# ON THE JOINT LAW OF THE $L^1$ AND $L^2$ NORMS OF A 3-DIMENSIONAL BESSEL BRIDGE

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ABSTRACT. We give an analytical expression for the joint Laplace transform of the $L^1$ and $L^2$ norms of a 3-dimensional Bessel bridge. We derive the results by using merely probabilistic arguments. More precisely we show that the law of this functional is closely connected with the one of the first passage time of an Ornstein Uhlenbeck process. The motivation for studying such problem are multiple. As an instance, they include the computation of the density of the first passage time of Brownian motion over some moving boundaries such as the square root and the quadratic ones.

## 1. Introduction

Let $(r_s, s \leq t)$ be a 3-dimensional Bessel bridge over the interval $[0,t]$ between $x$ and $y$, where $x$, $y$ are some positive reals and $t$ is a fixed time horizon. Introduce the couple of random variables

$$\left(N_t^{(1)}(r), N_t^{(2)}(r)\right) = \left(\int_0^t r_s ds, \int_0^t r_s^2 ds\right). \tag{1.1}$$

In this paper, we aim to compute explicitly its joint Laplace transform. Let $(W_t, t \geq 0)$ be a standard real-valued Brownian motion started at $x \in \mathbb{R}$ and set $H_a^{(\lambda)} = \inf\{s \geq 0;\, W_s = a\sqrt{1+2\lambda s}\}$, where $\lambda > 0$ and $a \in \mathbb{R}$. Doob's transform allows to relate $H_a^{(\lambda)}$ to the hitting time of the same level $a$ by an Ornstein-Uhlenbeck process with parameter $\lambda$. That is with $\sigma_a = \inf\{s \geq 0;\, U_s = a\}$ and

$$U_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s} dB_s\right), \quad t \geq 0, \tag{1.2}$$

where $B$ is another real-valued Brownian motion defined on the same probability space, we have $H_a = \frac{1}{2\lambda}\log\left(1 + 2\lambda\sigma_a\right)$ almost surely. We shall see that the determination of the distribution of $\sigma_a$, or equivalently that of $H_a$, amounts to the study of the joint distribution of the $L^1$ and $L^2$ norms of a 3-dimensional Bessel bridge. While we are interested in the joint law, we mention that there is a substantial literature devoted to the study of the law of the $L^1$ norm of the Brownian excursion, that is when $x = y = 0$, see e.g. [18],[9], [21] and [12]. The $L^2$ norm of

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the Bessel bridge, which is closely related to the Lévy stochastic area formula, has been also intensively studied by many authors including for instance [22], [6] and the references therein.

Then, we establish a relationship between the first passage times of the Brownian motion to a large class of (smooth) curves to the linear or quadratic ones. As a by-product, we establish some connections between certain stochastic objects and some special functions. We will show that this device applies to continuous time stochastic processes.

The paper is organized as follows. The next section is devoted to some recalls concerning Bessel and Ornstein-Uhlenbeck processes. In particular, we give a probabilistic construction of the cylinder parabolic function which characterizes the Laplace transform of the first hitting time of fixed level by an Ornstein-Uhlenbeck process. In section 3, we derive the sought joint law in terms of transforms via stochastic techniques for the case $y = 0$. For any $y > 0$, we resort to the Feynman-Kac formula. Then, we show some relationship between stopping times for general stochastic processes which we apply to the Brownian motion. This link allows to get some asymptotic results for the parabolic cylinder functions. We end up this paper by making some connections between the studied law and the one of some other functionals.

## 2. Preliminaries and recalls

Let $(B_t, t \geq 0)$ be a one dimensional Brownian motion starting from 0. The 3-dimensional Bessel process, denoted by $R$, is defined to be the unique strong solution to

$$dR_t = dB_t + \frac{1}{R_t}dt, \quad R_0 = x \geq 0.$$

This is a linear diffusion with speed measure given by $m(dy) = 2y^2dy$. Its semi-group is absolutely continuous with respect to $m$ with density

$$q_t(x,y) = \frac{1}{2\sqrt{2\pi t}}\frac{1}{yx}\left(e^{-\frac{1}{2t}(x-y)^2} - e^{-\frac{1}{2t}(x+y)^2}\right), \quad x, y, t > 0$$

and by passage to the limit as $x$ tends to zero we obtain

$$q_t(0,x) = \frac{1}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}}, \quad x, t > 0.$$

We shall denote by $\mathbb{Q}_x$ the law of $R$ when it is started at $x$ and we simply write $\mathbb{Q}$ for $x = 0$. Next, for $y$ and $t \geq 0$, the conditional measure $\mathbb{Q}^t_{x,y} = \mathbb{Q}_x[\,.\mid R_t = y]$, viewed as a probability measure on $\mathcal{C}([0,t],[0,\infty))$, stands for the law of the 3-dimensional Bessel bridge starting at $x$ and ending at $y$ at time $t$. Since $R$ is transient, we have $\mathbb{Q}^t_{x,y} = \mathbb{Q}_x[\,.\mid L_y = t]$ where $L_y = \sup\{s \geq 0;\ R_s = y\}$. Williams' time reversal relationship states that, for $R_0 = 0$, $B_0 = x > 0$, the processes $(R_{L_x - s}, s \leq L_x)$ and $(B_s, s \leq T_0)$ are equivalent, where $T_0 = \inf\{s \geq 0;\ B_s = 0\}$.

We continue by providing some recalls on Ornstein-Uhlenbeck processes (for short OU process). For an OU process, with parameter $\lambda \in \mathbb{R}$, the realization given by (1.2) is also the unique strong solution to

$$dU_t = dB_t - \lambda U_t dt, \quad U_0 = x \in \mathbb{R}. \tag{2.1}$$

Denote by $\mathbb{P}^{(\lambda)}_x$ the law of $U$ when $U_0 = x \in \mathbb{R}$ and write simply $\mathbb{P}^{(\lambda)}$ for $\mathbb{P}^{(\lambda)}_0$. By Girsanov's theorem, $\mathbb{P}^{(\lambda)}_x$ is absolutely continuous with respect to the Wiener measure $\mathbb{P}_x$ via

$$d\mathbb{P}^{(\lambda)}_{x|\mathcal{F}_t} = e^{-\frac{\lambda}{2}(B_t^2 - x^2 - t) - \frac{\lambda^2}{2}\int_0^t B_u^2\, du}\, d\mathbb{P}_{x|\mathcal{F}_t}, \quad t > 0, \tag{2.2}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $B$. We obviously can write

$$\int_0^t e^{\lambda s}\, dB_s = W_{\tau(t)} \text{ and } \quad W_t = \int_0^{A(t)} e^{\lambda s}\, dB_s, \quad t \geq 0,$$

where $\tau(t) = \frac{1}{2\lambda}\left(e^{2\lambda t} - 1\right)$, $A(t) = \frac{1}{2\lambda}\log(1 + 2\lambda t)$, and $W$ is a Brownian motion thanks to the Lévy's characterization theorem, see [23]. Hence, Doob's representation

$$U_t = e^{-\lambda t}\left(x + W_{\tau(t)}\right), \quad t \geq 0, \tag{2.3}$$

holds. The relation between the stopping times $\sigma_a$ and $H^{(\lambda)}_a$, discovered by Breiman [3] and recalled in the introduction, is a straightforward consequence of this fact. The process $U$ is a linear diffusion. Moreover, when $\lambda > 0$, it is positively recurrent and its semi-group has a unique invariant measure which is the law of a centered Gaussian random variable with variance $1/2\lambda$. Next, for a fixed $a \in \mathbb{R}$, introduce the random variable $\sigma_a = \inf\{s \geq 0;\, U_s = a\}$. It is a stopping time which law is absolutely continuous with respect to the Lebesgue measure with a probability density function $p^{(\lambda)}_{x\to a}$ i.e. $\mathbb{P}^{(\lambda)}_x(\sigma_a \in dt) = p^{(\lambda)}_{x\to a}(t)\, dt$. For the Brownian motion, recovered by letting $\lambda$ tend to 0, we recall that

$$p_{x\to a}(t) = \frac{|a - x|}{\sqrt{2\pi t^3}} e^{-\frac{(a-x)^2}{2t}}.$$

We are now ready to derive the expression of the Laplace transform of $\sigma_a$. This is a well-known result which could be found in Breiman [3]. However, we give a proof which relies on probabilistic arguments.

**Proposition 2.1.** *For any $x, a \in \mathbb{R}$ and $\beta \geq 0$, we have*

$$\mathbb{E}_x\left[e^{-\beta\sigma_a}\right] = \frac{e^{\lambda x^2/2} D_{-\beta/\lambda}(\varepsilon x\sqrt{2\lambda})}{e^{\lambda a^2/2} D_{-\beta/\lambda}(\varepsilon a\sqrt{2\lambda})}, \tag{2.4}$$

*where $\varepsilon = sgn(x - a)$ and $D_\nu$ stands for the parabolic cylinder function which admits the following integral representation*

$$D_\nu(z) \;=\; \frac{2^{\frac{\nu+1}{2}} e^{-z^2/4}}{\Gamma\left(\frac{1-\nu}{2}\right)} \int_0^\infty \left(t^2 + z^2\right)^{\nu/2} t^{-\nu} e^{-t^2/2}\, dt, \tag{2.5}$$

*where $Re(\nu) < 1, |arg(z)| < \frac{\pi}{2}$.*

**Proof.** Doob's transformation implies the identity $H_{x\to a} = \tau(\sigma_a)$ almost surely, where $H_{x\to a} = \inf\{s \geq 0; W_s + x = a\sqrt{1 + 2\lambda s}\}$. Specializing on $a = 0$ we deduce that $p^{(\lambda)}_{x\to 0}(t) = \tau'(t) p_{x\to 0}(\tau(t))$. Hence, the expression

$$p^{(\lambda)}_{x\to 0}(t) = \frac{|x|}{\sqrt{2\pi}} \exp\left(-\frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}\right)\left(\frac{\lambda}{\sinh(\lambda t)}\right)^{3/2}. \tag{2.6}$$

It follows that

$$\begin{aligned}\mathbb{E}_x\left[e^{-\beta\sigma_0}\right] &= \int_0^\infty e^{-\beta t}\tau'(t)p_{x\to 0}(\tau(t))\,dt\\ &= \frac{|x|}{\sqrt{2\pi}}\int_0^\infty (1+2\lambda t)^{-\beta/2\lambda}\,t^{-3/2}e^{-x^2/2t}\,dt\\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty \left(t^2+\lambda x^2\right)^{-\beta/2\lambda}t^{\beta/\lambda}e^{-t^2}\,dt.\end{aligned}$$

The strong Markov property yields the following identity

$$\sigma_{x\to 0}\overset{(d)}{=}\sigma_{x\to a}+\hat{\sigma}_{a\to 0},\quad x\le a\le 0,$$

where $\hat{\sigma}_{a\to 0}$ is an independent copy of $\sigma_{a\to 0}$. It follows that

$$\mathbb{E}_x\left[e^{-\beta\sigma_a}\right] = \frac{\int_0^\infty \left(t^2+\lambda x^2\right)^{-\beta/2\lambda}t^{\beta/\lambda}e^{-t^2}\,dt}{\int_0^\infty \left(t^2+\lambda a^2\right)^{-\beta/2\lambda}t^{\beta/\lambda}e^{-t^2}\,dt}.$$

By using the integral representation of the cylinder parabolic function (2.5), we get

$$\mathbb{E}_x\left[e^{-\beta\sigma_a}\right]=\frac{e^{\lambda x^2/2}D_{-\beta/\lambda}(-x\sqrt{2\lambda})}{e^{\lambda a^2/2}D_{-\beta/\lambda}(-a\sqrt{2\lambda})},\quad x\le a\le 0.$$

Next, we observe that the symmetry of $B$ in (1.2) allows to recover the case $x\ge a\ge 0$. The proof is then completed since we have computed the two functions, the increasing and decreasing one, which characterized the Laplace transform of $\sigma_a$, see Itô and McKean [11]. □

## 3. On the law of $\left(N_t^{(1)}(r),N_t^{(2)}(r)\right)$

For any $\beta>0$, we introduce the resolvent kernel, or the Green's function, $G_\beta$ given, for $\alpha\ge 0$ and $\lambda$ real, by

$$G_\beta(x,y)dy=\int_0^\infty e^{-\beta t}\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}N_t^{(2)}(R)-\alpha N_t^{(1)}(R)},R_t\in dy\right]dt,\ x,y\ge 0.$$

As we shall see below we have that $G_\beta(x,y)=w_\beta^{-1}m(y)\phi_\beta(x\wedge y)\psi_\beta(x\vee y)$ where $\phi_\beta$ (resp. $\psi_\beta$) is the unique, up to some multiplicative positive constants, decreasing, positive and bounded at $+\infty$ solution (res. increasing, positive and bounded at 0 solution) of the Sturm-Liouville equation

$$2^{-1}\varphi''(x)+x^{-1}\varphi'(x)-\left(2^{-1}\lambda^2x^2+\alpha x+\beta\right)\varphi(x)=0,\quad x>0. \tag{3.1}$$

Note that for the case $\lambda=0$ (resp. $\alpha=0$), the corresponding Green function is already known, see e.g. [2, Formula 5.1.8.5 ], (resp. Formula 5.1.9.5). For a fixed $t\ge 0$, let us introduce the notation

$$\Pi^{\lambda,\alpha}_{x\to y}(t)=\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}N_t^{(2)}(R)-\alpha N_t^{(1)}(R)}\mid R_t=y\right],\quad x,y,\alpha\ge 0,\lambda\in\mathbb{R}.$$

We denote simply $\Pi^{\lambda,\alpha}_x(t)$ (resp. $\Pi^{\lambda,\alpha}(t)$) for $\Pi^{\lambda,\alpha}_{x\to 0}(t)$ (resp. $\Pi^{\lambda,\alpha}_{0\to 0}(t)$).

**Remark 3.1.** We point out that, thanks to the scaling property of Bessel processes, we have the identity $\Pi^{\lambda,\alpha}_{x\to y}(t) = \Pi^{\lambda t^2, \alpha t^{3/2}}_{\frac{x}{\sqrt{t}}\to\frac{y}{\sqrt{t}}}(1)$.

3.1. **Stochastic approach for the case** $y = 0$**.** In here we show how to exploit the results of the former section in order to compute $\Pi^{\lambda,\alpha}_x(t)$.

**Proposition 3.2.** *For* $x, \lambda, \beta > 0$ *and* $\alpha \geq 0$*, we have*

$$\int_0^\infty e^{-\beta t} q_t(x,0)\Pi^{\lambda,\alpha}_x(t)\, dt = \frac{1}{x}\frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}.$$

*Consequently, we have*

$$\int_0^\infty \left(e^{-\beta t}-1\right)\Pi^{\lambda,\alpha}(t)\,\frac{dt}{\sqrt{2\pi t^3}} =$$
$$\sqrt{2\lambda}\left(\frac{D^{(x)}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} - \frac{D^{(x)}_{\frac{\alpha^2}{2\lambda^3}-\frac{1}{2}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{D_{\frac{\alpha^2}{2\lambda^3}-\frac{1}{2}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\right),$$

*where* $D^{(x)}_\nu(y) = \frac{\partial D_\nu(x)}{\partial x}|_{x=y}$.

**Proof.** We fix $a = \alpha/\lambda^2$, observe that

$$\Pi^{\lambda,a\lambda^2}_x(t) = e^{a^2\lambda^2 t/2}\,\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u+a)^2\,du}\,\middle|\, R_t = 0\right], \tag{3.2}$$

and recall that $L_x = \sup\{s \geq 0;\ R_s = x\}$ and $T_a = \inf\{s \geq 0;\ B_s = a\}$. Following a line of reasoning similar to [7], we get

$$\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u+a)^2\,du}\,\middle|\, R_t = 0\right] = \mathbb{E}_{x+a}\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\,du}\,\middle|\, T_a = t\right], \tag{3.3}$$

where we used the properties of Bessel bridges recalled in section 2. Now, thanks to the absolute-continuity relationship (2.2), we can write

$$p^{(\lambda)}_{x+a\to a}(t) \;=\; e^{\frac{\lambda}{2}(x^2+2ax+t)}\mathbb{E}_{x+a}\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\,du}\,\middle|\, T_a = t\right] p_{x\to 0}(t)\ . \tag{3.4}$$

A combination of (3.2), (3.3) and (3.4) leads to

$$e^{(\frac{1}{2}a^2\lambda^2-\frac{\lambda}{2})t}p^{(\lambda)}_{x+a\to a}(t) = e^{\frac{\lambda}{2}x^2+a\lambda x}p_{x\to 0}(t)\,\Pi^{\lambda,\alpha}_x(t).$$

By taking the Laplace transform with respect to the variable $t$ on both sides and making use of (2.4) we get the first assertion. To prove the second one, it is enough to let $x$ tend to 0 in the following formula

$$\int_0^\infty \left(e^{-\beta t}-1\right)e^{-x^2/2t}\Pi^{\lambda,\alpha}_x(t)\,\frac{dt}{\sqrt{2\pi t^3}} =$$
$$\frac{1}{x}\left(\frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} - \frac{D_{-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D_{-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\right).$$

□

Below, we give a straightforward reformulation of the previous result, which is based on the Laplace transform inversion formula. To this end, we recall the expression of the density of $\sigma_a$ as a series expansion which can be found for instance in [1] and [17]. That is, for $x$ and $a$ reals, we have

$$p^{(\lambda)}_{x\to a}(t) = -\lambda e^{\lambda(x^2-a^2)/2} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,\varepsilon\sqrt{2\lambda}a}}(\varepsilon\sqrt{2\lambda}x)}{D^{(\nu)}_{\nu_{n,\varepsilon\sqrt{2\lambda}a}}(\varepsilon\sqrt{2\lambda}a)} \, e^{-\lambda\nu_{n,\varepsilon\sqrt{2\lambda}a}t}, \tag{3.5}$$

where we set $\varepsilon = \operatorname{sgn}(x-a)$, $D^{(\nu)}_{\nu_{n,b}}(b) = \frac{\partial D_\nu(b)}{\partial \nu}|_{\nu=\nu_{n,b}}$ and the sequence $(\nu_{j,b})_{j\geq 0}$ stands for the ordered positive zeros of the function $\nu \to D_\nu(b)$.

**Corollary 3.3.** *For $\lambda, x, t > 0$ and $\alpha \geq 0$, we have*

$$\Pi^{\lambda,\alpha}_x(t) = -\frac{\lambda}{x}\sqrt{2\pi t^3} e^{\left(\frac{\alpha^2}{\lambda^2}-\lambda\right)t/2 + x^2/2t} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,c}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D^{(\nu)}_{\nu_{n,c}}(\sqrt{2}\alpha\lambda^{-3/2})} \, e^{-t\lambda\nu_{n,c}}, \tag{3.6}$$

*where we set $c = \sqrt{2}\alpha\lambda^{-3/2}$.*

The proof is omitted and left to the reader.

3.2. **Extension to $y > 0$ using Feynman-Kac formula.** Our aim here is to provide an extension of the previous result to any positive reals $y$ by using the Feynman-Kac formula.

**Proposition 3.4.** *For $y, x, \beta, \lambda > 0$ and $\alpha \geq 0$, we have*

$$\int_0^\infty e^{-\beta t} q_t(x,y)\, \Pi^{\lambda,\alpha}_{x\to y}(t)\, dt = \frac{\Gamma(\frac{\beta}{\lambda}+\frac{1}{2}-\frac{\alpha^2}{2\lambda^3})y}{\sqrt{\lambda\pi} D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\alpha\lambda^{-2/3}\right) x} \times$$
$$S_{-\frac{\beta}{2\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x\wedge y+\frac{\alpha}{\lambda^2}), \sqrt{2}\alpha\lambda^{2/3}\right) D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x\vee y+\frac{\alpha}{\lambda^2})\right),$$

*where $S_\alpha(x,y) = D_\alpha(-x)D_\alpha(y) - D_\alpha(x)D_\alpha(-y)$.*

**Proof.** We shall prove our statement by following a method which is similar to that used by Shepp [26]. Set $F^y_\epsilon(x) = \frac{1}{2\epsilon}\mathbb{I}_{\{|x-y|<\epsilon\}}$ and $a(x) = \left(\frac{\lambda^2}{2}x^2 + \alpha x + \beta\right)$. First, note that

$$\lim_{\epsilon\to 0} \mathbb{E}_x\left[\int_0^\infty e^{-\beta t} e^{-\int_0^t a(R_s)\,ds} F^y_\epsilon(R_t)\,dt\right] = \int_0^\infty e^{-\beta t} q_t(x,y)\, \Pi^{\lambda,\alpha}_{x\to y}(t)\,dt.$$

Then, the Feynman-Kac formula states that

$$u_\epsilon(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\beta t} e^{-\int_0^t a(R_s)\,ds} F^y_\epsilon(R_t)\,dt\right]$$

is the bounded solution of

$$\frac{1}{2}u''_\epsilon(x) + \frac{1}{x}u'_\epsilon(x) - a(x)u_\epsilon(x) = F^y_\epsilon(x), \quad x > 0. \tag{3.7}$$

In order to solve this equation, we first consider the following homogeneous one

$$\frac{1}{2}u''(x) + \frac{1}{x}u'(x) - a(x)u(x) = 0, \quad x > 0.$$

Setting $u(x) = x^{-1}v(x)$, we get that $v$ satisfies the Weber equation

$$\frac{1}{2}v''(x) = \left(\frac{\lambda^2}{2}\bar{x}^2 - \frac{\alpha^2}{2\lambda^2} + \beta\right) v(x), \quad x > 0, \tag{3.8}$$

where $\bar{x} = x + \frac{\alpha}{\lambda^2}$. A fundamental solution of (3.8) is expressed in terms of the parabolic cylinder function $D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\bar{x}\right)$, see e.g. [8]. Thus, the solution of (3.8) which is positive and decreasing is given by

$$\varphi(x) \quad = \quad x^{-1}D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\bar{x}\right), \quad x > 0.$$

The solution of (3.8) which is positive and increasing has the form

$$\psi(x) \quad = \quad x^{-1}\left(c_1 D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(-\sqrt{2\lambda}\bar{x}\right) + c_2 D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\bar{x}\right)\right),$$

where $c_1$ and $c_2$ are constants. Choosing $c_1 = D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\right)$ and
$c_2 = -D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(-\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\right)$, we check that $\psi(x)$ is bounded at 0. The two solutions are linearly independent and their Wronskian, normalized by the derivative of the scale function $s'(x) = x^{-2}$, is given by

$$w_\beta \quad = \quad D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\right) w_\beta^D$$

where $w_\alpha^D = \frac{2\sqrt{\lambda\pi}}{\Gamma(\frac{\beta}{\lambda}+\frac{1}{2}-\frac{\alpha^2}{2\lambda^3})}$ is the Wronskian of the cylinder parabolic functions. Next, we recall the Green formula for the solution of the nonhomogeneous ode (3.7), that is with second member given by $F_\epsilon^y$

$$u_\epsilon(x) \quad = \quad \frac{1}{w_\beta}\left(\varphi(x)\int_0^x \psi(r)F_\epsilon^y(r)\, m(dr) + \psi(x)\int_x^\infty \varphi(r)F_\epsilon^y(r)\, m(dr)\right),$$

where we recall that the speed measure $m$ of the 3-dimensional Bessel process is $m(dr) = 2r^2\, dr$. The proof is then completed by passing to the limit as $\epsilon$ tends to 0. □

**Remark 3.5.** Observing that $\lim_{x\to 0} x^{-1}S_\alpha(x,y) = w_\alpha^D$, we recover the result of Proposition 3.2.

**Remark 3.6.** In the same vein than Corollary 3.3, it is possible to derive an expression of the joint Laplace transform $\Pi_{x\to y}^{\lambda,\alpha}(t)$ as a series expansion.

## 4. **Connection between the law of first passage times**

Let $\lambda > 0$ and introduce the function $f_\delta(\lambda t) = \delta g(\lambda t) - \mu\lambda t - y$ where $g$ is a twice continuously differentiable function on a neighborhood of 0, and $\alpha, \mu$ and $y$ are some reals. Let $Z$ be a continuous time stochastic process. Introduce the stopping times

$$\begin{aligned} T^\delta_{y,\mu} &= \inf\{s \geq 0;\ Z_s = f_\delta(\lambda s)\} \\ L^\alpha &= \inf\{s \geq 0;\ Z_s = \alpha s\} \\ S^\alpha &= \inf\{s \geq 0;\ Z_s = -\frac{\alpha}{2}s^2\}. \end{aligned}$$

We shall describe a device which allows to connect the law of the first passage times $T^\delta_{y,\mu}$, simply denoted by $T^\delta_y$ for $\mu = 0$, to the linear boundary and to the quadratic one. As an application, we shall apply this technique to the Brownian motion case and derive some limits results of the cylinder parabolic functions. This limit result can also be used as a test for checking the validity of the hitting time densities.

**Proposition 4.1.** *Let $\delta^{(1)}_\lambda = \alpha/\lambda$. Assume $g'(0) \neq 0$, then*

$$\lim_{\lambda\to 0} T^{\delta^{(1)}_\lambda}_{\delta^{(1)}_\lambda g(0)} = L^{\alpha g'(0)} \quad a.s.. \tag{4.1}$$

*Next, let $\delta^{(2)}_\lambda = \alpha/\lambda^2$. Assume $g''(0) \neq 0$, then*

$$\lim_{\lambda\to 0} T^{\delta^{(2)}_\lambda}_{\delta^{(2)}_\lambda g(0), \delta^{(2)}_\lambda g'(0)} = S^{-\alpha g''(0)} \quad a.s.. \tag{4.2}$$

**Proof.** The assertions follows from the following expansion

$$f_{\delta_\lambda}(\lambda t) = \delta_\lambda g(0) - y + \lambda(\delta_\lambda g'(0) - \mu)t + \frac{\lambda^2}{2}\delta_\lambda g''(0)t^2 + o(\lambda^2).$$

□

### 4.1. **Brownian motion and the square root boundary.**

We apply the previous technique to the first passage time of the Brownian motion over the curve $f_\delta(\lambda t) = \delta\sqrt{1 + 2\lambda t} - \mu\lambda t - y$ in order to evaluate some well known limits of the ratio of parabolic cylinder functions.

4.1.1. *Linear case.* In this case, we set $\mu = 0$ and $\delta = y = \alpha/\lambda$ and state the following result.

**Corollary 4.2.** *Let $\beta > 0$, $x, \alpha \in \mathbb{R}$ then we have*

$$\lim_{\lambda\to 0} \frac{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2\lambda}(x + \frac{\alpha}{\lambda})\right)}{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2}\alpha\lambda^{-1/2}\right)} = e^{-|x|\sqrt{\alpha^2+2\beta}}.$$

*As a consequence, we also have*

$$\lim_{\lambda\to 0} \lambda e^{\lambda(x^2 - 2\frac{\alpha}{\lambda}x)/2} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,\frac{\alpha}{\lambda}\sqrt{2\lambda}}}(\sqrt{2\lambda}x)}{D^{(\nu)}_{\nu_{n,\frac{\alpha}{\lambda}\sqrt{2\lambda}}}(\sqrt{2}\alpha\lambda^{-1/2})} e^{-\lambda\nu_{n,\varepsilon a\sqrt{2\lambda}}t} = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}(x-\alpha t)^2}.$$

**Proof.** First, by combining Doob's transformation with Proposition 2.1, we recover the result of Breiman [3] about the Melin transform of $T_\delta^\delta$

$$\mathbb{E}_x\left[(1+2\lambda T_\delta^\delta)^{-\beta/2\lambda}\right] = e^{\alpha x}\frac{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda})\right)}{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2}\alpha\lambda^{-1/2}\right)}.$$

Next, recall that the Laplace transform of $L^\alpha$ is specified by, see e.g. [13, p.197],

$$\mathbb{E}_x\left[e^{-\beta L^\alpha}\right] = e^{\alpha x-|x|\sqrt{\alpha^2+2\beta}}.$$

The statement follows readily from Proposition 4.1. □

4.1.2. *Quadratic case.* In what follows, we investigate the second order expansion. We start by computing the law of $S^\alpha$, the first passage time of the Brownian motion over the second order boundary. In the case $x\alpha > 0$, its law has been computed by Groeneboom [9] and Salminen [24] in terms of the Airy function, see e.g. [15]. For the sake of completeness we recall their approach.

**Lemma 4.3.** *For $\beta$ and $\alpha, x > 0$, hold the relations*

$$\mathbb{E}_x\left[e^{-\beta S^\alpha}G(S^\alpha)\right] = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}$$

*where $G(t) = e^{\frac{1}{6}\alpha^2 t^3}$ and*

$$\mathbb{P}_x(S^\alpha \in dt) = (2\alpha^2)^{1/3}e^{-\frac{1}{6}\alpha^2 t^3}\sum_{k=0}^{\infty}\frac{Ai\left(z_k-(2\alpha)^{1/3}\right)}{Ai'(z_k)}e^{2^{-1/3}\alpha^{2/3}z_k t}\,dt,$$

*where $(z_k)_{k\geq 0}$ is the decreasing sequence of negative zeros of the Airy function.*

**Proof.** Let us denote by $\mathbb{P}^\alpha$ the law of the process $\left(B_t + \frac{\alpha}{2}t^2,\ t \geq 0\right)$. We have the following absolute continuity relationship

$$\begin{aligned} d\mathbb{P}^\alpha_{x|\mathcal{F}_t} &= e^{\alpha\int_0^t s\,dB_s - \frac{\alpha^2}{6}t^3}\,d\mathbb{P}_{x|\mathcal{F}_t} \\ &= e^{\alpha t B_t - \alpha\int_0^t B_s\,ds - \frac{\alpha^2}{6}t^3}\,d\mathbb{P}_{x|\mathcal{F}_t},\quad t > 0, \end{aligned}$$

where the last line follows from Itô's formula. An application of the Doob's optional stopping theorem yields

$$\mathbb{E}_x\left[e^{-\beta S^\alpha}G(S^\alpha)\right] = \mathbb{E}_x\left[e^{-\beta T_0 - \alpha\int_0^{T_0}B_s ds}\right].$$

As in the previous section, the expectation on the right hand side can be computed via the Feynman-Kac formula. It is the solution to the boundary value problem

$$\begin{gathered} \frac{1}{2}u''(x) - (\alpha x+\beta)u(x) = 0,\quad x > 0, \\ u(0) = 1,\quad \lim_{x\to\infty}u(x) = 0, \end{gathered}$$

which is given in terms of the Airy function, see e.g. [12]. The expression of the density is a consequence of the Laplace transform inversion formula and the residues theorem, see [9] or [24] for more details. □

**Remark 4.4.** By analogy to the results of section 3, we have

$$\lim_{\lambda\to 0}\int_0^\infty e^{-\beta t}q_t(x,0)\Pi_x^{\lambda,\alpha}(t)\,dt = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)},$$

$$\Pi_x^{0,\alpha}(t) = \sqrt{2\pi t^3}e^{\frac{x^2}{2t}}(2\alpha^2)^{1/3}\sum_{k=0}^{\infty}\frac{Ai\left(z_k-(2\alpha)^{1/3}\right)}{Ai'(z_k)}e^{(\frac{\alpha^2}{2})^{1/3}z_k t}\,dt$$

and finally

$$\int_0^\infty \left(e^{-\beta t}-1\right)\Pi^{0,\alpha}(t)\,\frac{dt}{\sqrt{2\pi t^3}} = (2\alpha)^{1/3}\left(\frac{Ai'\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)} - \frac{Ai'(0)}{Ai(0)}\right).$$

**Remark 4.5.** We mention that the other case, i.e. $\alpha x < 0$, has been studied by Martin-Löf [19].

Next, we define the process $(U_t^{(\mu)},\, t\geq 0)$ as the solution to the stochastic differential equation

$$dU_t^{(\mu)} = \left(-\lambda U_t^{(\mu)} + \mu e^{\lambda t}\right)dt + dB_t, \qquad U_0^{(\mu)} = x\in\mathbb{R}.$$

Note that $X^{(\mu)}$ can also be expressed as follows

$$U_t^{(\mu)} = e^{-\lambda t}\left(x - \frac{\mu}{2\lambda} + \frac{\mu}{2\lambda}e^{2\lambda t} + \int_0^t e^{\lambda s}dB_s\right), \qquad t\geq 0.$$

For reals numbers $x$ and $a$, we introduce the stopping time $\sigma_a^{(\mu)} = \inf\{s\geq 0;\, U_s^{(\mu)} = a\}$ and denote by $p_{x\to a}^{(\lambda,\mu)}(t)$ its density. Let us also introduce the function $G_\lambda(t) = e^{\frac{\mu^2}{2}\tau_t - \mu e^{\lambda t}a}$, $t\geq 0$. The law of $\sigma_a^{(\mu)}$ is characterized in the following.

**Proposition 4.6.** *For $\beta > 0$, we have*

$$\mathbb{E}_x\left[e^{-\beta\sigma_a^{(\mu)}}G_\lambda(\sigma_a^{(\mu)})\right] = \frac{e^{\lambda x^2/2-\mu x}D_{-\frac{\beta}{\lambda}}\left(\varepsilon x\sqrt{2\lambda}\right)}{e^{\lambda a^2/2}D_{-\frac{\beta}{\lambda}}\left(\varepsilon a\sqrt{2\lambda}\right)}. \tag{4.3}$$

*where we set $\varepsilon = sgn(x-a)$. In particular,*

$$p_{x\to 0}^{(\lambda,\mu)}(t) = \frac{\mid x\mid}{\sqrt{2\pi}}\,e^{-\mu e^{\lambda t}(\frac{\mu}{2}\sinh(\lambda t)-a)-\mu x-\frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)}+\frac{\lambda t}{2}}\left(\frac{\lambda}{\sinh(\lambda t)}\right)^{3/2}. \tag{4.4}$$

**Proof.** The first assertion follows from the following absolutely continuity relationship

$$d\mathbb{P}_{x|\mathcal{F}_t}^{(\lambda,\mu)} = e^{\mu e^{\lambda t}X_t-\mu x-\frac{\mu^2}{2}\tau_t}\,d\mathbb{P}_{x|\mathcal{F}_t}^{(\lambda)},\quad t>0 \tag{4.5}$$

and the application of the Doob's optional stopping theorem. We point out that the exponential martingale is the one associated with the Gaussian martingale $\left(B_{\tau(t)}, t\geq 0\right)$. The expression of the density in the case $a=0$ is obtained from the Laplace inversion formula of the parabolic cylinder function, see formula (2.6). □

**Remark 4.7.** An expression of the density $p^{(\lambda,\mu)}_{x\to a}(t)$ is given in Daniels [4] as a contour integral. The author used a technique suggested by Shepp [25].

Let us now introduce the stopping times $H^{(\lambda,\mu)}_{x\to a} = \inf\{s \geq 0;\ x + B_s + \mu s = a\sqrt{1+2\lambda s}\}$ and $S^\alpha_x = \inf\{s \geq 0;\ B_s + x = -\frac{\alpha}{2}s^2\}$. We denote by $p^{(\lambda,\mu)}_{x\to a}$ (resp. $q^\alpha_x$) the density of $H^{(\lambda,\mu)}_{x\to a}$ (resp. $S^\alpha_x$). We proceed by giving some relationships between these different hitting times.

$$H^{(\lambda,\mu)}_{x\to a} = \tau(\sigma^{(\lambda,\mu)}_{x\to a}) \quad a.s., \tag{4.6}$$

$$\lim_{\lambda\to 0} H^{(\lambda,\frac{\alpha}{\lambda})}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}} = S^\alpha_x \quad a.s.. \tag{4.7}$$

We are now ready to state the following limit result which can be found for instance in [5].

**Corollary 4.8.** *For $\beta, \alpha$ and $x > 0$, we have*

$$\lim_{\lambda\to 0} \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}$$

**Proof.** Substituting $\beta$ by $\beta - \frac{\alpha^2}{2\lambda^2}$, $x$ by $x + \frac{\alpha}{\lambda^2}$ and setting $a = \frac{\alpha}{\lambda^2}$ and $\mu = \frac{\alpha}{\lambda}$ in (4.3), we get

$$\frac{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} = e^{-\frac{\lambda}{2}x^2+\frac{\alpha^2}{\lambda^3}}$$
$$\int_0^\infty e^{-(\beta-\frac{\alpha^2}{2\lambda^2})t+\frac{\alpha^2}{2\lambda^2}\tau_t-\frac{\alpha^2}{\lambda^3}e^{\lambda t}} p^{(\lambda,\frac{\alpha}{\lambda})}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}(t)\, dt$$

Note that $\tau\left(H^{(\lambda,\alpha/\lambda)}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}\right) \to S^\alpha_x$ a.s., as $\lambda \to 0$. Thus, we have

$$\begin{aligned}
\lim\nolimits_{\lambda\to 0} \quad & e^{-\frac{\lambda}{2}x^2+\frac{\alpha^2}{\lambda^3}} \int_0^\infty e^{-(\beta-\frac{\alpha^2}{2\lambda^2})t+\frac{\alpha^2}{2\lambda^2}\tau_t-\frac{\alpha^2}{\lambda^3}e^{\lambda t}} p^{(\lambda,\frac{\alpha}{\lambda})}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}(t)\, dt \\
= \quad & \int_0^\infty e^{-\beta t+\frac{1}{6}\alpha^2 t^3} q^\alpha_x(t)\, dt \\
= \quad & \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}.
\end{aligned}$$

where the last line follows from Lemma 4.3. □

**Remark 4.9.** We mention that Lachal [14] get the following identity

$$\mathbb{E}_x\left[e^{-\beta\sigma_0-\alpha\int_0^{\sigma_0} U_s\, ds}\right] = e^{\frac{\lambda}{2}x^2} \frac{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}$$

which gives the following relationship

$$\int_0^\infty e^{-\beta t - x^2/2t} t^{-3/2} \Pi_x^{\lambda,\alpha,(1)}(t)\, dt \quad = \quad e^{-\frac{\lambda}{2}x^2} \mathbb{E}_x \left[ e^{-(\beta+\frac{\lambda}{2})\sigma_0 - \alpha \int_0^{\sigma_0} U_s\, ds} \right].$$

We also indicate that the author computed the limit as $\lambda \to 0$ to recover the result of Lefebvre [16] stating that

$$\mathbb{E}_x \left[ e^{-\beta T_0 - \alpha \int_0^{T_0} B_s\, ds} \right] \quad = \quad \frac{Ai\left(2^{1/3} \frac{\beta + \alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3} \frac{\beta}{\alpha^{2/3}}\right)}.$$

In order to compute the expression of the limit of the Laplace transform, he used an asymptotic result of the cylinder parabolic function which has been derived by the steepest descent method in [5].

4.2. **Another limit.** From Proposition 3.2, we readily derive

$$\lim_{\alpha \to 0} \int_0^\infty e^{-\beta t} x e^{-x^2/2t} \Pi_x^{\lambda,\alpha}(t) \frac{dt}{\sqrt{2\pi t^3}} = \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}}\left(\sqrt{2\lambda}x\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}}(0)}.$$

We recall and show the following well known results regarding the Laplace transform of the $L^2$ norm of Bessel bridges. In conjunction with (2.6), we extract the relationship

$$\Pi_x^\lambda(t) \quad = \quad \left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{3}{2}} e^{-\frac{x^2}{2t}(\lambda t \coth(\lambda t) - 1)}. \tag{4.8}$$

Since in this case the zeros of the function $\nu \mapsto D_\nu(0) = 2^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\nu}{2})}$ correspond to the odd poles of the $\Gamma$ function, we also have

$$\Pi_x^\lambda(t) \quad = \quad -\frac{\lambda}{x}\sqrt{2\pi t^3} e^{\frac{x^2}{2t}} \sum_{n=1}^\infty \frac{D_{2n+1}(x\sqrt{2\lambda})}{D_{2n+1}^{(\nu)}(0)}\, e^{-2(n+1)\lambda t}.$$

We precise that from the expression (4.8), it is easy to derive the Generalized Lévy stochastic area formula, see e.g. [22]. Indeed for any $\delta > 0$, denoting by $\Pi_x^{\lambda,(\delta)}$ the Laplace transform of the $L^2$ norm of a $\delta$-dimensional Bessel process, thanks to the additivity property of the squared Bessel processes, we have

$$\Pi_x^{\lambda,(\delta)}(t) \quad = \quad \left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{\delta}{2}} e^{-\frac{x^2}{2t}(\lambda t \coth(\lambda t) - 1)}. \tag{4.9}$$

In [6] the inverse of the Laplace transform $\Pi_x^{\lambda,(\delta)}(t)$ is given in terms of the parabolic cylinder functions.

## 5. Comments and some applications

Our aim here is first to examine the law of the studied functional when the fixed time $T$ is replaced by some interesting stopping times. To a stopping time $S$ we associate the following notation

$$\Sigma_x^{(\delta)}(S) = \mathbb{E}_x^{(\delta)}\left[e^{-\beta S - \frac{\lambda^2}{2}\int_0^S R_u^2\,du - \alpha\int_0^S R_u\,du}\right],$$

where $\beta, \lambda > 0$, $\alpha \geq 0$ and $\mathbb{E}_x^{(\delta)}$ denotes the expectation operator derived from $\mathbb{Q}_x^\delta$, the law of the $\delta$-dimensional Bessel process starting from $x \geq 0$.

Next, with $H_y = \inf\{s \geq 0;\ R_s = y\}$ and $S = H_y$, we state the following result.

**Proposition 5.1.** *Let $x \geq y > 0$.*

$$\Sigma_x^{(3)}(H_y) = \frac{y}{x}\frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}. \tag{5.1}$$

**Proof.** First, we recall the following absolute continuity relationship

$$d\mathbb{P}_{x|\mathcal{F}_t} = (R_t/x)^{-1}\ d\mathbb{Q}^{(3)}_{x|\mathcal{F}_t}, \quad \text{on } \{H_0 > t\},$$

Then observe that $H_y < H_0$ a.s. since $x \geq y$. Next, denote by $\sigma_x^{(\mu)}$ the first passage time to a fixed level $x \in \mathbb{R}$ of the OU process when the Brownian motion in the SDE (2.1) is replaced by a Brownian motion with drift $\mu \in \mathbb{R}$. The determination of its density, denoted by ${}^{(\mu)}p^{(\lambda)}_{x\to a}(t)$, can be reduced to the case $\mu = 0$ as follows

$${}^{(\mu)}p^{(\lambda)}_{x\to a}(t) = p^{(\lambda)}_{x-\frac{\mu}{\lambda}\to a-\frac{\mu}{\lambda}}(t), \quad t > 0.$$

Thus, we have

$$\begin{aligned}
\Sigma_x^{(3)}(H_y) &= \mathbb{E}_x^{(3)}\left[e^{-\beta H_y - \frac{\lambda^2}{2}\int_0^{H_y} R_s^2\,ds - \alpha\int_0^{H_y} R_s\,ds}\right] \\
&= \frac{y}{x}\mathbb{E}_x\left[e^{-\beta T_y - \frac{\lambda^2}{2}\int_0^{T_y} B_s^2\,ds - \alpha\int_0^{T_y} B_s\,ds}\right] \\
&= \frac{y}{x}e^{\frac{\lambda}{2}(y^2-x^2)}\mathbb{E}_x\left[e^{-(\beta+\frac{\lambda}{2})\sigma_y - \alpha\int_0^{\sigma_y} U_s\,ds}\right] \\
&= \frac{y}{x}e^{\frac{\lambda}{2}(y^2-x^2)+\frac{\alpha}{\lambda}(y-x)}\ \mathbb{E}_x\left[e^{-(\beta+\frac{\lambda}{2}-\frac{\alpha^2}{2\lambda^2})\sigma_y^{(\frac{\alpha}{\lambda})}}\right] \\
&= \frac{y}{x}\frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}.
\end{aligned}$$

□

**Corollary 5.2.** *For any $x \geq y > 0$, we have*

$$\Sigma_x^{(1)}(H_y) = \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}. \tag{5.2}$$

**Proof.** The result follows from the absolute continuity relationship

$$d\mathbb{Q}^{(1)}_{x|\mathcal{F}_t} = (R_t/x)^{-1}\, d\mathbb{Q}^{(3)}_{x|\mathcal{F}_t}, \quad \text{on } \{H_0 > t\},$$

where $\mathbb{Q}^1$ stands for the law of the reflected Brownian motion and $H_0$ is the first time when the canonical process hits 0. □

Next, let $(\tau_t, t \geq 0)$ be defined as the right continuous inverse process of the local time $(l_t, t \geq 0)$ at 0 of the reflected Brownian motion. It is a stable subordinator, its Laplace exponent is given by

$$\mathbb{Q}^{(1)}\left[e^{-\beta\tau_t}\right] = e^{-t\sqrt{2\beta}}.$$

We denote by $n$ and $(e_u, 0 \leq u \leq V)$ the Itô's measure associated with $R^1$ and the generic excursion process under $n$ respectively. We recall that with the choice of the normalization of the local time via the occupation formula with respect to the speed measure, we have $n(V \in dt) = \frac{dt}{\sqrt{2\pi t^3}}$, see e.g [10].

**Proposition 5.3.** *Let $\alpha, \beta \geq 0$ and $\lambda > 0$.*

$$-\log\left(\Sigma^{(1)}(\tau_1)\right) = \sqrt{2\lambda}\frac{D^{(x)}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}. \tag{5.3}$$

**Proof.** From the exponential formula of excursions theory, see e.g. [23] and the fact that conditionally on $V = t$ the process $(e_u, u \leq V)$ is a 3-dimensional Bessel bridge over $[0, t]$ between 0 and 0. We get

$$\begin{aligned}
-\log\left(\Sigma^{(1)}(\tau_1)\right) &= \int n(de)\left(1 - e^{-\beta V - \frac{\lambda^2}{2}\int_0^V e_u^2\,du - \alpha\int_0^V e_u\,du}\right)\\
&= \int_0^\infty \left(1 - e^{-\beta t}\Pi^{\lambda,\alpha}(t)\right)\frac{dt}{\sqrt{2\pi t^3}}.
\end{aligned}$$

Next, set $K(\beta) = \int_0^\infty \left(1 - e^{-\beta t}\Pi^{\lambda,\alpha}(t)\right)\frac{dt}{\sqrt{2\pi t^3}}$. Thus, we have

$$K(\beta) - K(0) = \int_0^\infty \left(1 - e^{-\beta t}\right)\Pi^{\lambda,\alpha}(t)\frac{dt}{\sqrt{2\pi t^3}}.$$

The statement follows from Proposition 3.2. □

Finally, we shall extend the above computations to the radial part of a $\delta$-dimensional Ornstein-Uhlenbeck process, denoted by $X$, with parameter $\theta \in \mathbb{R}^+$. The law of this process, when started at $x > 0$, is denoted by $\mathbb{P}^{(\theta),\delta}_x$. Girsanov's theorem gives

$$d\mathbb{P}^{(\theta),\delta}_{x|\mathcal{F}_t} = e^{-\frac{\theta}{2}(R_t^2 - x^2 - \delta t) - \frac{\theta^2}{2}\int_0^t R_u^2\,du}\, d\mathbb{Q}^{\delta}_{x|\mathcal{F}_t}, \quad t > 0. \tag{5.4}$$

We also shall need the densities of its semi-group which are given by, see [2],

$$\begin{aligned}
p_t^{(3)}(0,x) &= \frac{\theta^{3/2}e^{\frac{3}{2}\theta t}}{\sqrt{2\pi}(\sinh(\theta t))^{3/2}}e^{-\frac{\theta x^2 e^{-\theta t}}{2\sinh(\theta t)}}\\
p_t^{(1)}(0,x) &= \frac{\theta^{1/2}e^{\frac{1}{2}\theta t}}{\sqrt{2\pi}(\sinh(\theta t))^{1/2}}e^{-\frac{\theta x^2 e^{-\theta t}}{2\sinh(\theta t)}}, \quad x > 0.
\end{aligned}$$

With the obvious notations, for a fixed $t \geq 0$, we set

$$\Lambda_{x\to y}^{\lambda,\alpha,(\delta)}(t) = \mathbb{E}_x^\delta \left[ e^{-\frac{\lambda^2}{2}\int_0^t X_u^2\,du - \alpha\int_0^t X_u\,du} \mid X_t = y \right], \quad \lambda, x \text{ and } \alpha \geq 0.$$

**Proposition 5.4.** *Set $\kappa = \lambda^2 + \theta^2$, $\omega_1 = \beta + \frac{\theta}{2}$ and $\omega_3 = \beta + \frac{3\theta}{2}$. For $x$ and $\beta > 0$, we have*

$$\int_0^\infty e^{-\beta t} p_t^{(1)}(0,x) \Lambda_x^{\lambda,\alpha,(1)}(t)\,dt = e^{-\frac{\theta}{2}x^2} \frac{D_{-\frac{\omega_1}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2\kappa}(x+\frac{\alpha}{\kappa^2})\right)}{D_{-\frac{\beta}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2}\alpha\kappa^{-3/2}\right)}$$

*and*

$$\int_0^\infty e^{-\beta t} p_t^{(3)}(0,x) \Lambda_x^{\lambda,\alpha,(3)}(t)\,dt = e^{-\frac{\theta}{2}x^2} x \frac{D_{-\frac{\omega_3}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2\kappa}(x+\frac{\alpha}{\kappa^2})\right)}{D_{-\frac{\beta}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2}\alpha\kappa^{-3/2}\right)}.$$

**Proof.** From the absolute continuity relationship (5.4), we have

$$\mathbb{E}_x^\delta \left[ e^{-\frac{\lambda^2}{2}\int_0^t X_s^2\,ds - \alpha\int_0^t X_s\,ds} \right] = \mathbb{E}_x^{(\delta)} \left[ e^{-\frac{\theta}{2}(R_t^2 - x^2 - \delta t)} e^{-(\frac{\lambda^2+\theta^2}{2})\int_0^t R_s^2\,ds - \alpha\int_0^t R_s\,ds} \right].$$

The results follow by the same reasoning that for the proof of Proposition 3.2. □

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