X)$  admits a unique recurrent extension that hits and leaves 0 continuously a.s. and which is a  $\frac{1}{\alpha}$ -semi-stable process on  $[0, \infty)$ . With a slight abuse of notation, we write  $(\mathbb{P}_x)_{x>0}$  for the family of laws of such a recurrent extension. We gather the different possibilities in the following.

**H0.**  $E = [0, \infty)$  and either  $b > 0$  or (1.3) holds with  $\theta < \alpha$ . Moreover, if  $\xi$  is spectrally negative, the case  $b = 0$  is also allowed.

We will also use the following hypothesis

**H1.**  $E \subseteq [0, \infty)$ .  $\xi$  has finite exponential moments of arbitrary positive orders, i.e.  $\psi(m) < \infty$  for every  $m \geq 0$  where  $\psi(m) = -\Psi(-im)$ .

If  $\xi$  is spectrally negative, excluding the degenerate cases, then **H1** holds and for  $b \in \mathbb{R}$ , we write  $\theta_0$  for the largest root of the equation  $\psi(u) = 0$ . Then, being continuous and increasing on  $[\theta_0, \infty)$ ,  $\psi$  has a well-defined inverse function  $\phi : [0, \infty) \rightarrow [\theta_0, \infty)$  which is also continuous and increasing.

For  $r > 0$ , we write  $P_t^r = e^{-rt}P_t$ . We say that a non-negative function,  $\mathcal{I}_r$ , is  $r$ -excessive (resp.  $r$ -invariant) for  $P_t$  if for any  $t > 0$ ,

$$P_t^r \mathcal{I}_r(x) \leq \mathcal{I}_r(x),$$

(resp. if we have  $=$  in place of  $\leq$ ) and  $\lim_{t \downarrow 0} P_t^r \mathcal{I}_r(x) = \mathcal{I}_r(x)$  pointwise. Taking  $r = 1$  and writing simply  $\mathcal{I} = \mathcal{I}_1$ , we have  $P_t^1 \mathcal{I}(x) = \mathcal{I}(x)$ . The self-similarity property (1.1) then yields

$$(1.4) \quad P_t^r \left( d_{r^{-\frac{1}{\alpha}}} \mathcal{I} \right) (r^{-\frac{1}{\alpha}} x) = \mathcal{I}(x),$$

which entails the identity  $\mathcal{I}_r(x) = \mathcal{I}(r^{\frac{1}{\alpha}} x)$  for all  $x \in E$  and  $r > 0$ . We denote the convex cone of 1-excessive (resp. 1-invariant) functions for  $X$  by  $\mathfrak{E}(X)$  (resp.  $\mathfrak{I}(X)$ ).

For any  $\lambda > 0$ , the Ornstein-Uhlenbeck (for short OU) semigroup,  $(Q_t)_{t \geq 0}$ , is defined, for  $f \in \mathfrak{B}(E)$ , by

$$(1.5) \quad Q_t f(x) = P_{e_\lambda(t)} \left( d_{e'_\lambda(-t)} f \right) (x), \quad x \in E, t \geq 0,$$

where  $e_\lambda(t) = \frac{e^{\lambda t} - 1}{\lambda}$ ,  $\chi = \alpha\lambda$  and we write  $v_\lambda(\cdot)$  for the continuous increasing inverse function of  $e_\lambda(\cdot)$ . We mention that such a deterministic transformation of self-similar processes traces back to Doob [10] who studied the generalized OU processes driven by symmetric stable Lévy processes. Moreover, Carmona et al. [7, Proposition 5.8] showed that  $(Q_t)_{t \geq 0}$  is a Feller semigroup with infinitesimal generator, for  $f \in \mathfrak{D}(\mathbf{A})$ , given by

$$\mathbf{U}f(x) = \mathbf{A}f(x) - \lambda x f'(x).$$

Let  $U$  be the realization of the Feller semigroup  $(Q_t)_{t \geq 0}$ . It follows from (1.5) that

$$(1.6) \quad U_t = e'_\lambda(-t) X_{e_\lambda(t)}, \quad t \geq 0.$$

We deduce, with obvious notation, that  $T_0^U = v_\chi(T_0^X)$  a.s.. If  $E = \mathbb{R}$  (resp. otherwise), we call  $U$  a self-similar (resp. semi-stable) OU process. We denote by  $(\mathbb{Q}_x)_{x > 0}$  the family of probability measures of a *semi-stable* OU process. We deduce, from the Lamperti mapping (1.2) and the discussions above the following.

**Proposition 1.1.** *For any  $x > 0$ , there exists a one to one mapping between the law of a Lévy process starting from  $\log(x)$  and the law of a semi-stable OU process starting from  $x$ . More precisely, we have*

$$(1.7) \quad U_t = e^{-\lambda t} e^{\xi \Delta_t}, \quad t < T_0^U,$$

where  $\Delta_t = \int_0^t U_s^{-\alpha} ds$ . Note that for  $b > 0$ , the previous identity holds for any  $t \geq 0$ .

Moreover, if (1.3) holds with  $0 < \theta < \alpha$ , then the minimal process  $(U, T_0^U)$  admits a recurrent extension which hits and leaves 0 continuously a.s. which is the OU process associated to  $(X, \mathbb{P}_x)$ . We write its family of laws by  $(\mathbb{Q}_x)_{x > 0}$ .

Under the condition **H1**, for any  $\gamma > 0$ , we denote by  $\mathbb{Q}_x^{(\gamma)}$  the law of the semi-stable OU process, starting at  $x \in \mathbb{R}^+$ , associated to the Lévy process having Laplace exponent  $\psi_\gamma(u) = \psi(u + \gamma) - \psi(\gamma)$ ,  $u \geq 0$ .

We say that a probability measure  $m$  is invariant for  $U$  if it satisfies, for any  $f \in \mathfrak{B}(E)$ ,

$$\int_E Q_t f(x) m(dx) = \int_E f(x) m(dx).$$

Finally, let  $G_q$  be a gamma random variable independent of  $X$ , with parameter  $q > 0$ , whose law is given by  $\gamma(dr) = \frac{e^{-r} r^{q-1}}{\Gamma(q)} dr$ . We are now ready to state the following.

**Theorem 1.2.** *If  $E = \mathbb{R}$  or **H0** holds then the Feller process  $U$  is positively recurrent and its unique invariant measure is  $\chi \mathbb{P}_0(X_1 \in dx)$ .*

*Next, assume that  $\mathcal{I} \in L^1(\gamma(dr))$ . For any  $q > 0$ , we introduce the function  $\mathcal{I}(q; x)$  defined by*

$$(1.8) \quad \mathcal{I}(q; x) = \chi^{\frac{q}{\alpha}} \mathbb{E} \left[ \mathcal{I} \left( \left( \chi G_{\frac{q}{\alpha}} \right)^{\frac{1}{\alpha}} x \right) \right], \quad x \in E.$$

*Then, if  $\mathcal{I} \in \mathfrak{J}(X)$  (resp.  $\mathfrak{E}(X)$ ) then  $\mathcal{I}(q; x) \in \mathfrak{J}^q(U)$  (resp.  $\mathfrak{E}^q(U)$ ).*

Consequently, if  $\mathcal{I} \in \mathfrak{I}(X)$ , we have, for any  $q > 0$ ,

$$(1.9) \quad (1 + \chi t)^{-\frac{q}{\alpha}} P_t \left( d_{(1+\chi t)^{-\frac{1}{\alpha}}} \mathcal{I} \right) (q; x) = \mathcal{I}(q; x), \quad x \in E.$$

*Remark 1.3.* (1) We call the multiplication operator (1.8) associated to a fractional power of a Gamma random variable, the  $\Gamma$ -transform.

(2) The characterization of time-space invariant functions of the form (1.9), associated to self-similar processes, has been first identified by Shepp [26] in the case of the Brownian motion and by several authors for some specific processes: Yor [27] for the Bessel processes, Novikov [19] and Patie [22] for the one sided-stable processes. Whilst in the mentioned papers, the authors made use of specific properties of the studied processes to derive the time-space martingales, we provide a proof which is based simply on the self-similarity property.

We proceed by investigating the process  $Y$ , defined, for any  $x, \beta \in \mathbb{R}$  and  $\xi_0 = 0$  a.s., by

$$(1.10) \quad Y_t = e^{\alpha \xi t} \left( x + \beta \int_0^t e^{-\alpha \xi s} ds \right), \quad t \geq 0.$$

We call  $Y$  the Lévy OU process. We mention that this generalization of the OU process is a specific instance of the continuous analogue of random recurrence equations, as shown by de Haan and Karandikar [12]. They have been also well-studied by Carmona et al. [6], Erickson and Maller [11], Bertoin et al. [1] and by Kondo et al. [14]. In Carmona et al. [6], it is proved that  $Y$  is a homogeneous Markov process with respect to the filtration generated by  $\xi$ . Moreover, they showed, from the stationarity and the independency of the increments of  $\xi$ , that, for any fixed  $t \geq 0$ ,

$$Y_t \stackrel{(d)}{=} x e^{\alpha \xi t} + \beta \int_0^t e^{\alpha \xi s} ds.$$

Then, if  $\mathbb{E}[\xi_1] < 0$ , they deduced that, as  $t \rightarrow \infty$ ,  $\xi_t \xrightarrow{(a.s.)} -\infty$  and  $Y_t \xrightarrow{(d)} \beta V_\infty = \int_0^\infty e^{\alpha \xi s} ds$ . We refer to Bertoin and Yor [4] for a thorough survey on the exponential functional of Lévy processes. In the spectrally negative case, it is well known that the law of  $V_\infty$  is self-decomposable, hence absolutely continuous and unimodal. Moreover, under the additional assumption that  $\theta < \alpha$ , its law has been computed in term of the Laplace transform by Patie [20]. Now, we introduce the process  $Z$  defined, for any  $x \neq 0, \beta \in \mathbb{R}$  and  $\xi_0 = 0$  a.s., by

$$(1.11) \quad Z_t = e^{\alpha \xi t} \left( x + \beta \int_0^t e^{\alpha \xi s} ds \right)^{-1}, \quad t \geq 0.$$

Before stating the next result, we introduce some notation. Let  $B$  be a Borel subset of  $E$  and we write  $T_B^U$  for the first exit time from  $B$  by  $U$ . With a slight abuse of terminology, we say that for any  $x \in E$ , a non-negative function  $\mathcal{H}$  is a  $(q\Delta, B)$ -harmonic function for  $(U, \mathbb{Q}_x)$  if

$$(1.12) \quad \mathbb{E}_x \left[ e^{-q\Delta T_B^U} \mathcal{H}(U_{T_B^U}) \mathbb{1}_{\{T_B^U < T_0^U\}} \right] = \mathcal{H}(x).$$

When  $\Delta_t$  is replaced by  $t$  in the previous expression, we simply say that  $\mathcal{H}$  is a  $(q, B)$ -harmonic function for  $(U, \mathbb{Q}_x)$ . We are ready to state the following.

**Theorem 1.4.** Set  $\beta = \alpha\lambda x$  in (1.11). Then, to a process  $Z$  starting from  $\frac{1}{x}$ , with  $x \neq 0$ , one can associate a semi-stable OU process  $(U, \mathbb{Q}_1)$  such that

$$(1.13) \quad Z_t = x^{-1}U_{\nabla_t}^\alpha, \quad t < T_0^U,$$

where  $\nabla_t = \int_0^t Z_s ds$  and its inverse is given by  $\Delta_t = \int_0^t U_s^{-\alpha} ds$ . Note that for  $b > 0$ , the previous identity holds for any  $t \geq 0$ . Consequently, with  $x > 0$ ,  $Z$  is a Feller process on  $(0, \infty)$ .

Moreover, let  $q > 0$ ,  $0 \leq a < b \leq +\infty$  and  $x \in (a, b)$ . Then, a  $(q\Delta, T_{(ax, bx)}^U)$ -harmonic function for  $(U, \mathbb{Q}_1)$  is a  $(q, T_{(a^\alpha, b^\alpha)}^Z)$ -harmonic function for the process  $Z$  starting from  $x^{-\alpha}$ . Similarly, a  $(q\Delta, T_{(\frac{x}{b}, \frac{x}{a})}^U)$ -harmonic function for  $(U, \mathbb{Q}_1)$  is a  $(q, T_{(a^\alpha, b^\alpha)}^{\hat{Y}})$ -harmonic function for the process  $\hat{Y}$  starting from  $x^\alpha$ , the Lévy OU process associated to the Lévy process  $\hat{\xi} = -\xi$ , the dual of  $\xi$  with respect to the Lebesgue measure.

Finally, assume that **H1** holds and write  $p_q(x) = x^q$  for  $x, q > 0$ . If the function  $\mathcal{H}$  is  $(\lambda\phi(q), B)$ -harmonic function for  $(U, \mathbb{Q}_x^{(\phi(q))})$  then the function  $p_{\phi(q)}\mathcal{H}$  is  $(q\Delta, B)$ -harmonic function for  $(U, \mathbb{Q}_x)$ .

## 2. Proofs

**2.1. Proof of Theorem 1.2.** The description of the unique invariant measure is a refinement of Carmona et al. [7, Proposition 5.7] where therein the proof is provided for  $\mathbb{R}$ -valued self-similar processes and can be extended readily for the  $\mathbb{R}^+$ -valued case under the condition **H0**, which ensures that  $(X, \mathbb{P}_x)$  admits an entrance law at 0.

Next, let us assume that  $\mathcal{I} \in L^1(\gamma(dr)) \cap \mathfrak{J}(X)$ . We need to show that for any  $q > 0$ ,  $e^{-qt}Q_t\mathcal{I}(q; x) = \mathcal{I}(q; x)$ . For  $x \in E$ , we deduce from the definition of  $(Q_t)_{t \geq 0}$  that

$$\begin{aligned} \Gamma\left(\frac{q}{\chi}\right) e^{-qt}Q_t\mathcal{I}(q; x) &= \chi^{\frac{q}{x}} e^{-qt} \mathbb{E}_x \left[ \int_0^\infty \mathcal{I}\left((\chi r)^{\frac{1}{\alpha}} U_t\right) e^{-r} r^{\frac{q}{x}-1} dr \right] \\ &= \chi^{\frac{q}{x}} e^{-qt} \mathbb{E}_x \left[ \int_0^\infty \mathcal{I}\left((\chi r)^{\frac{1}{\alpha}} e'_\lambda(-t) X_{e_\chi(t)}\right) e^{-r} r^{\frac{q}{x}-1} dr \right]. \end{aligned}$$

Using the change of variable  $u = \chi e'_\lambda(-t)r$ , Fubini theorem and (1.4), we get

$$\begin{aligned} e^{-qt}Q_t\mathcal{I}(q; x) &= \mathbb{E}_x \left[ \int_0^\infty e^{-ue_\chi(t)} \mathcal{I}\left(u^{\frac{1}{\alpha}} X_{e_\chi(t)}\right) e^{-\frac{u}{x}} u^{\frac{q}{x}-1} du \right] \\ &= \int_0^\infty \mathcal{I}\left(u^{\frac{1}{\alpha}} x\right) e^{-\frac{u}{x}} u^{\frac{q}{x}-1} du \\ &= \mathcal{I}(q; x) \end{aligned}$$

where the last line follows after the change of variable  $u = \chi r$ . The case  $\mathcal{I} \in L^1(\gamma(dr)) \cap \mathfrak{E}(X)$  is obtained by following the same line of reasoning. The last assertion is deduced from (1.5) and (1.8) by performing the change of variable  $u = v_\chi(t)$ , with  $v_\chi(t) = \frac{1}{\chi} \log(1 + \chi t)$ .

2.2. **Proof of Theorem 1.4.** Setting  $\beta = \alpha\lambda x$ , the Lamperti mapping (1.2) yields

$$\begin{aligned} Z_t &= x^{-1} e^{\alpha\xi t} \left( 1 + \alpha\lambda \int_0^t e^{\alpha\xi s} ds \right)^{-1} \\ &= x^{-1} \left( \frac{1}{(1 + \alpha\lambda \cdot)^{\frac{1}{\alpha}}} X \cdot \right)_{V_t}^\alpha \\ &= x^{-1} (U_{v_\chi(\cdot)})_{V_t}^\alpha \end{aligned}$$

where the last identity follows from (1.5). The proof of the assertion (1.13) is completed by observing that

$$(v_\chi(V_t))' = e^{\alpha\xi t} \left( 1 + \alpha\lambda \int_0^t e^{\alpha\xi s} ds \right)^{-1}.$$

Moreover, since the mapping  $x \mapsto x^\alpha$  is a homeomorphism of  $\mathbb{R}^+$ , the Feller property follows from its invariance by "nice" time change of Feller processes, see Lamperti [15, Theorem 1]. We also obtain the following identities

$$\begin{aligned} T_{(a,b)}^Z &= \inf \{u \geq 0; Z_u \notin (a, b)\} \\ &= \inf \{u \geq 0; U_{\nabla_u}^\alpha \notin (ax, bx)\} \\ &= \Delta \left( \inf \left\{ u \geq 0; U_u \notin \left( (ax)^\frac{1}{\alpha}, (bx)^\frac{1}{\alpha} \right) \right\} \right). \end{aligned}$$

The characterization of the harmonic functions of  $Z$  follows. The characterization of the harmonic functions of  $\hat{Y}$  are readily deduced from the ones of  $Z$  and the identity

$$\hat{Y}_t = \frac{1}{Z_t}, \quad t \geq 0.$$

The proof of Theorem is then completed by using the following Lemma together with an application of the optional stopping theorem.

**Lemma 2.1.** *Assume that **H1** holds, then for  $\gamma, \delta \geq 0$  and  $x > 0$ , we have*

$$(2.1) \quad d\mathbb{Q}_x^{(\gamma)} = \left( \frac{U_t}{x} \right)^{\gamma-\delta} e^{\lambda(\gamma-\delta)t - (\psi(\gamma) - \psi(\delta))\Delta t} d\mathbb{Q}_x^{(\delta)}, \quad \text{on } F_t^U \cap \{t < T_0^U\}.$$

Note that for  $b > 0$  the condition " on  $\{t < T_0^U\}$ " can be omitted. For the particular case  $\gamma = \theta$  and  $\delta = 0$ , the absolute continuity relationship (2.1) reduces to

$$d\mathbb{Q}_x^{(\theta)} = \left( \frac{U_t}{x} \right)^\theta e^{\lambda\theta t} d\mathbb{Q}_x, \quad \text{on } F_t^U \cap \{t < T_0^U\}.$$

*Proof.* We start by recalling that in Patie [20], the following power Girsanov transform has been derived, under **H1**, for  $\gamma, \delta \geq 0$  and  $x > 0$ , with obvious notation,

$$d\mathbb{P}_x^{(\gamma)} = \left( \frac{X_t}{x} \right)^{\gamma-\delta} e^{-(\psi(\gamma) - \psi(\delta))A_t} d\mathbb{P}_x^{(\delta)}, \quad \text{on } F_t^X \cap \{t < T_0^X\}.$$

The assertion (2.1) follows readily by time change and recalling that

$$A_{e_\chi(t)} = \int_0^t U_u^{-\alpha} du.$$

We complete the proof by recalling that for  $b > 0$ ,  $U$  does not reach 0 a.s.. □

### 3. Applications

In this section, we illustrate our results to some new interesting examples.

**3.1. First passage times and overshoot of stable OU processes.** Let  $X$  be an  $\alpha$ -stable Lévy process whose characteristic exponent satisfy, for  $u \in \mathbb{R}$ ,

$$\Psi(iu) = -c|u|^\alpha \left( 1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\alpha\pi}{2}\right) \right)$$

where  $1 < \alpha < 2$  and for convenience we take  $c = (1 + \beta^2 \tan^2(\frac{\alpha\pi}{2}))^{-1/2}$ . Then, we introduce the constant  $\rho = \mathbb{P}(X_1 > 0)$  which was evaluated by Zolotarev [28] as follows

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1} \left( \beta \tan\left(\frac{\alpha\pi}{2}\right) \right).$$

Following Doney [9], we introduce, for any integers  $k, l$ , the class  $C_{k,l}$  of stable processes such that

$$\rho + k = l\tilde{\alpha}$$

where  $\tilde{\alpha} = \frac{1}{\alpha}$ . For  $m \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $z \in \mathbb{C}$ , introduce the function

$$f_m(x, z) = \prod_{i=0}^m \left( z + e^{ix(m-2i)\pi} \right).$$

Next, we recall from the Wiener-Hopf factorization of Lévy processes due to Rogozin [25], that the law of the first passage times  $\tau_0^X$  and the over(under)shoot of  $X$  at the level 0 is described by the following identities, for  $\delta, r > 0$  and  $p \geq 0$ ,

$$\begin{aligned} \int_0^\infty e^{-\delta x} \mathbb{E}_{-x} \left[ e^{-r\tau_0^X - pX_{\tau_0^X-}} \right] dx &= \frac{1}{\delta - p} \left( 1 - \frac{\Psi^+(-r\frac{1}{\alpha}\delta)}{\Psi^+(-r\frac{1}{\alpha}p)} \right) \\ \int_0^\infty e^{-\delta x} \mathbb{E}_x \left[ e^{-r\tau_0^X - pX_{\tau_0^X-}} \right] dx &= \frac{1}{\delta - p} \left( 1 - \frac{\Psi^-(r\frac{1}{\alpha}\delta)}{\Psi^-(r\frac{1}{\alpha}p)} \right) \end{aligned}$$

where  $(1 - \Psi(\delta))^{-1} = \Psi^-(\delta)\Psi^+(\delta)$ . Here  $\Psi^+(\delta)$  (resp.  $\Psi^-(\delta)$ ) is analytic in  $\Re(\delta) < 0$  (resp.  $\Re(\delta) > 0$ ) continuous and nonvanishing on  $\Re(\delta) \leq 0$  (resp.  $\Re(\delta) \geq 0$ ). In Doney [9], the author computed the Wiener-Hopf factors for stable processes in  $C_{k,l}$  as follows

$$\begin{aligned} \Psi^+(z) &= \frac{f_{k-1}(\alpha, (-1)^l(-z)^\alpha)}{f_{l-1}(\tilde{\alpha}, (-1)^{k+1}z)}, \quad \mathfrak{A}rg(z) \neq 0, \\ \Psi^-(z) &= \frac{f_{l-1}(\tilde{\alpha}, (-1)^{k+1}z)}{f_k(\alpha, (-1)^l z^\alpha)}, \quad \mathfrak{A}rg(z) \neq -\pi, \end{aligned}$$

where  $z^\beta$  stands for  $\sigma^\beta e^{i\beta\phi}$  when  $z = \sigma e^{i\phi}$  with  $\sigma > 0$  and  $-\pi < \phi \leq \pi$ . Observe also that  $\Psi^+(-x\frac{1}{\alpha}) \sim x^{-\rho}$  for large real  $x$ . Moreover, using the fact that the function  $\mathbb{E}_x \left[ e^{-r\tau_0^X - pX_{\tau_0^X-}} \right]$ ,  $x \in \mathbb{R}$ , is  $r$ -excessive for the semigroup of  $X$ , we deduce from the  $\Gamma$ -transform the following.

**Corollary 3.1.** For any  $q, \delta > 0, p \geq 0$ , and for any integers  $k, l$  such that  $X \in C_{k,l}$ , we have

$$\int_{-\infty}^0 e^{\delta x} \mathbb{E}_x \left[ e^{-q\tau_0^U - pU_{\tau_0^U}^-} \right] dx = \frac{1}{\delta - p} \left( \chi^{\frac{q}{\alpha}} - \frac{1}{\Gamma(\frac{q}{\alpha})} \int_0^\infty \frac{\Psi^+(-r^{\frac{1}{\alpha}}\delta)}{\Psi^+(-r^{\frac{1}{\alpha}}p)} e^{-\frac{r}{\alpha} r^{\frac{q}{\alpha}-1}} dr \right)$$

$$\int_0^\infty e^{-\delta x} \mathbb{E}_x \left[ e^{-q\tau_0^U - pU_{\tau_0^U}^-} \right] dx = \frac{1}{\delta - p} \left( \chi^{\frac{q}{\alpha}} - \frac{1}{\Gamma(\frac{q}{\alpha})} \int_0^\infty \frac{\Psi^-(r^{\frac{1}{\alpha}}\delta)}{\Psi^-(r^{\frac{1}{\alpha}}p)} e^{-\frac{r}{\alpha} r^{\frac{q}{\alpha}-1}} dr \right).$$

**3.2. First passage times of one-sided semi-stable- and Lévy-OU processes.** We now fix  $(P_t)_{t \geq 0}$  to be the semigroup of a spectrally negative  $\frac{1}{\alpha}$ -semi-stable process  $X$ .  $X$  is then associated via the Lamperti mapping (1.2) to a spectrally negative Lévy process,  $\xi$ , which we assume to have a finite mean  $b$ . Its characteristic exponent  $\psi$  has the well known Lévy-Khintchine representation

$$(3.1) \quad \psi(u) = bu + \frac{\sigma}{2}u^2 + \int_{-\infty}^0 (e^{ur} - 1 - ur)\nu(dr), \quad u \geq 0,$$

where  $\sigma \geq 0$  and the measure  $\nu$  satisfies the integrability condition  $\int_{-\infty}^0 (r \wedge r^2)\nu(dr) < +\infty$ . In Patie [20], the author computed the Laplace transform of the first passage times above of  $X$  as follows. For any  $r \geq 0$  and  $0 \leq x \leq a$ ,

$$(3.2) \quad \mathbb{E}_x \left[ e^{-rT_a^X} \right] = \frac{\mathcal{I}_{\alpha,\psi}(rx^\alpha)}{\mathcal{I}_{\alpha,\psi}(ra^\alpha)}$$

where the entire function,  $\mathcal{I}_{\alpha,\psi}$ , is given, for  $\gamma \geq 0$  and  $\alpha > 0$ , by

$$\mathcal{I}_{\alpha,\psi}(z) = \sum_{n=0}^{\infty} a_n(\psi; \alpha) z^n, \quad z \in \mathbb{C}$$

and

$$a_n(\psi; \alpha)^{-1} = \prod_{k=1}^n \psi(\alpha k), \quad a_0 = 1.$$

Using the  $\Gamma$ -transform, we introduce the following power series

$$(3.3) \quad \mathcal{I}_{\alpha,\psi}(q; z) = \sum_{n=0}^{\infty} a_n(\psi; \alpha)(q)_n z^n$$

where  $(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}$  is the Pochhammer symbol and we have used the integral representation of the gamma function  $\Gamma(q) = \int_0^\infty e^{-r} r^{q-1} dr$ ,  $\Re(q) > 0$ . By means of the following asymptotic formula of ratio of gamma functions, see e.g. Lebedev [17, p.15], for  $\delta > 0$ ,

$$(3.4) \quad (z+n)_\delta = z^\delta \left[ 1 + \frac{\delta(2n+\delta-1)}{2z} + O(z^{-2}) \right], \quad |\arg z| < \pi - \epsilon,$$

we deduce that  $\mathcal{I}_{\alpha,\psi}(q; z)$  is an entire function in  $z$  and is analytic on the domain  $\{q \in \mathbb{C}; \Re(q) > -1\}$ . For  $b < 0$ , we recall that there exists  $\theta > 0$  such that  $\psi(\theta) = 0$  and thus  $\psi_\theta(u) = \psi(\theta + u)$ . In this case, by setting  $\theta_\alpha = \frac{\theta}{\alpha}$ , it is shown in Patie [20] that there exists a positive constant  $C_{\theta_\alpha}$  such that

$$\mathcal{I}_{\alpha,\psi}(x^\alpha) \sim C_{\theta_\alpha} x^\theta \mathcal{I}_{\alpha,\psi_\theta}(x^\alpha) \quad \text{as } x \rightarrow \infty.$$

We also introduce the function  $\mathcal{N}_{\alpha,\psi,\theta}(q; x^\alpha)$  defined by

$$(3.5) \quad \mathcal{N}_{\alpha,\psi,\theta}(q; x^\alpha) = \mathcal{I}_{\alpha,\psi}(q; x^\alpha) - C_{\theta_\alpha} x^\theta \frac{\Gamma(q + \theta_\alpha)}{\Gamma(q)} \mathcal{I}_{\alpha,\psi_\theta}(q + \theta_\alpha; x^\alpha), \quad \Re(x) \geq 0.$$

Moreover, if we assume that there exists  $\beta \in [0, 1]$  and a positive constant  $a_\beta$  such that  $\lim_{u \rightarrow \infty} \psi(u)/u^{1+\beta} = a_\beta$ , then  $C_{\theta_\alpha}$  is characterized by

$$C_{\theta_\alpha} = \begin{cases} \frac{\Gamma(1-\theta_\alpha)}{\alpha} \frac{(\theta_\alpha-1)!}{\prod_{k=1}^{\theta_\alpha-1} \psi(\alpha k)}, & \text{if } \theta_\alpha \text{ is a positive integer,} \\ \frac{\Gamma(1-\theta_\alpha)}{\alpha} a_\beta^{-\theta_\alpha} e^{E_\gamma \beta \theta_\alpha} \prod_{k=1}^{\infty} e^{-\frac{\beta \theta_\alpha}{k} \frac{(k+\theta_\alpha)\psi(\alpha k)}{k\psi(\alpha k+\theta_\alpha)}}, & \text{otherwise,} \end{cases}$$

where  $E_\gamma$  stands for the Euler-Mascheroni constant. We recall, also from Patie [20], that, for  $r, x \geq 0$ ,

$$\mathbb{E}_x \left[ e^{-rT_0^X} \right] = \mathcal{I}_{\alpha,\psi}(rx^\alpha) - C_{\theta_\alpha} (r^{\frac{1}{\alpha}} x)^\theta \mathcal{I}_{\alpha,\psi_\theta}(rx^\alpha).$$

We deduce from Theorems 1.2 and 1.4 the following.

**Corollary 3.2.** *Let  $q \geq 0$  and  $0 < x \leq a$ . Then,*

$$\mathbb{E}_x \left[ e^{-qT_a^U} \right] = \frac{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi x^\alpha\right)}{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi a^\alpha\right)}$$

and

$$\mathbb{E}_x \left[ \left(1 + \chi T_{(\alpha)}^X\right)^{-\frac{q}{\chi}} \right] = \frac{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi x^\alpha\right)}{\mathcal{I}_{\alpha,\psi}\left(\frac{q}{\chi}; \chi a^\alpha\right)}$$

where  $T_{(\alpha)}^X = \inf\{u \geq 0; X_u = a(1 + \chi u)^{\frac{1}{\alpha}}\}$ . We also deduce that

$$\mathbb{E}_x \left[ e^{-q\Delta_{T_a^U}} \mathbb{I}_{\{T_a^U < T_0^U\}} \right] = \left(\frac{x}{b}\right)^\gamma \frac{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi x^\alpha\right)}{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi a^\alpha\right)}.$$

Moreover, assume  $b > 0$  and set  $\beta = \alpha\lambda x$  and  $\gamma = \phi(q)$ . Then,

$$\mathbb{E}_{\frac{1}{x}} \left[ e^{-qT_a^Z} \right] = \left(\frac{1}{bx}\right)^\gamma \frac{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi\right)}{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi(ax)^\alpha\right)}, \quad 0 < \frac{1}{x} \leq a,$$

$$\mathbb{E}_a \left[ e^{-qT_x^{\hat{Y}}} \right] = \left(\frac{x}{a}\right)^{\frac{\gamma}{\alpha}} \frac{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \chi^{\frac{1}{\alpha}}\right)}{\mathcal{I}_{\alpha,\psi_\gamma}\left(\frac{\gamma}{\alpha}; \left(\frac{\chi a}{x}\right)^{\frac{1}{\alpha}}\right)}, \quad 0 < x \leq a.$$

Finally, if  $b < 0$  and  $0 < \theta < \alpha$ , we have

$$\mathbb{E}_x \left[ e^{-qT_0^U} \right] = \frac{\mathcal{N}_{\alpha,\psi,\theta}\left(\frac{q}{\chi}; \chi x^\alpha\right)}{\mathcal{N}_{\alpha,\psi,\theta}\left(\frac{q}{\chi}; \chi x^\alpha\right)}.$$

*Remark 3.3.* From the strong Markov property and the absence of positive jumps, we easily get that first passage times above for the processes  $U$  and  $Z$  are infinitely divisible random variables. Hence, we obtain from Corollary 3.2, that the functions (3.3) and (3.5) are Laplace transforms, with respect to the parameter  $q$ , of infinitely divisible distributions concentrated on the positive real line.

We end up by investigating some special cases which allow to make some connections between the power series introduced and some well-known or new special functions.

**3.2.1. The confluent hypergeometric functions.** We first consider a Brownian motion with drift  $-\nu$ , i.e.  $\psi(u) = \frac{1}{2}u^2 - \nu u$ . Setting  $\alpha = 2$ , we have  $\theta = 2\nu$  and therefore we assume  $\nu < 1$ . Its associated semi-stable process is well known to be a Bessel process of index  $\nu$  and thus the associated Ornstein-Uhlenbeck process is, in the case  $n = 2\nu + 1 \in \mathbb{N}$ , the radial norm of  $n$ -dimensional Ornstein-Uhlenbeck process. We get

$$\mathcal{I}_{2,\psi}(x) = (x/2)^{\nu/2} \Gamma(-\nu + 1) I_{-\nu}(\sqrt{2x})$$

where  $I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}$  stands for the modified Bessel function of index  $\nu$ , see e.g. Lebedev [17, 5.], and

$$\begin{aligned} \mathcal{I}_{2,\psi}(q; x^2) &= \Phi\left(q, 1 - \nu, \frac{x^2}{2}\right) \\ \mathcal{I}_{2,\psi_{2\nu}}(q; x^2) &= \Phi\left(q + \nu, \nu + 1, \frac{x^2}{2}\right) \end{aligned}$$

where  $\Phi(q, \nu, x) = \sum_{n=0}^{\infty} \frac{(q)_n}{(\nu)_n n!} x^n$  stands for the confluent hypergeometric function of the first kind, see e.g. Lebedev [17, 9.9]. Using the asymptotic behavior of the Bessel function

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \rightarrow \infty,$$

we deduce that  $C_{2\nu} = -\frac{\Gamma(-\nu)}{\Gamma(\nu)}$ . Hence,

$$\begin{aligned} \mathcal{N}_{\alpha,\psi_{2\nu}}(q; x^2) &= \left( \Phi\left(q, 1 - \nu, \frac{x^2}{2}\right) + x^{2\nu} \frac{\Gamma(-\nu)\Gamma(q + \nu)}{\Gamma(\nu)\Gamma(q)} \Phi\left(q, 1 - \nu, \frac{x^2}{2}\right) \right) \\ &= \frac{\Gamma(q)\Gamma(q + \nu)}{\Gamma(\nu)} \Lambda\left(q, \nu + 1, \frac{x^2}{2}\right) \end{aligned}$$

where  $\Lambda(q, \nu + 1, \frac{x^2}{2})$  is the confluent hypergeometric of the second kind. We mention that, in this case, the results of Corollary 3.2 are well-known and can be found in Matsumoto and Yor [18] and in Borodin and Salminen [5, II.8.2].

**3.2.2. Some generalization of the Mittag-Leffler function.** Patie [21] introduced a new parametric family of one-sided Lévy processes which are characterized by the following Laplace exponents, for any  $1 < \alpha < 2$ , and  $\gamma > 1 - \alpha$ ,

$$(3.6) \quad \psi_\gamma(u) = \frac{1}{\alpha} ((u + \gamma - 1)_\alpha - (\gamma - 1)_\alpha).$$

Its characteristic triplet are  $\sigma = 0$ ,  $\nu(dy) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \frac{e^{(\alpha+\gamma-1)y}}{(1-e^y)^{\alpha+1}} dy$ ,  $y < 0$ , and  $b_\gamma = (\gamma)_\alpha (\Upsilon(\gamma-1) + \alpha) - \Upsilon(\gamma-1)$  where  $\Upsilon(\lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$  is the digamma function. In particular, if  $\gamma_0$  denotes the zero of the function  $\gamma \rightarrow b_\gamma$ , then for  $\gamma \geq \gamma_0 \in (1-\alpha, 0)$ ,  $b \geq 0$ .

The case  $\gamma = 0$ . (3.6) reduces to  $\psi(u) = \frac{1}{\alpha}(u-1)_\alpha$ . Observe that  $\theta = 1$ ,  $\psi'(1) = \frac{\Gamma(\alpha)}{\alpha}$  and

$$a_n(\psi; \alpha)^{-1} = \frac{\Gamma(\alpha(n+1))}{\Gamma(\alpha)}, \quad a_0 = 1.$$

The series (3.3) can be written as follows

$$\begin{aligned} \mathcal{I}_{1, \psi_1}(q; x) &= \Gamma(\alpha) \mathcal{M}_{\alpha, \alpha}^q(\alpha x) \\ \mathcal{I}_{1, \psi}(q; x) &= \Gamma(\alpha-1) \mathcal{M}_{\alpha, \alpha-1}^q(\alpha x) \end{aligned}$$

where

$$\mathcal{M}_{\alpha, \beta}^q(z) = \sum_{n=0}^{\infty} \frac{(q)_n z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C},$$

stands for the Mittag-Leffler function of parameter  $\alpha, \beta, q > 0$ , which was introduced by Prabhakar [23]. Moreover, we have, see e.g. Patie [20],

$$\sum_{n=0}^{\infty} \frac{z^{\alpha n}}{\Gamma(\alpha n + \beta)} \sim \frac{1}{\alpha} e^x x^{1-\beta} l(x^\alpha) \quad \text{as } x \rightarrow \infty,$$

with  $l$  a slowly varying function at infinity. Thus,  $C_{\frac{1}{\alpha}} = \frac{\alpha}{\alpha-1}$  and

$$\mathcal{N}_{\alpha, \psi_1}(q; x^\alpha) = \mathcal{M}_{\alpha, \alpha-1}^q(x^\alpha) - \frac{\alpha x}{\alpha-1} \frac{\Gamma(q + \frac{1}{\alpha})}{\Gamma(q)} \mathcal{M}_{\alpha, \alpha}^q(x^\alpha).$$

As concluding remarks, we first mention that in the diffusion case, i.e. when  $U$  is the Ornstein-Uhlenbeck process associated to a Bessel process, the law of the first passage time above can be expressed as an infinite convolution of exponential distributions with parameters given by the sequence of positive zeros of the confluent hypergeometric function, see Kent [13] for more details. Beside this case, we do not know whether such a representation is available. For instance, the location of the zeros of the generalized Mittag-Leffler functions, considered in the second example treated above, is still an open problem, see e.g. Craven and Csordas [8].

## References

- [1] J. Bertoin, A. Lindner and R. Maller. On continuity properties of the law of integrals of Lévy processes. *To appear in Séminaire de Probabilités* (2007).
- [2] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.* **17** (4), 389–400 (2002).
- [3] J. Bertoin and M. Yor. On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse Math.* **11** (1), 19–32 (2002).
- [4] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.* **2**, 191–212 (2005).

- [5] A.N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, 2<sup>nd</sup> edition (2002).
- [6] Ph. Carmona, F. Petit and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In M. Yor (ed.) *Exponential functionals and principal values related to Brownian motion*. *Biblioteca de la Rev. Mat. Iberoamericana* pages 73–121 (1997).
- [7] Ph. Carmona, F. Petit and M. Yor. Beta-gamma random variables and intertwining relations between certain Markov processes. *Rev. Mat. Iberoamericana* **14** (2), 311–368 (1998).
- [8] T. Craven and G. Csordas. The Fox-Wright functions and Laguerre multiplier sequences. *J. Math. Anal. Appl.* **314** (1), 109–125 (2006).
- [9] R.A. Doney. On Wiener-Hopf factorisation and the distribution of extrema for certain stable processes. *Ann. Prob.* **15** (4), 1352–1362 (1987).
- [10] J.L. Doob. The Brownian movement and stochastic equations. *Ann. of Math.* **43** (2), 351–369 (1942).
- [11] K.B. Erickson and R.A. Maller. Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals. In *Séminaire de Probabilités XXXVIII*, volume 1857 of *Lecture Notes in Math.*, pages 70–94. Springer, Berlin (2005).
- [12] L. de Haan and K.L. Karandikar. Embedding a stochastic difference equation into a continuous-time process. *Stochastic Process. Appl.* **32** (2), 225–235 (1989).
- [13] J.T. Kent. Eigenvalue expansions for diffusion hitting times. *Z. Wahrsch. Verw. Gebiete* **52**, 309–319 (1980).
- [14] H. Kondo, M. Maejima and K.I. Sato. Some properties of exponential integrals of Lévy processes and examples. *Electron. Comm. Probab.* **11**, 291–303 (2006).
- [15] J. Lamperti. On random time substitutions and the Feller property. In *Markov Processes and Potential Theory (Proc Sympos. Math. Res. Center, Madison, Wis., 1967)*, pages 87–101. Wiley, New York (1967).
- [16] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Geb.* **22**, 205–225 (1972).
- [17] N.N. Lebedev. *Special Functions and their Applications*. Dover Publications, New York (1972).
- [18] H. Matsumoto and M. Yor. An analogue of Pitman’s 2M-X theorem for exponential Wiener functionals, Part I: A time-inversion approach. *Nagoya Math. J.* **159**, 125–166 (2000).
- [19] A. A. Novikov. The martingale approach in problems on the time of the first crossing of nonlinear boundaries. *Trudy Mat. Inst. Steklov.* **158**, 130–152 (1981 (English version 1983)).
- [20] P. Patie. Infinitely divisibility of solutions to some semi-stable integro-differential equations and exponential functionals of Lévy processes. *Submitted* (2007).
- [21] P. Patie. A new family of spectrally negative Lévy processes, their exponential functionals via self-similar continuous state branching processes with immigration. *Submitted* (2007).
- [22] P. Patie. Two sided-exit problem for a spectrally negative  $\alpha$ -stable Ornstein-Uhlenbeck process and the Wright’s generalized hypergeometric functions. *Electron. Comm. Probab.* **12**, 146–160 (electronic) (2007).
- [23] T.R. Prabhakar. A singular integral equation with a generalized Mittag- Leffler function in the kernel. *Yokohama Math. J.* **19**, 7–15 (1971).
- [24] V. Rivero. Recurrent extensions of positive self similar markov processes and Cramér’s condition II. *Bernoulli* **13** (4), 1053–1070 (2007).

- [25] B.A. Rogozin. On the distribution of functionals related to boundary problems for processes with independent increments. *Theory Probab. Appl.* **11**, 580–591 (1966).
- [26] L.A. Shepp. A first passage problem for the Wiener process. *Ann. Math. Stat.* **38**, 1912–1914 (1967).
- [27] M. Yor. On square-root boundaries for Bessel processes and pole seeking Brownian motion. *Stochastic analysis and applications (Swansea, 1983). Lecture Notes in Mathematics* **1095**, 100–107 (1984).
- [28] V.M. Zolotarev. Mellin-Stieltjes transform in probability theory. *Theory Probab. Appl.* **2**, 433–460 (1957).

DEPARTMENT OF MATHEMATICAL STATISTICS AND ACTUARIAL SCIENCE, UNIVERSITY OF BERN, SIDLER-STRASSE, 5, CH-3012 BERN, SWITZERLAND

*E-mail address:* patie@stat.unibe.ch