A REFINED FACTORIZATION OF THE EXPONENTIAL LAW

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Abstract. Let ξ be a (possibly killed) subordinator with Laplace exponent ϕ and denote by $I_\phi = \int_0^\infty e^{-\xi_s} ds$, the so-called exponential functional. Consider the positive random variable I_{ψ_1} whose law, according to Bertoin and Yor [6], is determined by its negative entire moments as follows

$$\mathbb{E}[I_{\psi_1}^{-n}] = \prod_{k=1}^{n} \phi(k),\ n = 1, 2 \ldots.$$

In this note, we show that I_{ψ_1} is a positive self-decomposable random variable whenever the Lévy measure of ξ is absolutely continuous with a monotone decreasing density. In fact, I_{ψ_1} is identified as the exponential functional of a spectrally negative (for short *sn*) Lévy process. We deduce from [6] the following factorization of the exponential law $\mathbf{e}$

$$I_\phi / I_{\psi_1} \stackrel{(d)}{=} \mathbf{e}$$

where I_{ψ_1} is taken independent of I_ϕ. We proceed by showing an identity in distribution between the entrance law of a sn self-similar positive Feller process and the reciprocal of the exponential functional of sn Lévy processes. As by-product, we get, some new examples of the law of the exponential functionals, a new factorization of the exponential law and some interesting distributional properties of some random variables. For instance, we obtain that $S(\alpha)^\alpha$ is a self-decomposable random variable where $S(\alpha)$ is a positive stable random variable of index $\alpha \in (0, 1)$.

1. Introduction

Let $\xi = (\xi_t, t \geq 0)$ be a possibly killed subordinator starting from 0, i.e. a $[0, \infty)$-valued (∞ serves as absorbing state) Lévy process such that $\xi_0 = 0$. The law of ξ is well-known to be characterized by its Laplace exponent ϕ which admits the following Lévy-Khintchine representation, for any $u \geq 0$,

$$\phi(u) = bu + \int_0^\infty (1 - e^{-ur}) \nu(dr) + q \tag{1.1}$$

where $q \geq 0$ is the killing rate, $b \geq 0$ is the drift and the Lévy measure ν satisfies the integrability condition $\int_{\mathbb{R}+} (1 \wedge r)\, \nu(dr) < \infty$. Note that functions of the form (1.1) are also named, in the literature, as Bernstein functions. We refer to the monographs [5],[15], respectively to [4] [13], for a detailed account on Lévy processes, respectively on Bernstein functions. Next, consider

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the so-called exponential functional associated to ξ, which is defined as

$$I_\phi = \int_0^{\mathbf{e}_q} e^{-\xi_s} ds$$

where $\mathbf{e}_q$ is an independent exponential random variable with parameter q and we understand that $\mathbf{e}_0 = +\infty$. Note that, for any $q \geq 0$, $I_\phi < \infty$ a.s.. We refer to the survey paper of Bertoin and Yor [9] for a thorough description of the properties of this positive random variable and of the motivations for studying its law. In particular, we mention that the law of I_ϕ has been determined through its positive entire moments by Carmona et al.[11] as follows

$$\mathbb{E}[I_\phi^n] = \frac{\Gamma(n+1)}{\prod_{k=1}^n \phi(k)}, \ n = 1, 2, \ldots \tag{1.2}$$

where Γ stands for the Gamma function. Then, Bertoin and Yor [6], see also [9, Theorem 2] for the case $q > 0$, show that there exists a positive random variable J whose law is determined by its positive entire moments as follows

$$\mathbb{E}[J^n] = \prod_{k=1}^n \phi(k), \quad n = 1, 2, \ldots$$

such that, when J is taken independent of I_ϕ, one has the following factorization of the exponential law

$$I_\phi J \stackrel{(d)}{=} \mathbf{e} \tag{1.3}$$

where $\stackrel{(d)}{=}$ means identity in distribution and $\mathbf{e} = \mathbf{e}_1$.

Let us now point out that, since the random variable J is defined on the half-line and its law is uniquely determined by its positive entire moments, the sequence $(s_n = \prod_{k=1}^n \phi(k))_{n\geq 0}$ corresponds to a determinate normalized Stieltjes moment sequence. In this direction, we mention that Berg [1] generalizes the above fact by showing that for any $c > 0$, the sequence $(s_n^c)_{n\geq 0}$ associated to a measure on the half-line ρ_c is also a Stieltjes moment sequence which is determinate for $c \leq 2$. Then, he deduces that there exists a unique product convolution semigroup $(\rho_c)_{c>0}$ such that the moments of ρ_c are given by s_n^c for any $c > 0$. Moreover, in [2], Berg characterizes the set of normalized Stieltjes moment sequences for which this power stability property still holds. In the same vein, Berg and Durán [3] study a more general mapping which allows, in particular, to construct a Stieltjes moment sequence of the form $(s_n)_{n\geq 0}$ with the Bernstein function ϕ replaced by a completely monotone function.

The first aim of this note is to show that the random variable $1/J$, is, actually a positive self-decomposable random variable as soon as the Lévy measure ν in (1.1) admits a monotone decreasing density. This will be achieved by identifying the random variable $1/J$ as the exponential functional of a spectrally negative Lévy process which we now introduce. Let $\Xi = (\Xi_t, t \geq 0)$ be a conservative spectrally negative Lévy process with a non-negative mean m and starting from 0, i.e. a Lévy process having only negative jumps such that $0 \leq m = \mathbb{E}[\Xi_1] < \infty$. Its law is characterized by its Laplace exponent ψ which admits, in this case, the following Lévy-Khintchine representation, for any $u \geq 0$,

$$\psi(u) = \sigma u^2 + mu + \int_{-\infty}^0 (e^{ur} - 1 - ur)\Pi(dr) \tag{1.4}$$

where $\sigma \geq 0$ is the Gaussian coefficient and the Lévy measure satisfies the condition $\int_{-\infty}^{0}(|r| \wedge r^2)\Pi(dr) < \infty$. The exponential functional associated to Ξ, denoted by I_ψ, is finite a.s whenever $m > 0$. Its law has been determined through its negative entire moments by Bertoin and Yor [8] as follows

$$\mathbb{E}[I_\psi^{-n}] \quad = \quad m\frac{\prod_{k=1}^{n-1}\psi(k)}{\Gamma(n)},\ n = 1, 2, \dots \tag{1.5}$$

with the convention that the right-hand side is m when $n = 1$.

Now, we recall that Lamperti [18], interested in limit theorems for stochastic processes, shows, in particular, that, for any $x > 0$ the process $X = (X_t, t \geq 0)$ defined, for any $t \geq 0$, by

$$X_t \quad = \quad x\exp\left(\Xi_{A_{t/x}}\right),\ A_t = \inf\{s \geq 0;\ \int_0^s e^{\Xi_u}du > t\}, \tag{1.6}$$

starting from x at time 0 is a self-similar Feller process on $(0, \infty)$ having only negative jumps. The Lamperti transformation is actually one-to-one and extends to any Lévy processes. Bertoin and Yor [8, Proposition 1] show that the family of probability measures $(\mathbb{Q}_x^{(\psi)})_{x>0}$ of X, as defined in (1.6), converges, as $x \downarrow 0$, in the sense of finite-dimensional distribution, to a probability measure $\mathbb{Q}_0^{(\psi)}$, see also Caballero and Chaumont [10] for the weak convergence in the Skorohod's topology. Thus, X is a Feller on $[0, \infty)$ and they determine the law of the random variable $J_\psi = (X_1, \mathbb{Q}_0^{(\psi)})$, the entrance law of X at time 1, in terms of its positive entire moments as follows

$$\mathbb{E}[J_\psi^n] \quad = \quad \frac{\prod_{k=1}^{n}\psi(k)}{\Gamma(n+1)},\ n = 1, 2, \dots \tag{1.7}$$

They also deduce, in the case $m > 0$ and ξ is the ascending ladder height process of the dual process of Ξ, see e.g. [5, Chap. VI], that the random variable J, in (1.3), is J_ψ, i.e.

$$I_\phi J_\psi \overset{(d)}{=} \mathbf{e}. \tag{1.8}$$

The second aim of this note is to relate, in simple way, the law of J_ψ, for any $m \geq 0$, with the exponential functional of a spectrally negative Lévy process.

Finally, as observed by Rivero [22], the study of the exponential functional is also motivated by its connection to some interesting random equations. Indeed, from the strong Markov property for Lévy processes, which entails that for any finite stopping time T in the natural filtration $(\mathcal{F}_t, t \geq 0)$ of ξ, the process $(\xi_{t+T} - \xi_T, t \geq 0)$ is independent of $\mathcal{F}_T$ and has the same distribution than ξ, we deduce readily that the random variable I_ϕ, in the case $q = 0$, is solution to the random affine equation

$$I_\phi \overset{(d)}{=} \int_0^T e^{-\xi_s}ds + e^{-\xi_T}I'_\phi \tag{1.9}$$

where, on the right-hand side, I'_ϕ is an independent copy of I_ϕ. Note that this type of random equations have been studied by Kesten [14] and Goldie [12]. By means of a similar argument but using the absence of positive jumps of Ξ, see Rivero [21, Proposition 4] for more details, one gets that I_ψ is solution to the random affine equation, for any $y > 0$,

$$I_\psi \overset{(d)}{=} \int_0^{T_y} e^{-\Xi_s}ds + e^{-y}I'_\psi \tag{1.10}$$

where $T_y = \inf\{s > 0;\ \Xi_s \geq y\}$ and, on the right-hand side, I'_ψ is an independent copy of I_ψ. Hence I_ψ is a positive self-decomposable random variable and, in particular, its law is absolutely continuous and unimodal, see e.g. Sato [23] and Steutel and van Harn [24] for an excellent account on this set of probability measures.

2. Main results

2.1. **Factorization of the exponential law with exponential functionals.** In this part, we suppose that ξ is a subordinator starting from 0 with Laplace exponent given by (1.1). We introduce the following hypothesis on the Lévy measure of ξ.

Assumption 2.1. There exists a monotone decreasing function f such that $\nu(dx) = f(x)dx$.

We recall that under this condition $-df(x)$ is a Stieltjes measure on $(0,\infty)$. We also use the notation $-df(-x)$ for the image of the positive measure $-df(x)$ by the map $x \mapsto -x$. For instance if f is in addition differentiable then $-df(-x) = -f'(-x)dx$. We are now ready to derive our refinement of the factorization of the exponential law.

Theorem 2.2. *Let ξ a subordinator with Laplace exponent ϕ given by* (1.1)*. If Assumption 2.1 holds, then there exists an independent spectrally negative Lévy process with a positive mean and Laplace exponent ψ_1, analytical in the domain $C = \{u \in \mathbb{C}; \mathfrak{Re}(u) > -1\}$, with $\psi_1(-1) = -\phi(0)$, given by*

$$\psi_1(u) = bu^2 + \phi(1)u + \int_{-\infty}^{0} (e^{ur} - 1 - ur)\Pi(dr),\ u \in C,$$

where $\Pi(dr) = e^r\left(f(-r)dr - df(-r)\right)$ is a Stieltjes measure on $(-\infty, 0)$. Moreover, the law of the positive self-decomposable random variable I_{ψ_1} is determined by its negative entire moments as follows

$$\mathbb{E}[I_{\psi_1}^{-n}] \;=\; \prod_{k=1}^{n} \phi(k),\ n = 1, 2, \ldots.$$

and the exponential law admits the following factorization

$$I_\phi / I_{\psi_1} \stackrel{(d)}{=} \mathbf{e} \tag{2.1}$$

where $\mathbf{e}$ stands for an exponential random variable of parameter 1.

Conversely, if ψ is of the form (1.4) *with $m > 0$ and analytical in the domain C with $\psi(-1) \leq 0$, then there exists an independent subordinator with Laplace exponent ϕ_{-1} given by*

$$\phi_{-1}(u) = -\psi(-1) + \sigma u + \int_{0}^{\infty} (1 - e^{-ur})e^r\Pi(-\infty, -r)dr,\ u \geq 0,$$

such that

$$I_{\phi_{-1}} / I_\psi \stackrel{(d)}{=} \mathbf{e}. \tag{2.2}$$

Remark 2.3. (1) We have several comments to offer on the identity (2.1) when compared to (1.8). First, our hypothesis are slightly less restrictive. Indeed, it is well known, see e.g. [5, Chap. VI], that the ascending ladder height process of the dual process of Ξ satisfies Assumption 2.1, and is a killed subordinator, thus in (1.8), q is necessary positive. More importantly, we have identified the mixture random variable of I_ϕ in the factorization of the exponential law as the reciprocal of a positive self-decomposable random variable. Finally, our identity allows to get further explicit examples for the law of the exponential functional of Lévy processes. All these facts will be illustrated in Section 3.

(2) The analycity property of ψ_1 means that the associated spectrally negative Lévy process Ξ^1 admits exponential moments of order $u \geq -1$, i.e. for any $u \geq -1$ we have

$$\mathbb{E}[e^{u\Xi^1_1}] < \infty.$$

We shall show, in Proposition 2.4 below, how to construct a spectrally negative Lévy process with such a property from any spectrally negative Lévy process with a non-negative mean.

Proof. Let us write $\psi_1(u) = u\phi(u+1)$. Then, recalling that ϕ is analytical in the right-half plane, we deduce readily that the mapping $u \mapsto \psi_1(u)$ is analytical in C, with $\psi_1(0) = 0$ and $\psi_1(-1) = -\phi(0)$. Let us now show that under Assumption 2.1, ψ_1 is the Laplace exponent of a spectrally negative Lévy process with a positive mean. On the one hand, since $r \mapsto f(r)$ is monotone decreasing on $(0,\infty)$ then $\Pi(dr) = e^r\left(f(-r)dr - df(-r)\right)$ is clearly a Stieltjes measure on $(-\infty, 0)$. On the other hand, by integration by parts and a change of variable, we have, for any $u \geq 0$,

$$\begin{aligned}
\psi_1(u) &= u\left(b(u+1) + \int_0^\infty (1 - e^{-(u+1)r})f(r)dr + q\right) \\
&= bu^2 + \left(b + q + \int_0^\infty (1 - e^{-r})f(r)dr\right)u + u\int_0^\infty (1 - e^{-ur})e^{-r}f(r)dr \\
&= bu^2 + \phi(1)u + \int_0^\infty (e^{-ur} - 1 + ur)e^{-r}(f(r)dr - df(r)) \\
&= bu^2 + \phi(1)u + \int_{-\infty}^0 (e^{ur} - 1 - ur)\Pi(dr).
\end{aligned}$$

Checking, by integration by parts, that $\int_{-\infty}^0 (|r| \wedge r^2)\Pi(dr) < \infty$, we get that ψ_1 is the Laplace exponent of a spectrally negative Lévy process with a positive mean since $\psi_1(0^+) = \phi(1) > 0$. Then, by means of the identity (1.5), we obtain, for any $n = 1, 2, \ldots$,

$$\begin{aligned}
\mathbb{E}[I_{\psi_1}^{-n}] &= \phi(1)\frac{\prod_{k=1}^{n-1}\psi_1(k)}{\Gamma(n)} \\
&= \phi(1)\frac{\prod_{k=1}^{n-1}k\phi(k+1)}{\Gamma(n)} \\
&= \prod_{k=1}^{n}\phi(k)
\end{aligned}$$

where we have used the identity $\Gamma(n) = \prod_{k=1}^{n-1} k$. The self-decomposability of I_{ψ_1} was discussed in the introduction and the factorization of the exponential law follows readily from the independence of the random variables I_{ψ_1} and I_ϕ and the identity (1.2). The converse follows by means of a similar reasoning. We only need to check that $\phi_{-1}(u) = \frac{\psi(u-1)}{u-1}$ is Laplace exponent of a subordinator. From the conditions imposed on ψ, one easily deduces that ϕ_{-1} is well defined on

$\mathbb{R}^+$ with $\phi_{-1}(0) = -\psi(-1) \geq 0$. Moreover, one gets, by integrations by parts,

$$\begin{aligned}
\phi_{-1}(u) &= \sigma u - (\sigma - m) + \frac{1}{u-1}\int_{-\infty}^{0} (e^{(u-1)r} - 1 - (u-1)r)\Pi(dr) \\
&= \sigma u - (\sigma - m) - \int_{-\infty}^{0} (e^{(u-1)r} - 1)\Pi(-\infty, r)dr \\
&= \sigma u - (\sigma - m) - \int_{-\infty}^{0} (e^{ur} - 1)e^{-r}\Pi(-\infty, r)dr - \int_{-\infty}^{0} (e^{-r} - 1 + r)\Pi(dr) \\
&= -\psi(-1) + \sigma u + \int_{0}^{\infty} (1 - e^{-ur})e^{r}\Pi(-\infty, -r)dr.
\end{aligned}$$

The proof of the Theorem is completed. □

As a direct consequence of Theorem 2.2, we have the following fact. By self-decomposability, the law of I_{ψ_1} is absolutely continuous and thus the random variable J in (1.3) admits a density with respect to the Lebesgue measure, which according to Bertoin and Yor [6], is a 1-harmonic function for the self-similar process associated to ξ in the Lamperti mapping (1.6). Thus writing p_1 for the density of I_{ψ_1}, the mapping $x \mapsto x^{-2}p_1\left(x^{-1}\right)$ on $\mathbb{R}^+$ is 1-harmonic function for the self-similar process associated to ξ.

We complete this part with the following observation. Let us suppose that there exists a subordinator with Laplace exponent $\widehat{\phi}$ such that

$$\phi(u)\widehat{\phi}(u) = u, \quad u \geq 0.$$

Under such a condition, ϕ is called a special Bernstein function and we refer to Kyprianou and Rivero [17] and references therein for more information on this function. Note that such an identity occurs in fluctuation theory for Lévy processes and a sufficient condition for ϕ to be a special Bernstein function is that Assumption 2.1 holds, $d = 0$ and the mapping $x \mapsto f(x)\left(\int_x^\infty f(y)dy\right)^{-1}$ is a decreasing function on $(0, \infty)$. As remarked in [6], one has in this case, with the obvious notation,

$$I_{\widehat{\phi}} I_\phi \stackrel{(d)}{=} \mathbf{e}.$$

We deduce readily, from Theorem 2.2, that if ϕ is a special Bernstein function satisfying Assumption 2.1, then

$$I_{\widehat{\phi}} \stackrel{(d)}{=} 1/I_{\psi_1}.$$

Moreover, if in addition $\widehat{\phi}$ also satisfies Assumption 2.1 then $1/I_\phi$ is a positive self-decomposable random variable. Note that if $\widehat{\phi}(0) = 0$ then $I_{\widehat{\phi}}$ is solution to the random affine equation (1.9) and thus in this case solving this equation reduces to solve the random affine equation with constant coefficient (1.10).

2.2. **Exponential functionals and entrance laws J_ψ.** We now assume that Ξ is a spectrally negative Lévy process with a non-negative mean m. Its Laplace exponent has the form (1.4). We recall that the positive random variable J_ψ is the entrance law at time 1 of the self-similar Feller process associated to Ξ via the Lamperti mapping (1.6). We also mention that, when

$m > 0$, Bertoin and Yor [7] show that the distribution of $1/I_\psi$ is the so-called length-biased distribution of J_ψ, i.e. using the identity $\mathbb{E}[1/I_\psi] = m$,

$$\mathbb{E}[g(J_\psi)] = m^{-1}\mathbb{E}[1/I_\psi g(1/I_\psi)],$$

for any measurable function $g : \mathbb{R}^+ \to \mathbb{R}^+$. We refine this connection in the following.

Proposition 2.4. *Let ψ be the Laplace exponent of a spectrally negative Lévy process with mean $m \geq 0$, then the mapping defined by*

$$\psi_2(u) = \frac{u}{u+1}\psi(u+1)$$

is analytical in $C = \{u \in \mathbb{C};\ \mathfrak{Re}(u) > -1\}$ and is the Laplace exponent of a spectrally negative Lévy process with a positive mean $\psi(1)$. Moreover, the following identity in distribution

$$J_\psi \stackrel{(d)}{=} 1/I_{\psi_2}$$

holds. Consequently the law of J_ψ is absolutely continuous for any $m \geq 0$.

Proof. First, since it is well-known that ψ is analytical in the right-half plane, it is plain that the mapping $\psi_2(u) = \frac{u}{u+1}\psi(u+1)$ is analytical in C. Next, we recall that ψ has the following form

$$\psi(u) = \sigma u^2 + mu + \int_{-\infty}^{0} (e^{ur} - 1 - ur)\Pi(dr)$$

where $\int_{-\infty}^{0}(|r| \wedge r^2)\Pi(dr) < \infty$ and $\sigma \geq 0$. Thus, by means of integration by parts and writing $f(r) = \Pi(-\infty, r)$ for the tail of the Lévy measure, one gets

$$\begin{aligned}
\frac{u}{u+1}\psi(u+1) &= \sigma u^2 + (m+\sigma)u + \frac{u}{u+1}\int_{-\infty}^{0}(e^{(u+1)r} - 1 - (u+1)r)\Pi(dr) \\
&= \sigma u^2 + (m+\sigma)u - u\int_{-\infty}^{0}(e^{(u+1)r} - 1)f(r)dr \\
&= \sigma u^2 + (m+\sigma)u - u\left(\int_{-\infty}^{0}(e^{ur} - 1)e^r f(r)dr + \int_{-\infty}^{0}(e^{r} - 1)f(r)dr\right) \\
&= \sigma u^2 + \left(m + \sigma + \int_{-\infty}^{0}(e^{r} - 1 - r)\Pi(dr)\right)u \\
&\quad + \int_{-\infty}^{0}(e^{ur} - 1 - ur)e^r(f(r)dr + \Pi(dr)) \\
&= \sigma u^2 + \psi(1)u + \int_{-\infty}^{0}(e^{ur} - 1 - ur)e^r(f(r)dr + \Pi(dr)) \qquad (2.3)
\end{aligned}$$

where we recognize the Laplace exponent of a spectrally negative Lévy process. Finally, observing that $\lim_{u\to 0}\frac{d}{du}\frac{u}{u+1}\psi(u+1) = \psi(1) > 0$ since ψ is increasing on $(0, \infty)$, we have, from (1.5) and any $n = 1, 2 \ldots$,

$$\begin{aligned}
\mathbb{E}[I_{\psi_2}^{-n}] &= \psi(1)\frac{\prod_{k=1}^{n-1} \frac{k}{k+1}\psi(k+1)}{\Gamma(n)} \\
&= \frac{\prod_{k=1}^{n} \psi(k)}{\Gamma(n+1)} \\
&= \mathbb{E}[J_{\psi}^{n}]
\end{aligned}$$

where the last identity follows from (1.7). The absolute continuity property of the law of J_ψ follows from the one of I_{ψ_2} as a self-decomposable random variable. □

We mention that the random variable J_ψ appears in the study of the so-called Ornstein-Uhlenbeck process associated to X. Indeed, if one considers the stochastic process $U = (U_t, t \geq 0)$ defined, for any $t \geq 0$, by

$$U_t = e^{-t} X_{e^t - 1}$$

then U is a stationary Feller process on $[0, \infty)$ and its unique invariant measure is the law of J_ψ, see e.g. [19, Theorem 1.2]. The above Proposition tells us that the invariant measure is absolutely continuous for any $m \geq 0$.

We also indicate that the transformation of the Laplace exponent of a spectrally negative Lévy process used in the proof of Proposition 2.4 is a specific instance of more general mappings of characteristic exponents of Lévy processes introduced and studied by Kyprianou and Patie [16].

3. **Some examples**

In this part we will make use of the identities presented in Section 2 to obtain new explicit examples of the law of the exponential functional associated to subordinators or spectrally negative Lévy processes, new factorization of the exponential law and to prove the self-decomposability property of some positive random variables.

In [6], the authors study the connection between the law of the exponential functional of some subordinators and the following factorization of the exponential law

$$\mathbf{e}^{\alpha} S(\alpha)^{-\alpha} \stackrel{(d)}{=} \mathbf{e} \tag{3.1}$$

where $\alpha \in (0, 1)$ and $S(\alpha)$ is a positive α-stable random variable and is independent of $\mathbf{e}$. We split this example into two parts.

3.1. On the one hand, they show that

$$I_\phi \stackrel{(d)}{=} S(\alpha)^{-\alpha}$$

with

$$\phi(u) = \frac{\alpha\Gamma(\alpha u + 1)}{\Gamma(\alpha(u-1)+1)} = \int_0^\infty (1 - e^{-ur}) f(r) dr \tag{3.2}$$

where

$$f(r) = \frac{e^{-r/\alpha}}{\Gamma(1-\alpha)(1-e^{-r/\alpha})^{\alpha+1}}, \ r > 0. \tag{3.3}$$

(1) We start by applying the first part of Theorem 2.2. It is easy to check to that the mapping $r \mapsto f(r)$ is decreasing on $(0,\infty)$ and thus from Theorem 2.2 and using the recurrence relation $\Gamma(u+1) = u\Gamma(u), u > 0$, one gets that

$$\psi_1(u) \quad = \quad \frac{\Gamma(\alpha(u+1)+1)}{\Gamma(\alpha u)},$$

which after some easy calculations yields

$$\psi_1(u) \quad = \quad \alpha\Gamma(\alpha+1)u + \int_{-\infty}^{0} (e^{ur} - ur - 1)\frac{(\alpha+1)e^{(\alpha+1)r/\alpha}}{\alpha\Gamma(1-\alpha)(1-e^{r/\alpha})^{\alpha+2}}dr. \tag{3.4}$$

Hence, from (3.1), we deduce that

$$I_{\psi_1} \stackrel{(d)}{=} \mathbf{e}^{-\alpha}.$$

This result, up to a multiplicative constant, was actually obtained by Patie in [20, Theorem 4.1], where it is shown that the law of I_{ψ_1} is related to the distribution of the absorption time of the α-self-similar continuous state branching process.

(2) Moreover, let us define, as in Proposition 2.4,

$$\begin{aligned}\psi_2(u) \quad &= \quad \frac{u}{u+1}\psi_1(u+1) \\ &= \quad \alpha u \frac{\Gamma(\alpha(u+2)+1)}{\Gamma(\alpha(u+1)+1)}\end{aligned}$$

and after observing that

$$\frac{\partial}{\partial r}\left(\frac{e^{(\alpha+1)r/\alpha}}{(1-e^{r/\alpha})^{\alpha+1}}\right) = \frac{(\alpha+1)e^{(\alpha+1)r/\alpha}}{\alpha(1-e^{r/\alpha})^{\alpha+2}},$$

one obtains, from (2.3) and (3.4),

$$\psi_2(u) \quad = \quad \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha)}u + \int_{-\infty}^{0} (e^{ur} - ur - 1)\frac{e^{(2\alpha+1)r/\alpha}}{\Gamma(1-\alpha)(1-e^{r/\alpha})^{\alpha+2}}\left(\frac{2\alpha+1}{\alpha} - e^{r/\alpha}\right)dr.$$

Hence, from (1.5), we get for any $n = 1, 2, \dots,$

$$\begin{aligned}\mathbb{E}[I_{\psi_2}^{-n}] \quad &= \quad \alpha^{n-1}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha n+\alpha+1)}{\Gamma(2\alpha+1)} \\ &= \quad \alpha^n \frac{\Gamma(\alpha n+\alpha+1)}{\Gamma(\alpha+1)}\end{aligned}$$

and by moment identification, one has

$$\alpha I_{\psi_2} \stackrel{(d)}{=} G^{-\alpha}(\alpha+1)$$

where $G(a)$ stands for a Gamma random variable of parameter $a > 0$. We deduce that, for any $\alpha \in (0,1)$, the random variable $G^{-\alpha}(\alpha+1)$ is a positive self-decomposable random variable.

(3) Next, we apply the converse part of Theorem 2.2 to ψ_2. To this end, we introduce the subordinator with Laplace exponent ϕ_{-1} defined by

$$\begin{aligned}\phi_{-1}(u) &= \frac{1}{u-1}\psi_2(u-1) \\ &= \frac{\psi_1(u)}{u} \\ &= \alpha\frac{\Gamma(\alpha(u+1)+1)}{\Gamma(\alpha u+1)}.\end{aligned}$$

To get the Lévy-Khintchine representation of ϕ_{-1} note, from (3.2), that

$$\phi_{-1}(u) = \phi(u+1) - \phi(1) + \phi(1).$$

That is ϕ_{-1} is the Laplace exponent of the Esscher transform of ϕ killed at a rate $\phi(1)$. Thus, using the expression (3.3),

$$\phi_{-1}(u) = \alpha\Gamma(\alpha+1) + \int_0^\infty (1-e^{-ur})e^{-r} f(r)dr.$$

We have, from (1.2),

$$\mathbb{E}[I_{\phi_{-1}}^n] = \alpha^{-n}\frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha n+\alpha+1)},\ n = 1,2,\ldots$$

In order to characterize the law of $I_{\phi_{-1}}$, let us denote by U an uniform random variable on $(0,1)$ and by $S_1^{-\alpha}(\alpha)$ a random variable distributed according to the length-biased distribution of $S^{-\alpha}(\alpha)$, i.e. for any measurable function $g:\mathbb{R}^+\to\mathbb{R}^+$ one has

$$\mathbb{E}\left[g\left(S_1^{-\alpha}(\alpha)\right)\right] = \frac{\mathbb{E}\left[S^{-\alpha}(\alpha)g\left(S^{-\alpha}(\alpha)\right)\right]}{\mathbb{E}\left[S^{-\alpha}(\alpha)\right]}.$$

Recalling that for any $n = 1,2,\ldots$,

$$\mathbb{E}\left[S^{-\alpha n}(\alpha)\right] = \frac{\Gamma(n+1)}{\Gamma(\alpha n+1)},$$

one gets, by taking the random variable U independent of $S_1^{-\alpha}(\alpha)$,

$$\begin{aligned}\mathbb{E}\left[U^n S_1^{-\alpha n}(\alpha)\right] &= \frac{\mathbb{E}\left[U^n\right]\mathbb{E}\left[S^{-\alpha(n+1)}(\alpha)\right]}{\mathbb{E}\left[S^{-\alpha}(\alpha)\right]} \\ &= \frac{\Gamma(\alpha+1)\Gamma(n+2)}{(n+1)\Gamma(\alpha n+\alpha+1)} \\ &= \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha n+\alpha+1)}.\end{aligned}$$

By moment identification, we deduce the following identity

$$\alpha I_{\phi_{-1}} \stackrel{(d)}{=} U S_1^{-\alpha}(\alpha).$$

This yields the following factorization of the exponential law

$$U S_1^{-\alpha}(\alpha) G^{\alpha}(\alpha+1) \stackrel{(d)}{=} \mathbf{e}$$

where the three random variables on the left-hand side are assumed independent.

3.2. On the other hand, Bertoin and Yor [6] also observed that

$$I_{\widehat{\phi}} \stackrel{(d)}{=} \mathbf{e}^{\alpha}$$

with

$$\begin{aligned}\widehat{\phi}(u) &= u\frac{\Gamma(\alpha(u-1)+1)}{\Gamma(\alpha u+1)} \\ &= \int_0^{\infty}(1-e^{-ur})\frac{(1-\alpha)^2 e^{r/\alpha}}{\alpha\Gamma(\alpha+1)(e^{r/\alpha}-1)^{2-\alpha}}dr.\end{aligned}$$

Verifying easily that the density is decreasing, we obtain, appealing to obvious notation, that

$$\widehat{\psi}_1(u) = u\frac{\Gamma(\alpha u+1)}{\alpha\Gamma(\alpha(u+1))}$$

which yields after some easy manipulations

$$\widehat{\psi}_1(u) = \Gamma^{-1}(\alpha+1)u + \int_{-\infty}^{0}(e^{ur}-ur-1)\frac{(1-\alpha)^2 e^{-r/\alpha}}{\alpha^2\Gamma(\alpha+1)(e^{-r/\alpha}-1)^{2-\alpha}}\left(1-\alpha+\frac{2-\alpha}{(1-e^{r/\alpha})}\right)dr.$$

Thus, from the identity (3.1) and Theorem 2.2, we deduce that

$$\mathcal{I}_{\widehat{\psi}_1} \stackrel{(d)}{=} S(\alpha)^{\alpha}.$$

Hence $S(\alpha)^{\alpha}$ is a positive self-decomposable random variable.

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