

# EXPONENTIAL FUNCTIONAL OF A NEW FAMILY OF LÉVY PROCESSES AND SELF-SIMILAR CONTINUOUS STATE BRANCHING PROCESSES WITH IMMIGRATION

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ABSTRACT. We first introduce and derive some basic properties of a two-parameters  $(\alpha, \gamma)$  family of one-sided Lévy processes, with  $1 < \alpha < 2$  and  $\gamma > -\alpha$ . Their Laplace exponents are given in terms of the Pochhammer symbol as follows

$$\psi^{(\gamma)}(\lambda) = c((\lambda + \gamma)_\alpha - (\gamma)_\alpha), \quad \lambda \geq 0,$$

where  $c$  is a positive constant,  $(\lambda)_\alpha = \frac{\Gamma(\lambda + \alpha)}{\Gamma(\lambda)}$  stands for the Pochhammer symbol and  $\Gamma$  for the gamma function. These are a generalization of the Brownian motion, since in the limit case  $\alpha \rightarrow 2$ , we end up to the Laplace exponent of a Brownian motion with drift  $\gamma + \frac{1}{2}$ . Then, we proceed by computing the density of the law of the exponential functional associated to some elements of this family (and their dual) and some transformations of these elements. More precisely, we shall consider the Lévy processes which admit the following Laplace exponent, for any  $\delta > \frac{\alpha-1}{\alpha}$ ,

$$\psi^{(0,\delta)}(\lambda) = \psi^{(0)}(\lambda) - \frac{\alpha\delta}{\lambda + \alpha - 1} \psi^{(0)}(\lambda), \quad \lambda \geq 0.$$

These densities are expressed in terms of the Wright hypergeometric functions. By means of probabilistic arguments, we derive some interesting properties enjoyed by these functions. On the way, we also characterize explicitly the semi-group of the family of self-similar continuous state branching processes with immigration.

RÉSUMÉ. Nous introduisons et étudions quelques propriétés élémentaires d'une famille de processus de Lévy complètement asymétriques. Leurs lois sont caractérisées par leurs exposants de Laplace qui s'expriment en termes du symbole de Pochhammer. Ensuite, nous calculons la loi de la fonctionnelle exponentielle associée à certains éléments de cette famille et d'une transformation de ces éléments. Ces lois s'avèrent absolument continues et leurs densités s'expriment en termes des fonctions hypergéométriques de Wright. En utilisant des arguments probabilistes, nous déduisons que ces fonctions possèdent des propriétés analytiques intéressantes. Lors du déroulement de la preuve, nous caractérisons également le semi-groupe des processus auto-similaires de branchement avec immigration.

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### 1. Introduction

Let  $\zeta$  be a Lévy process with a negative first moment, the exponential functional associated to  $\zeta$  is defined as follows

$$\int_0^\infty e^{\zeta_s} ds.$$

This positive random variable plays an important role from both theoretical and applied perspectives. Indeed, it appears in various fields such as diffusion processes in random environments, fragmentation and coalescence processes, the classical moment problems, mathematical finance, astrophysics ... We refer to the paper of Bertoin and Yor [7] for a thorough survey on this topic and a description of cases when the law of such a functional is known explicitly. In particular, we mention that Bertoin and Yor [6] determine, through their negative entire moments, the law of the exponential functional associated to spectrally positive Lévy processes and Patie [32] expresses its Laplace transform for some spectrally negative Lévy processes. We also refer, in the case of the Brownian motion with a negative drift, to the two survey papers of Matsumoto and Yor [26] and [27] where the law of the exponential functional allows to characterize several interesting stochastic processes and to develop stochastic analysis related to Brownian motions on hyperbolic spaces. Finally, we indicate that Bertoin et al. [3] have derived the law of the exponential functional of a Poisson point process by means of  $q$ -calculus and have made an interesting connection to the indeterminate moment problem associated to a log-normal random variable.

In this paper, we start by introducing through both their Laplace exponents and their characteristics triplets, a two  $(\alpha, \gamma)$ -parameters family of spectrally negative Lévy processes, with  $1 < \alpha \leq 2$  and  $\gamma > -\alpha$ . Their Laplace exponents have the following form

$$(1.1) \quad \psi^{(\gamma)}(\lambda) = c((\lambda + \gamma)_\alpha - (\gamma)_\alpha), \quad \lambda \geq 0,$$

where  $c$  is a positive constant,  $(\lambda)_\alpha = \frac{\Gamma(\lambda + \alpha)}{\Gamma(\lambda)}$  stands for the Pochhammer symbol and  $\Gamma$  for the Gamma function. It turns out that this family possesses several interesting properties. Indeed, their Lévy measures behave around 0 as the Lévy measure of a stable Lévy process of index  $\alpha$ . Moreover, a member of this family has its law at a fixed time which belongs to the domain of attraction of a stable law of index  $\alpha$ . They include the family of Brownian motion with positive drifts but also negative since they admit negative exponential moments. After studying some further basic properties of this family, we compute the density of the law of the exponential functional associated to some elements of this family (and their dual) and some transformations of these elements. More specifically, we shall consider the Lévy processes which admit the following Laplace exponent, for any  $\delta > \frac{\alpha-1}{\alpha}$ ,

$$(1.2) \quad \psi^{(0,\delta)}(\lambda) = \psi^{(0)}(\lambda) - \frac{\alpha\delta}{\lambda + \alpha - 1} \psi^{(0)}(\lambda), \quad \lambda \geq 0.$$

The densities of the corresponding exponential functionals are expressed in terms of the Wright hypergeometric function  ${}_1\Psi_1$ . As a specific instance, we obtain the inverse Gamma law and hence recover Dufresne's result [13] regarding the law of the exponential functional of a Brownian motion with negative drifts. As a limit case, we show that the family encompasses the inverse Linnik law.

The path we follow to derive such a law is as follows. The first stage consists on characterizing the family of Lévy processes associated, via the Lamperti mapping, to the family of self-similar continuous state branching processes with immigration (for short *cbip*). We mention that this family includes the family of squared Bessel processes and belong to the class of affine term structure models in mathematical finance, see the survey paper of Duffie et al. [12]. At a second stage, by using well-know results on *cbip*, we derive the spatial Laplace transforms of the semi-groups of this later family. Then, by means of inversion techniques, we compute, in terms of a power series, the density of these semi-groups. In particular, we get an expression of their entrance laws in terms of the Wright hypergeometric function  ${}_1\Psi_1$ . Finally, we end up our journey by related these entrance laws to the law of the exponential functionals associated to the family of Lévy processes (1.2).

On the way, we get an expression, as a power series, for the density of the semi-group of a family of self-similar positive Markov processes (for short *sspMp*). We mention, that beside the family of Bessel processes, such a semi-group is only known explicitly for the so-called saw-tooth processes (a piecewise linear *sspMp*) which were studied by Carmona et al. [11]. Therein, the authors use the multiplicative kernel associated to a gamma random variable to obtain an intertwining relationship between the semi-group of the Bessel processes and the family of piecewise linear *sspMp*.

The remaining part of the paper is organized as follows. In the next Section, we gather some preliminary results. In particular, we introduce a transformation which leaves invariant the class of sspMp without diffusion coefficients. We also provide the detailed computation of an integral which will appear several times in the paper. Section 3 concerns the definition and the study of some basic properties of the new family of one-sided Lévy processes (1.1). Section 4 is devoted to the statement and the proof of the law of the exponential functionals under study. Section 5 contains several remarks regarding some representations of the special functions which appear in this paper. Finally Section 6 is a summarize of the asymptotic behaviors of the Wright hypergeometric functions.

## 2. Preliminary results

**2.1. Self-similar positive Markov processes.** Let  $\zeta = (\zeta_t)_{t \geq 0}$  denotes a real-valued Lévy process starting from  $y \in \mathbb{R}$ . Then, for any  $\alpha, t > 0$ , introduce the time change process

$$X_t = e^{\zeta_{A_t}}$$

where

$$A_t = \inf\{s \geq 0; \Sigma_s := \int_0^s e^{\alpha \zeta_u} du > t\}.$$

Lamperti [22] showed that the process  $X = (X_t)_{t \geq 0}$  is a positive  $\alpha$ -self-similar Markov process starting from  $e^y$  and that the mapping is actually one to one. We shall refer to this time change transformation as the Lamperti mapping. Next, we recall that the infinitesimal generator,  $\mathbf{Q}$ , of the  $\alpha$ -sspMp  $X$  is given for any  $\alpha, x > 0$  and  $f$  a smooth function on  $\mathbb{R}^+$ , by

$$\begin{aligned} \mathbf{Q}f(x) &= x^{-\alpha} \left( \frac{\sigma}{2} x^2 f''(x) + b x f'(x) \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} ((f(e^r x) - f(x)) - x f'(x) r \mathbb{1}_{\{r \leq 1\}}) \nu(dr) \right) \end{aligned}$$

where the three parameters  $b \in \mathbb{R}$ ,  $\sigma \geq 0$  and the measure  $\nu$  which satisfies the integrability condition  $\int_{-\infty}^{+\infty} (1 \wedge |r|^2) \nu(dr) < +\infty$  form the characteristic triplet of the Lévy process  $\zeta$ , the image of  $X$  by the Lamperti mapping. Finally, let us denote by  $T_0^X = \inf\{s \geq 0; X_{s-} = 0, X_s = 0\}$  the lifetime of  $X$ . Note that  $T_0^X = \infty$  or  $< \infty$  a.s. according to  $\zeta$  drifts to  $+\infty$  or  $-\infty$  a.s. We proceed by stating a general result on a transformation between sspMp.

**Proposition 2.1.** *Let  $\zeta$ , with characteristic exponent  $\Upsilon$ , be the image via the Lamperti mapping of an  $\alpha$ -sspMp  $X$  and fix  $\beta > 0$ . Then, if  $\sigma = 0$ , the  $\frac{\alpha}{\beta}$ -sspMp  $X^\beta$  and the  $\alpha$ -sspMp  $X^{(\beta)}$ , obtained as the image via the Lamperti mapping of the Lévy process  $\beta\zeta$ , have the same image via this mapping. Otherwise, the characteristics exponent of the image via the Lamperti mapping of the  $\frac{\alpha}{\beta}$ -sspMp  $X^\beta$  is*

$$\Upsilon(\beta\lambda) + \frac{\sigma}{2} \beta(\beta - 1)\lambda, \quad \lambda \in \mathbb{R}.$$

The proof is split into the following two lemmas.

**Lemma 2.2.** For any  $\beta > 0$ , we have the following relationship between  $\text{sspMp}$ ,

$$(2.1) \quad X_t^{(\beta)} = X_t^\beta \int_0^t X_s^{(\beta)\alpha(\frac{1}{\beta}-1)} ds, \quad t < T_0^X,$$

where  $X^{(\beta)}$  is the  $\text{sspMp}$  associated to the Lévy process  $\beta\zeta$ .

**Proof.** Since  $\beta > 0$ , note that the lifetime of both processes are either finite a.s or infinite a.s. Then, set  $A_t = \int_0^t X_s^{-\alpha} ds$  and observe that the Lamperti mapping yields, for any  $t < T_0^X$ ,

$$\begin{aligned} \log(X_t^\beta) &= \beta\zeta_{A_t} \\ &= \log\left(X_{\int_0^{A_t} e^{\alpha\beta\xi_s} ds}^{(\beta)}\right) \\ &= \log\left(X_{\int_0^t X_s^{\alpha(\beta-1)} ds}^{(\beta)}\right) \end{aligned}$$

which completes the proof.  $\square$

**Lemma 2.3.** With the notation as above, for any  $\beta > 0$ , the Laplace exponent of the Lévy process associated to the  $\frac{\alpha}{\beta}$ - $\text{sspMp}$   $X^\beta$  is given by

$$\Upsilon(\beta\lambda) + \frac{\sigma}{2}\beta(\beta-1)\lambda, \quad \lambda \in \mathbb{R}.$$

**Proof.** Let us denote by  $\mathbf{Q}^\beta$  the infinitesimal generator of  $X^\beta$  and set  $p_\beta(x) = x^\beta$  for any  $\beta > 0$ . Since the function  $x \mapsto p_\beta(x)$  is a homeomorphism of  $\mathbb{R}^+$  into itself, we obtain, after some easy computations,

$$\begin{aligned} \mathbf{Q}^\beta f(x) &= \mathbf{Q}(f \circ p_\beta)(x^{1/\beta}) \\ &= x^{-\alpha/\beta} \left( \frac{\sigma}{2} \beta^2 x^2 f''(x) + \left( b\beta + \frac{\sigma}{2} \beta(\beta-1) \right) x f'(x) \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} \left( (f(e^{\beta r} x) - f(x)) - \beta x f'(x) r \mathbb{1}_{\{r \leq 1\}} \right) \nu(dr) \right). \end{aligned}$$

The proof is completed by identification.  $\square$

The proof of the Proposition follows by putting pieces together.

**2.2. An integral computation.** We proceed by computing an integral, expressed in terms of ratios of gamma functions, which will be very important for the sequel. Indeed, it will be useful for computing the characteristic triplet associated to the families (1.1) and (1.2), which will be provided in Section 3. We fix the following constants

$$\begin{aligned} c &= -\frac{1}{\alpha \cos\left(\frac{\alpha\pi}{2}\right)} > 0 \\ c_\alpha &= \frac{c}{\Gamma(-\alpha)} > 0. \end{aligned}$$

The motivation for the choice of these constants is given in Remark 3.2 below.

**Theorem 2.4.** For any  $\alpha, \lambda, \gamma \in \mathbb{C}$ , with  $1 < \Re(\alpha) < 2$ ,  $\Re(\lambda) \geq 0$  and  $\Re(\alpha + \gamma) > 0$ , we have

$$(2.2) \quad c_\alpha \int_0^1 \frac{(u^\lambda - 1)u^{\alpha+\gamma-1} - \lambda(u-1)}{(1-u)^{\alpha+1}} du = c((\lambda + \gamma)_\alpha - (\gamma)_\alpha) - \frac{c_\alpha \lambda}{\alpha - 1},$$

where we recall that  $(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)}$  is the Pochhammer symbol and  $c_\alpha = \frac{c}{\Gamma(-\alpha)} > 0$ .

**Proof.** In the sequel, we denote by  $\mathcal{F}_{\lambda, \alpha, \gamma}$  the integral of the left-hand side on (2.2) divided by  $c_\alpha$ . Before starting the proof, we recall the following integral representation of the Beta function, see e.g. Lebedev [23],

$$\begin{aligned} \mathcal{B}(\gamma, \alpha) &= \frac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma + \alpha)} \\ &= \int_0^1 (1-v)^{\gamma-1} v^{\alpha-1} dv, \quad \Re(\gamma) > 0, \Re(\alpha) > 0, \end{aligned}$$

and the recurrence relation for the gamma function  $\Gamma(\lambda + 1) = \lambda\Gamma(\lambda)$ . Then, reiteration of integrations by parts yield

$$\mathcal{F}_{\lambda, \alpha, \gamma} = -\frac{\lambda(\lambda + \gamma)_\alpha \Gamma(1 - \alpha)}{\alpha(\lambda + \gamma + \alpha - 1)} + \frac{\lambda}{\alpha - 1} - \frac{\alpha + \gamma - 1}{\alpha} \mathcal{F}$$

where we have set

$$\mathcal{F} := \int_0^1 (u^\lambda - 1)u^{\alpha+\gamma-2}(1-u)^{-\alpha} du$$

and we have used the condition  $\Re(\alpha + \gamma) > 0$ . Next, according to the binomial expansion, we have

$$\begin{aligned} \mathcal{F} &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \int_0^1 (u^\lambda - 1)u^{n+\alpha+\gamma-2} du \\ &= -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \int_0^1 (1-v)v^{\frac{n+\alpha+\gamma-1}{\lambda}-1} dv \\ &= -\lambda \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(n + \alpha + \gamma - 1 + \lambda)(n + \alpha + \gamma - 1)n!} \\ &= -\frac{\lambda}{\alpha + \gamma - 1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\alpha + \gamma - 1)_n}{(\alpha + \gamma)_n n! (n + \alpha + \gamma - 1 + \lambda)}. \end{aligned}$$

Using the power series of the  ${}_2\mathcal{F}_1$  hypergeometric functions, see e.g. [23, 9.1],

$${}_2\mathcal{F}_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n, \quad |z| < 1,$$

we observe that

$$\begin{aligned}\mathcal{F} &= -\frac{\lambda}{\alpha + \gamma - 1} \lim_{z \rightarrow 1^-} \int_0^z u^{\lambda + \alpha + \gamma - 2} {}_2\mathcal{F}_1(\alpha, \alpha + \gamma - 1; \alpha + \gamma; u) du \\ &= -\frac{\lambda}{\alpha + \gamma - 1} \frac{\Gamma(\lambda + \alpha + \gamma - 1)}{\Gamma(\lambda + \alpha + \gamma)} \lim_{z \rightarrow 1^-} {}_3\mathcal{F}_2\left(\begin{smallmatrix} \alpha, \alpha + \gamma - 1, \lambda + \alpha + \gamma - 1 \\ \alpha + \gamma, \lambda + \alpha + \gamma \end{smallmatrix}; z\right)\end{aligned}$$

where the last line follows from [16, 7.512(5)] and  ${}_3\mathcal{F}_2$  is the hypergeometric function of degree (3, 2). Note that this later representation holds for  $\Re(\alpha) < 2$ . We proceed by using a result of Milgram [28, (11)] regarding the limit of the  ${}_3\mathcal{F}_2$  function which is

$$\begin{aligned}\lim_{z \rightarrow 1^-} {}_3\mathcal{F}_2\left(\begin{smallmatrix} \alpha, \alpha + \gamma - 1, \lambda + \alpha + \gamma - 1 \\ \alpha + \gamma, \lambda + \alpha + \gamma \end{smallmatrix}; z\right) &= \frac{-(\alpha + \gamma - 1)\Gamma(\alpha + \gamma + \lambda)\Gamma(1 - \alpha)}{\lambda\Gamma(\gamma + \lambda)} \\ &+ \frac{(\alpha + \gamma + \lambda - 1)\Gamma(\alpha + \gamma)\Gamma(1 - \alpha)}{\Gamma(\gamma)\lambda}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{F} &= \frac{\lambda\Gamma(1 - \alpha)}{(\gamma + \alpha - 1)(\lambda + \gamma + \alpha - 1)} \left( \frac{(\gamma + \alpha - 1)(\lambda + \gamma)_\alpha}{\lambda} - \frac{(\lambda + \gamma + \alpha - 1)(\gamma)_\alpha}{\lambda} \right) \\ (2.3) \quad &= \Gamma(1 - \alpha) \left( \frac{(\lambda + \gamma)_\alpha}{\lambda + \gamma + \alpha - 1} - \frac{(\gamma)_\alpha}{(\gamma + \alpha - 1)} \right).\end{aligned}$$

Finally, we obtain

$$\mathcal{F}_{\lambda, \alpha, \gamma} = \frac{\Gamma(1 - \alpha)}{\alpha} \left( -\frac{(\lambda + \gamma)_\alpha(\lambda + \gamma + \alpha - 1)}{\lambda + \gamma + \alpha - 1} + (\gamma)_\alpha \right) + \frac{\lambda}{\alpha - 1}$$

which completes the proof by recalling that  $c_\alpha = -c \frac{\Gamma(1 - \alpha)}{\alpha}$ .  $\square$

### 3. Basic properties of the family $(\xi, \mathbb{P}^{(\gamma)})$

Let us denote by  $\mathbb{P}^{(\gamma)} = \left( \mathbb{P}_x^{(\gamma)} \right)_{x \in \mathbb{R}}$  the family of probability measures of the process  $\xi$  such that  $\mathbb{P}_x^{(\gamma)}(\xi_0 = x) = 1$  and recall that the Laplace exponent, denoted by  $\psi^{(\gamma)}$ , of the process  $(\xi, \mathbb{P}^{(\gamma)})$  has the following form

$$\psi^{(\gamma)}(\lambda) = c((\lambda + \gamma)_\alpha - (\gamma)_\alpha), \quad \lambda \geq 0,$$

where  $(\lambda)_\alpha = \frac{\Gamma(\lambda + \alpha)}{\Gamma(\lambda)}$  stands for the Pochhammer symbol and  $c = -\frac{1}{\alpha \cos(\frac{\alpha\pi}{2})}$ , and the parameters  $\alpha, \gamma$  belong to the set

$$\mathcal{K}_{\alpha, \gamma} = \{1 < \alpha < 2, \gamma > -\alpha\}.$$

We denote by  $\mathbb{E}_x^{(\gamma)}$  the expectation operator associated to  $\mathbb{P}_x^{(\gamma)}$  and write simply  $\mathbb{E}^{(\gamma)}$  for  $\mathbb{E}_0^{(\gamma)}$ . In what follow, we show that it is the Laplace exponent of a spectrally negative Lévy process, we also provide its characteristic triplet and derive some basic properties.

**Proposition 3.1.** (1) For  $\alpha, \gamma \in \mathcal{K}_{\alpha, \gamma}$ , the process  $(\xi, \mathbb{P}^{(\gamma)})$  is a spectrally negative Lévy process with finite quadratic variation. More specifically, we have, for any  $\lambda \geq 0$ ,

$$\psi^{(\gamma)}(\lambda) = \tilde{c}_\alpha \lambda + \int_{-\infty}^0 \left( e^{\lambda y} - 1 - \lambda y \mathbb{I}_{\{|y| < 1\}} \right) \nu(dy)$$

where

$$\nu(dy) = c_\alpha \frac{e^{(\alpha+\gamma)y}}{(1-e^y)^{\alpha+1}} dy, \quad y < 0,$$

and

$$\begin{aligned} \tilde{c}_\alpha = c_\alpha \sum_{k=1}^{\infty} \frac{1}{k(k-\alpha)} & \left( \mathcal{B} \left( \frac{e-1}{e}; k+1-\alpha, \alpha+\gamma-1 \right) - \right. \\ & \left. \left( \frac{e-1}{e} \right)^{k-\alpha} ((\alpha+\gamma-1)e^{1-\alpha-\gamma} - 1) \right) \end{aligned}$$

where  $\mathcal{B}(\cdot; \cdot, \cdot)$  stands for the incomplete Beta function.

(2) The random variable  $(\xi_1, \mathbb{P}^{(\gamma)})$  admits negative exponential moments of order lower than  $\gamma + \alpha$ , i.e. for any  $\lambda < \gamma + \alpha$ , we have

$$\mathbb{E}^{(\gamma)} \left[ e^{-\lambda \xi_1} \right] < +\infty.$$

(3) We have the following invariance property by Girsanov transform, i.e. for any  $\gamma \in \mathcal{K}_{\alpha, \gamma}$ ,

$$(3.1) \quad d\mathbb{P}_0^{(\gamma)}|_{F_t} = e^{\gamma \xi_t - \psi^{(0)}(\gamma)t} d\mathbb{P}_0^{(0)}|_{F_t}, \quad t > 0.$$

(4) The first moments have the following expressions

$$(3.2) \quad \mathbb{E}^{(\gamma)} [\xi_1] = c(\gamma)_\alpha (\Psi(\gamma + \alpha) - \Psi(\gamma)), \quad \gamma > -\alpha,$$

$$(3.3) \quad \mathbb{E}^{(0)} [\xi_1] = c\Gamma(\alpha)$$

$$(3.3) \quad \mathbb{E}^{(1-\alpha)} [\xi_1] = c \frac{1}{\Gamma(1-\alpha)} (-E_\gamma - \Psi(1-\alpha))$$

$$(3.4) \quad \mathbb{E}^{(-1)} [\xi_1] = -c\Gamma(\alpha - 1)$$

where  $\Psi(\lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$  is the digamma function and  $E_\gamma$  stands for Euler-Mascheroni constant.

(5) For any  $1 < \alpha < 2$ , there exists  $\gamma_\alpha$ , with  $-\alpha < \gamma_\alpha < 0$ , such that  $\mathbb{E}^{(\gamma_\alpha)} [\xi_1] = 0$ . Moreover, for any  $\gamma < \gamma_\alpha$  the Cramér condition holds, i.e. for any  $\gamma < \gamma_\alpha$ , there exists  $\lambda_\alpha > 0$  such that

$$(3.5) \quad \mathbb{E}^{(\gamma)} \left[ e^{\lambda_\alpha \xi_1} \right] = 1.$$

(6) Consequently, for any fixed  $1 < \alpha < 2$ , the process  $(\xi, \mathbb{P}^{(\gamma_\alpha)})$  oscillates and otherwise  $\lim_{t \rightarrow \infty} (\xi_t, \mathbb{P}^{(\gamma)}) = \text{sgn}(\gamma - \gamma_\alpha) \infty$  a.s.

(7) For  $\gamma = 0$  or  $-1$ , the scale function of  $(\xi, \mathbb{P}^{(\gamma)})$  is given by

$$\mathcal{W}^{(\gamma)}(x) = \frac{1}{\Gamma(\alpha)} e^{-\gamma x} (1 - e^{-x})^{\alpha-1}, \quad x > 0.$$

(8) Finally, we have the following two limits results:

- (a) For any  $\gamma \in \mathcal{K}_{\alpha, \gamma}$ , the process  $(\xi, \mathbb{P}^{(\gamma)})$  converges in distribution as  $\alpha \rightarrow 2$  to a Brownian motion with drift  $\gamma + \frac{1}{2}$ .
- (b) For any fixed  $1 < \alpha < 2$ , the process  $(\lambda^{\frac{1}{\alpha}} \xi_{\lambda t}, \mathbb{P}^{(0)})$  converges in distribution as  $\lambda \rightarrow \infty$  to a spectrally negative  $\alpha$ -stable process.

**Proof.** (1) First, from (2.2), we deduce that

$$\begin{aligned} \psi^{(\gamma)}(\lambda) &= c((\lambda + \gamma)_{\alpha} - (\gamma)_{\alpha}) \\ &= \frac{c_{\alpha} \lambda}{1 - \alpha} + \int_0^1 \left( (u^{\lambda} - 1) u^{\alpha + \gamma - 1} - \lambda(u - 1) \right) \frac{c_{\alpha} du}{(1 - u)^{\alpha + 1}} \\ &= \left( c_{\alpha} \left( \int_0^1 \frac{(\log(u) - (u - 1)) \mathbb{I}_{\{|\log(u)| < 1\}}}{(1 - u)^{\alpha + 1}} du + \int_{-\infty}^{\frac{1}{e}} (1 - u)^{-\alpha} du \right) \right. \\ &\quad \left. + c_{\alpha} \int_0^1 \frac{(u^{\alpha + \gamma - 1} - 1) \log(u) \mathbb{I}_{\{|\log(u)| < 1\}}}{(1 - u)^{\alpha + 1}} du \right) \lambda \\ &\quad + \int_0^1 \left( u^{\lambda} - 1 - \lambda \log(u) \mathbb{I}_{\{|\log(u)| < 1\}} \right) \frac{c_{\alpha} u^{\alpha + \gamma - 1} du}{(1 - u)^{\alpha + 1}} \\ &= \tilde{c}_{\alpha} \lambda + \int_0^{\infty} \left( e^{\lambda y} - 1 - \lambda y \mathbb{I}_{\{|y| < 1\}} \right) \frac{c_{\alpha} e^{(\alpha + \gamma)y} du}{(1 - e^y)^{\alpha + 1}}. \end{aligned}$$

where we have set

$$\begin{aligned} \tilde{c}_{\alpha} = c_{\alpha} \left( \int_0^1 \frac{(\log(u) - (u - 1)) \mathbb{I}_{\{|\log(u)| < 1\}}}{(1 - u)^{\alpha + 1}} du + \int_{-\infty}^{\frac{1}{e}} (1 - u)^{-\alpha} du \right. \\ \left. + \int_0^1 \frac{(u^{\alpha + \gamma - 1} - 1) \log(u) \mathbb{I}_{\{|\log(u)| < 1\}}}{(1 - u)^{\alpha + 1}} du \right) \end{aligned}$$

Hence, we recognize the Lévy-Khintchine representation of a one-sided Lévy process. Moreover, the quadratic finite variation property follows from the following asymptotic behavior of the Pochhammer symbol, see e.g. [23],

$$(3.6) \quad (z + \gamma)_{\alpha}^{-1} = z^{-\alpha} \left[ 1 + \frac{(-\alpha)(2\gamma - \alpha - 1)}{2z} + O(z^{-2}) \right], \quad |\arg z| < \pi - \delta, \quad \delta > 0,$$

and the condition  $1 < \alpha < 2$ . It remains to compute the constant  $\tilde{c}_{\alpha}$ . Performing the change of variable  $v = 1 - u$ , we get, for the first term on the left-hand side of

the previous identity,

$$\begin{aligned} \int_0^{\frac{e-1}{e}} (\log(1-v) + v) \frac{c_\alpha}{v^{\alpha+1}} dv &= -c_\alpha \sum_{k=2}^{\infty} \frac{1}{k} \int_0^{\frac{e-1}{e}} v^{k-\alpha-1} dv \\ &= -c_\alpha \sum_{k=2}^{\infty} \frac{1}{k(k-\alpha)} \left( \frac{e-1}{e} \right)^{k-\alpha}. \end{aligned}$$

Moreover, proceeding as above, we have

$$\begin{aligned} &\int_0^1 \frac{(u^{\alpha+\gamma-1} - 1) \log(u) \mathbb{I}_{\{|\log(u)| < 1\}}}{(1-u)^{\alpha+1}} du \\ &= -c_\alpha \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\frac{e-1}{e}} v^{k-\alpha-1} ((1-v)^{\alpha+\gamma-1} - 1) dv \\ &= c_\alpha \sum_{k=1}^{\infty} \frac{\alpha + \gamma - 1}{k(k-\alpha)} \left( \int_0^{\frac{e-1}{e}} v^{k-\alpha} (1-v)^{\alpha+\gamma-2} dv \right. \\ &\quad \left. - \left( \frac{e-1}{e} \right)^{k-\alpha} e^{1-\alpha-\gamma} \right) \\ &= c_\alpha \sum_{k=1}^{\infty} \frac{\alpha + \gamma - 1}{k(k-\alpha)} \left( \mathcal{B} \left( \frac{e-1}{e}; k+1-\alpha, \alpha+\gamma-1 \right) - \right. \\ &\quad \left. \left( \frac{e-1}{e} \right)^{k-\alpha} e^{1-\alpha-\gamma} \right). \end{aligned}$$

Putting pieces together, one gets

$$\begin{aligned} \tilde{c}_\alpha &= c_\alpha \sum_{k=1}^{\infty} \frac{1}{k(k-\alpha)} \left( \mathcal{B} \left( \frac{e-1}{e}; k+1-\alpha, \alpha+\gamma-1 \right) - \right. \\ &\quad \left. \left( \frac{e-1}{e} \right)^{k-\alpha} ((\alpha+\gamma-1)e^{1-\alpha-\gamma} - 1) \right) \end{aligned}$$

which completes the description of the characteristic triplet of  $(\xi, \mathbb{P}^{(\gamma)})$ .

- (2) This item follows from the fact that the mapping  $\lambda \rightarrow \psi^{(\gamma)}(\lambda)$  is well defined on  $(-\gamma + \alpha, \infty)$ .
- (3) It is simply the Esscher transform.
- (4) The expressions for the first moment of  $(\xi_1, \mathbb{P}^{(\gamma)})$  is obtained from the formula

$$\frac{\partial}{\partial \lambda} (\lambda + \gamma)_\alpha = (\lambda + \gamma)_\alpha (\Psi(\lambda + \gamma + \alpha) - \Psi(\lambda + \gamma)).$$

Moreover, for  $\gamma = 0$  and  $\gamma = -1$ , we use the recurrence formula for the digamma function,  $\Psi(u+1) = \frac{1}{u} + \Psi(u)$ , see [23, Formula 1.3.3] and for the Gamma function.

- (5) Since, for any  $\gamma \in \mathcal{K}_{\alpha, \gamma}$ , the mapping  $\lambda \mapsto \psi^{(\gamma)}(\lambda)$  is convex and continuously differentiable on  $(\alpha + \gamma, \infty)$ , its derivative is continuous and increasing. Hence, the

mapping  $\gamma \mapsto \psi^{(\gamma)}(0^+)$  is continuous and increasing on  $\mathcal{K}_{\alpha,\gamma}$ . Moreover, noting that  $\mathbb{E}^{(-1)}[\xi_1] < 0 < \mathbb{E}^{(0)}[\xi_1]$  for any  $1 < \alpha < 2$ , we deduce that for each  $1 < \alpha < 2$ , there exists a unique  $-1 < \gamma_\alpha < 0$  such that  $\mathbb{E}^{(\gamma_\alpha)}[\xi_1] = 0$ . Since for any  $\gamma < \gamma_\alpha$ ,  $\mathbb{E}^{(\gamma)}[\xi_1]$  is negative and  $\lambda \mapsto \psi^{(\gamma)}(\lambda)$  is convex with  $\lim_{\lambda \rightarrow \infty} \psi^{(\gamma)}(\lambda) = \infty$ , we deduce that there exists  $\lambda_\alpha > 0$  such that  $\psi^{(\gamma)}(\lambda_\alpha) = 0$ .

- (6) The long time behavior of  $(\xi, \mathbb{P}^{(\gamma)})$  follows from the previous item and the strong law of large numbers.
- (7) The expression of the scale function is derived from the following identity, see [16, 3.312,1.], for  $\Re(\alpha), \Re(\lambda + \gamma) > 0$ ,

$$\int_0^\infty e^{-(\lambda+\gamma)x} (1 - e^{-x})^{\alpha-1} dx = \frac{\Gamma(\lambda + \gamma)\Gamma(\alpha)}{\Gamma(\lambda + \gamma + \alpha)}.$$

- (8) It is enough to show that the random variable  $(\xi_1, \mathbb{P}^{(\gamma)})$  converges in law to a normally distributed random variable with zero mean and variance 1. By continuity of the function  $\psi^{(\gamma)}(\lambda)$  in  $\alpha$ , we get the result after easy manipulation of the Gamma function. Similarly, for the second limit, we simply need to show that the random variable  $(\xi_1, \mathbb{P}^{(0)})$  belongs to the domain of attraction of a stable distribution, i.e.  $\lim_{\eta \rightarrow \infty} \eta \psi^{(0)}(\eta^{-\frac{1}{\alpha}} \lambda) = c\lambda^\alpha$ . The claim follows by means of the asymptotic of the ratio of Gamma functions, see (3.6). □

**Remark 3.2.** The choice of the constant  $c_\alpha$  is motivated by the item (8b). Actually, the coefficients of the Laplace exponent of a completely asymmetric stable random variable is given for any  $\iota > 0$  by  $c(\iota) = -\frac{\iota^\alpha}{\alpha \cos(\frac{\alpha\pi}{2})} > 0$ , see e.g. [37, Proposition 1.2.12]. Thus, we have made the choice  $\iota^\alpha = 1$ .

**Remark 3.3.** Note that Caballero and Chaumont [10] show, by characterizing its characteristic triplet, that the process  $(\xi, \mathbb{P}^{(0)})$  is the image via the Lamperti mapping of a spectrally negative regular  $\alpha$ -stable process conditioned to stay positive which we denote by  $\widehat{X}^\uparrow$ . Indeed, they show that the infinitesimal generator,  $\widehat{\mathbf{Q}}^{(0)}$ , of  $\widehat{X}^\uparrow$ , is given, for a smooth function  $f$  and any  $x > 0$ , by

$$\begin{aligned} \widehat{\mathbf{Q}}^{(0)} f(x) &= \frac{c_\alpha x^{1-\alpha}}{\alpha - 1} f'(x) + \\ & x^{-\alpha} \int_0^1 ((f(ux) - f(x))u^{\alpha-1} - x f'(x)(u - 1)) \frac{c_\alpha du}{(1 - u)^{\alpha+1}}. \end{aligned}$$

Thus, writing for  $x, \lambda > 0$ ,  $p_\lambda(x) = x^\lambda$ , we have  $\widehat{\mathbf{Q}}^{(0)} p_\lambda(x) = x^{\lambda-\alpha} \psi^{(0)}(\lambda)$ . For any  $\gamma \in \mathcal{K}_{\alpha,\gamma}$ , the expression of the Laplace exponent appears as an example in [32].

#### 4. Law of some exponential functionals via . . .

In this section, we derive the explicit law of the exponential functional associated to some elements of the family of Lévy processes introduced in the previous Section and to some transforms of these elements.

We proceed by introducing the Wright hypergeometric function which is defined, see e.g. Braaksma[9], by

$${}_p\Psi_q \left( \begin{matrix} (A_1, a_1), \dots, (A_p, a_p) \\ (B_1, b_1), \dots, (B_q, b_q) \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(A_i n + a_i) z^n}{\prod_{i=1}^q \Gamma(B_i n + b_i) n!}$$

where  $p, q$  are nonnegative integers,  $a_i \in \mathbb{C}$  ( $i = 1 \dots p$ ),  $b_j \in \mathbb{C}$  ( $j = 1 \dots q$ ) and the coefficients  $A_i \in \mathbb{R}^+$  ( $i = 1 \dots p$ ) and  $B_j \in \mathbb{R}^+$  ( $j = 1 \dots q$ ) are such that  $1 + \sum_{i=1}^q B_i - \sum_{i=1}^p A_i \geq 0$ . Under such conditions, it follows from the asymptotic formula of the ratio of Gamma functions, see (3.6), that  ${}_p\Psi_q(z)$  is an entire function with respect to  $z$ . We refer the reader to Section 6 for a description of this class of functions and for a detail account on their asymptotic behaviors.

**4.1. ... an important continuous state branching process.** In this part, we establish a connection between the dual of an element of the family of Lévy processes introduced in this paper and a self-similar continuous state branching process. As a byproduct, we derive the explicit law of the exponential functional associated to the dual of this element.

To this end, we recall that continuous state branching processes form a class of non negative valued Markov processes which appear as limits of integer valued branching processes. Lamperti [21] showed that when the units and the initial state size is allowed to tend to infinity, the limiting process is necessarily a self-similar continuous state branching process with index lower than 1. We denote this process by  $(Y, \mathbb{Q}^{(0)})$ , i.e.  $\mathbb{Q}^{(0)} = (\mathbb{Q}_x^{(0)})_{x>0}$  is a family of probability measures such that  $\mathbb{Q}_x^{(0)}(Y_0 = x) = 1$ . The associated expectation operator is  $\mathbf{E}^{(0)}$ . Moreover, Lamperti showed that the semi-group of  $(Y, \mathbb{Q}^{(0)})$  is characterized by its spatial Laplace transform as follows, for any  $\lambda, x \geq 0$ ,

$$(4.1) \quad \mathbf{E}_x^{(0)} \left[ e^{-\lambda Y_t} \right] = e^{-x d \lambda (1 + c t \lambda^\kappa)^{-1/\kappa}},$$

for some  $0 < \kappa \leq 1$  and some positive constants  $d, c$ . On the other hand, he also observed in [20] that a continuous state branching process can be obtained from a *spectrally positive* Lévy process,  $\zeta$ , by a time change. More precisely, consider  $\zeta$  started at  $x > 0$  and write

$$T_0^\zeta = \inf\{s \geq 0; \zeta_s = 0\}.$$

Next, let

$$\Lambda_t = \int_0^t \zeta_s^{-1} ds, \quad t < T_0^\zeta$$

and

$$V_t = \inf\{s \geq 0; \Lambda_s > t\} \wedge T_0^\zeta.$$

Then, the time change process  $\zeta \circ V$  is a continuous state branching process starting at  $x$  and the Laplace exponent  $\varphi$  of  $\zeta$  is called the branching mechanism. Finally, we recall that the law of the absorption time

$$i = \inf\{s \geq 0; \zeta \circ V_s = 0\}$$

has been computed explicitly by Grey [17]. More specifically, put  $\phi(0) = \inf\{s \geq 0; \varphi(s) > 0\}$  (with the usual convention that  $\inf \emptyset = \infty$ ) and assume that  $\int_0^\infty \varphi^{-1}(s)ds < \infty$  and  $\phi(0) < \infty$ , then

(4.2) the law of  $g(i)$  for  $\zeta \circ V$  starting at  $x > 0$  is exponential with parameter  $x$ ,

where  $g : (0, \infty) \rightarrow (\phi(0), \infty)$  is the inverse mapping of  $\int_t^\infty \varphi^{-1}(s)ds$ . We are now ready to state and proof the main result of this part.

**Theorem 4.1.** *For any  $0 < \kappa \leq 1$ , we have the identity in distribution*

$$\left( \int_0^\infty e^{-\kappa \xi_s} ds, \mathbb{P}_0^{(0)} \right) \stackrel{(d)}{=} c\varepsilon^{-\kappa}$$

where  $\varepsilon$  is an exponential random variable of parameter 1.

In the case  $\kappa = 1$ ,  $(\xi, \mathbb{P}^{(0)})$  is a Brownian motion with positive drift  $\frac{1}{2}$ , see item 8a in Proposition 3.1, and the result above corresponds to Dufresne's result [13]. We split the proof in several lemmas. First, let us denote by  $X$  a *spectrally positive*  $\alpha$ -stable Lévy process killed when it hits zero. It is plain that it is also a  $\alpha$ -sspMp. We have the following.

**Lemma 4.2.** *The Lévy process associated, via the Lamperti mapping, to  $X$  is the dual, with respect to the Lebesgue measure, of  $(\xi, \mathbb{P}^{(0)})$ , i.e.  $(-\xi, \mathbb{P}^{(0)})$ .*

**Proof.** First, by Hunt switching identity, see e.g. Gettoor and Sharpe [15], we have that  $X$  is in duality, with respect to the Lebesgue measure, with  $\widehat{X}$ , the spectrally negative  $\alpha$ -stable Lévy process killed when entering the negative real line. Moreover, it is well known, see e.g. Bertoin [2] that the spectrally negative  $\alpha$ -stable Lévy process conditioned to stay positive, which was denoted by  $\widehat{X}^\dagger$  in Remark 3.3, is obtained as  $h$ -transform, in the Doob sense, of  $\widehat{X}$  with  $h(x) = x^{\alpha-1}, x > 0$ . Thus, it is plain that  $\widehat{X}^\dagger$  is the dual, with respect to the reference measure  $y^{\alpha-1}dy$ , of  $X$ . Moreover, the Lévy process associated via the Lamperti mapping to  $\widehat{X}^\dagger$  is  $(\xi, \mathbb{P}^{(0)})$ , see Remark 3.3. Since  $X$  and  $\widehat{X}^\dagger$  are sspMp, the conclusion follows from Bertoin and Yor [6, Lemma 2].  $\square$

**Lemma 4.3.** *The spectrally positive Lévy process associated, via the Lamperti mapping, to the  $\kappa$ -ssMp  $(Y, \mathbb{Q}^{(0)})$  is  $(-\xi, \mathbb{P}^{(0)})$ .*

**Remark 4.4.** As mentioned above, in the case  $\kappa = 1$ ,  $(-\xi, \mathbb{P}^{(0)})$  corresponds to a Brownian motion with drift  $-\frac{1}{2}$ , then  $(2Y, \mathbb{Q}^{(0)})$  is a squared Bessel process of dimension 0 and we recover the well-known formula, see e.g. Revuz and Yor [34],

$$\mathbf{E}_x^{(0)} \left[ e^{-2\lambda Y_t} \right] = e^{-x\lambda(1+t2\lambda)^{-1}}, \quad \lambda, x, t > 0.$$

**Proof.** First, we show that the branching mechanism associated to  $(Y, \mathbb{Q}^{(0)})$  is the Laplace exponent of a spectrally positive  $\alpha$ -stable process. It is likely that this result is known. Since we did not find any reference and for sake of completeness we provide an easy proof. It is well-know, see e.g. Li [24], that the semi-group of a continuous state branching process with branching mechanism  $\varphi$  admits as spatial Laplace transform the following expression, for any  $\lambda \geq 0$ ,

$$e^{-x\vartheta_\lambda(t)}$$

where  $\vartheta : [0, \infty) \rightarrow [0, \infty)$  solves the following boundary valued differential equation

$$\vartheta'_\lambda(t) = \varphi(\vartheta_\lambda(t)), \quad \vartheta_\lambda(0) = \lambda.$$

It is then not difficult to check, in the case  $\varphi(\lambda) = -\frac{c}{\kappa}\lambda^\alpha$ , that

$$\vartheta_\lambda(t) = \lambda(1 + ct\lambda^\kappa)^{-1/\kappa}$$

which is the Laplace exponent of  $(Y, \mathbb{Q}^{(0)})$  given in (4.1) with  $d = 1$  and  $0 < \kappa = \alpha - 1 \leq 1$ . Then, we deduce that  $(Y, \mathbb{Q}^{(0)})$  is obtained from  $X$  by the random time change described above. Finally, from Lemma 4.2 and Lemma 2.2, by choosing  $\beta_\alpha$  such that  $\beta_\alpha = -\frac{\beta_\alpha}{\alpha(\beta_\alpha - 1)}$ , i.e.  $\beta_\alpha = \frac{\kappa}{\alpha}$ , we get that the image via the Lamperti mapping of  $(Y^{\beta_\alpha}, \mathbb{Q}^{(0)})$  is  $(\beta_\alpha \xi, \mathbb{P}^{(0)})$ . By using Lemma 2.3 and observing that  $(Y, \mathbb{Q}^{(0)})$  is  $\kappa$ -self-similar we complete the proof.  $\square$

The proof of the Theorem 4.1 follows readily by observing that for  $\varphi(\lambda) = -\frac{c}{\kappa}\lambda^\alpha$ ,  $g(t) = (ct)^{\frac{1}{1-\alpha}}$  and from the identity  $(i, \mathbb{Q}_x^{(0)}) \stackrel{(d)}{=} (x^\kappa \int_0^\infty e^{-\kappa \xi_s} ds, \mathbb{P}_0^{(0)})$ . In the following part, we shall provide the expression of the semi-group of  $(Y, \mathbb{Q}^{(0)})$  in terms of a power series.

**4.2. ... a family of continuous state branching processes with immigration.** We recall that  $\kappa = \alpha - 1$ , and for any  $\delta \in \mathbb{R}^+$  we write  $\mathbb{P}^{(0, \delta)} = \left( \mathbb{P}_x^{(0, \delta)} \right)_{x \in \mathbb{R}}$  for the family of probability measures of the process  $\xi$ , which admits the following Laplace exponent

$$\begin{aligned} \psi^{(0, \delta)}(\lambda) &= \psi^{(0)}(\lambda) - \frac{\alpha \delta}{\lambda + \kappa} \psi^{(0)}(\lambda), \quad \lambda \geq 0 \\ &= c(\lambda + \kappa - \alpha \delta) \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)}. \end{aligned}$$

**Proposition 4.5.** *For any  $\delta > 0$  and  $1 < \alpha < 2$ ,  $(\xi, \mathbb{P}^{(0, \delta)})$  is a spectrally negative Lévy process. Its characteristic triplet is given by  $\sigma^{(\delta)} = 0$ ,*

$$b^{(\delta)} = c\Gamma(\alpha) \left( 1 - \frac{\alpha \delta}{\kappa} \right),$$

and

$$\nu^{(\delta)}(dy) = c_\alpha \frac{e^{\alpha y}}{(1 - e^y)^{\alpha+1}} (1 + \delta(e^{-y} - 1)) dy, \quad y < 0,$$

where

$$\log \left( \mathbb{E}^{(0, \delta)} \left[ e^{-\lambda \xi_1} \right] \right) = b^{(\delta)} \lambda + \int_{-\infty}^0 (e^{\lambda y} - 1 - \lambda y) \nu^{(\delta)}(dy).$$

**Proof.** First, writing simply here  $\psi$  for  $\psi^{(0)}$ , we have from (2.3) by choosing  $\gamma = 0$  that

$$\begin{aligned}
\frac{\alpha\delta}{\lambda + \kappa}\psi(\lambda) &= \frac{c\alpha\delta}{\lambda + \kappa} \frac{\Gamma(\lambda + \alpha)}{\Gamma(\lambda)} \\
&= \frac{c\alpha\delta}{\Gamma(1 - \alpha)} \int_0^1 (u^\lambda - 1) u^{\alpha-2} (1-u)^{-\alpha} du \\
&= -\delta \int_{-\infty}^0 (e^{\lambda y} - 1) \frac{c_\alpha e^{\kappa y} dy}{(1 - e^y)^\alpha} \\
&= -\delta \left( \int_{-\infty}^0 (e^{\lambda y} - 1 - \lambda y) \frac{c_\alpha e^{\kappa y} dy}{(1 - e^y)^\alpha} + \lambda \int_{-\infty}^0 y \frac{c_\alpha e^{\kappa y} dy}{(1 - e^y)^\alpha} \right) \\
&= -\delta \int_{-\infty}^0 (e^{\lambda y} - 1 - \lambda y) \frac{c_\alpha e^{\kappa y} dy}{(1 - e^y)^\alpha} + \lambda c \Gamma(\alpha) \frac{\alpha\delta}{\kappa},
\end{aligned}$$

where we have used the identities  $c_\alpha = \frac{c}{\Gamma(-\alpha)} > 0$  and

$$\begin{aligned}
\int_{-\infty}^0 y \frac{c_\alpha e^{\kappa y} dy}{(1 - e^y)^\alpha} &= -\lim_{\lambda \rightarrow 1} \frac{\partial}{\partial \lambda} \int_{-\infty}^0 (e^{\lambda y} - 1 - \lambda y) \frac{c_\alpha e^{\kappa y} dy}{(1 - e^y)^{\alpha+1}} \\
&= -\lim_{\lambda \rightarrow 1} \frac{\partial}{\partial \lambda} \psi^{(-1)}(\lambda) - c \frac{\Gamma(\alpha)}{\kappa} \\
&= -\psi'(0) - c \frac{\Gamma(\alpha)}{\kappa}.
\end{aligned}$$

The proof is completed by putting pieces together.  $\square$

Next, set  $\delta_\kappa = \frac{\delta}{\kappa}$  and  $M_\delta = \mathbb{E}^{(0,\delta)}[\xi_1]$  and note from the previous proposition that

$$M_\delta = c\Gamma(\alpha)(1 - \alpha\delta_\kappa).$$

In particular, we have  $M_\delta < 0$  if  $\delta > \frac{\kappa}{\alpha}$ . Under such a condition, we write simply

$$\left( \Sigma_\infty, \mathbb{P}_0^{(0,\delta)} \right) = \left( \int_0^\infty e^{\kappa \xi_s} ds, \mathbb{P}_0^{(0,\delta)} \right).$$

We are now ready to state to the main result of this section.

**Theorem 4.6.** *Let  $0 < \kappa < 1$ .*

- (1) *For any  $\delta > \frac{\kappa}{\alpha}$ , the law of the positive random variable  $(\Sigma_\infty, \mathbb{P}_0^{(0,\delta)})$  is absolutely continuous with an infinitely continuously differentiable density on  $(0, \infty)$ , denoted by  $f_\infty^{(\delta)}$ , and*

$$\begin{aligned}
f_\infty^{(\delta)}(y) &= |M_\delta| y^{-\alpha\delta_\kappa} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \alpha\delta_\kappa)}{n! \Gamma(\kappa n + \alpha\delta)} y^{-n} \\
&= |M_\delta| y^{-\alpha\delta_\kappa} {}_1\Psi_1 \left( \begin{matrix} (1, \alpha\delta_\kappa) \\ (\kappa, \alpha\delta) \end{matrix} \middle| -y^{-1} \right), \quad y > 0.
\end{aligned}$$

Moreover, we have

$$f_\infty^{(\delta)}(y) \sim |M_\delta| \frac{\Gamma(\alpha\delta_\kappa)}{\Gamma(\alpha\delta)} y^{-\alpha\delta_\kappa} \quad \text{as } y \rightarrow \infty.$$

We deduce, from (6.1), the following asymptotic behavior

$$f_\infty^{(\delta)}(y) \sim |M_\delta| \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + \alpha\delta_\kappa)}{n! \Gamma(-\kappa n)} y^n \quad \text{as } y \rightarrow 0.$$

(2) For any  $\delta < \frac{\kappa}{\alpha}$ , we have

$$\left( \int_0^\infty e^{-\kappa \xi_s} ds, \mathbb{P}_0^{(0,\delta)} \right) \stackrel{(d)}{=} c\kappa B^{-1}(1 - \alpha\delta_\kappa, \alpha\delta_\kappa) \varepsilon^{-\kappa}$$

where  $B(\delta, \gamma)$  is a Beta random variable with parameter  $\delta, \gamma > 0$  and the random variables on the right-hand side are taken independent. Thus, the law of  $\left( \int_0^\infty e^{-\kappa \xi_s} ds, \mathbb{P}_0^{(0,\delta)} \right)$  is absolutely continuous with a density, denoted by  $\widehat{f}_\infty^{(\delta)}$ , which admits the following integral representation, for any  $y > 0$ ,

$$\widehat{f}_\infty^{(\delta)}(y) = \frac{\sin(\alpha\delta_\kappa\pi)}{c\kappa^2\pi(y/c\kappa)^{\frac{\alpha}{\kappa}}} \int_0^1 e^{-\left(\frac{y}{c\kappa}r\right)^{-1/\kappa}} r^{-\frac{\alpha\delta+1}{\kappa}} (1-r)^{\alpha\delta_\kappa-1} dr.$$

**Remark 4.7.** (1) For  $\delta > \frac{\kappa}{\alpha}$ , the law of  $(\Sigma_\infty, \mathbb{P}^{(0,\delta)})$  is a generalization of the inverse Gamma distribution. Indeed, specifying on  $\kappa = 1$ , i.e.  $\alpha = 2$ ,  $c = \frac{1}{2}$  and  $\delta > \frac{1}{2}$ , the expression above reduces to

$$f_\infty^{(\delta)}(y) = 2 \frac{y^{-2\delta}}{\Gamma(2\delta - 1)} e^{-\frac{1}{y}}$$

which corresponds to Dufresne's result [13], i.e.

$$\int_0^\infty e^{B_s - (\delta - \frac{1}{2})s} ds \stackrel{(d)}{=} \frac{1}{2G(2\delta - 1)}$$

where  $B$  is a standard Brownian motion. Note that Dufresne's identity holds also for  $\delta < \frac{\kappa}{\alpha}$ .

(2) Note also that, the densities  $|M_\delta^{-1}| f_\infty^{(\delta)}$  converge, as  $\delta \rightarrow \frac{\kappa}{\alpha}$  to a density probability distribution given by

$$(4.3) \quad f^{(\frac{\kappa}{\alpha})}(y) = y^{-1} \sum_{n=0}^{\infty} \frac{(-y)^{-n}}{\Gamma(\kappa(n+1))}, \quad y > 0,$$

which is the inverse of a positive Linnik law of parameters  $(\kappa, \kappa)$ , see [25].

(3) We also point out that in a recent paper, Bertoin et al. [4] provides several interesting distributional properties of the duration of a recurrent Bessel process of dimension  $d = 2(1 - \alpha)$ ,  $0 < d < 2$ , straddling an independent exponential time, denoted by  $\Delta_\alpha$ . In particular, they show that the law of  $\Delta_\alpha$  is expressed in term of ratios of independent Gamma and Beta random variables. It would be interesting to establish connections, if any, between the random variables  $\Delta_\alpha$  and  $(\Sigma_\infty, \mathbb{P}^{(0,\delta)})$ . We also refer to James et al. [18] for related results.

We start by proving the item 2 of the Theorem. It follows readily from the expression, found by Bertoin and Yor [5], of the negative entire moments of the law of the exponential functional of a spectrally positive Lévy process with a negative mean. Indeed, the Laplace

exponent of  $(\kappa\xi, \mathbb{P}_0^{(0,\delta)})$  is given by  $\psi^{(0,\delta)}(\kappa\lambda) = c\kappa(\lambda + 1 - \alpha\delta_\kappa) \frac{\Gamma(\kappa(\lambda+1))}{\Gamma(\kappa\lambda)}$ , and its mean by  $c\kappa\Gamma(\alpha)(1 - \alpha\delta_\kappa)$ . Thus, from [5, Proposition 2], for  $\delta < \frac{\kappa}{\alpha}$  and any integer  $n$ , we get, after some easy computation,

$$\begin{aligned} \mathbb{E}_0^{(0,\delta)} \left[ \left( \int_0^\infty e^{-\kappa\xi_s} ds \right)^{-n} \right] &= \frac{c\kappa\Gamma(\alpha)(1 - \alpha\delta_\kappa)}{\Gamma(n)} \prod_{j=1}^{n-1} \psi^{(0,\delta)}(\kappa j) \\ &= c^n \kappa^n \Gamma(\alpha)(1 - \alpha\delta_\kappa) \frac{\Gamma(\kappa n)\Gamma(n+1 - \alpha\delta_\kappa)}{\Gamma(\kappa)\Gamma(n)\Gamma(2 - \alpha\delta_\kappa)} \\ &= \kappa^n c^n \frac{\Gamma(\kappa n + 1)\Gamma(n+1 - \alpha\delta_\kappa)}{\Gamma(n+1)\Gamma(1 - \alpha\delta_\kappa)}. \end{aligned}$$

The result follows by identification with the moments of a Beta random variable. To obtain the integral representation of the density, we first recall that

$$\mathbb{P}(\varepsilon^{-\kappa} \in dy) = \frac{1}{\kappa} y^{-(1/\kappa+1)} e^{-y^{-1/\kappa}}, \quad y > 0,$$

and

$$\mathbb{P}(B(1 - \alpha\delta_\kappa, \alpha\delta_\kappa) \in dr) = \frac{1}{\Gamma(1 - \alpha\delta_\kappa)\Gamma(\alpha\delta_\kappa)} r^{-\alpha\delta_\kappa} (1-r)^{\alpha\delta_\kappa-1} dr, \quad 0 < r < 1.$$

Thus, we deduce that

$$\mathbb{P}(B^{-1}(1 - \alpha\delta_\kappa, \alpha\delta_\kappa)\varepsilon^{-\kappa} \in dy) = \frac{\sin(\alpha\delta_\kappa\pi)}{\kappa\pi y^{\frac{\alpha}{\kappa}}} \int_0^1 e^{-(yr)^{-1/\kappa}} r^{-\frac{\alpha\delta+1}{\kappa}} (1-r)^{\alpha\delta_\kappa-1} dr$$

where we have used the Euler's reflection formula  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ .

The proof of the remaining part of the Theorem is split in several intermediate results which we find worth being stated. We start with the following easy result which gives a complete characterization, in terms of their Laplace transforms, of self-similar cbip. To this end, we now recall the definition of a cbip with parameters  $[\varphi, \chi]$ . It is well known, see e.g. [24], that the semi-group of a cbip with branching mechanism  $\varphi$  and immigration mechanism  $\chi$ , where  $\chi$  is the Laplace exponent of a positive infinitely divisible random variable, admits as a spatial Laplace transform the following expression

$$(4.4) \quad \exp\left(-x\vartheta_\lambda(t) - \int_0^t \chi(\vartheta_\lambda(s)) ds\right), \quad x, t \geq 0.$$

**Lemma 4.8.** *A cbip is self-similar of index  $\kappa$  if and only if  $0 < \kappa \leq 1$  and it corresponds to the cbip with parameters  $[\varphi, \delta\chi]$  where  $\delta > 0$ ,  $\varphi(\lambda) = -\frac{c}{\kappa}\lambda^{\kappa+1}$  and  $\chi(\lambda) = c(\frac{\kappa+1}{\kappa})\lambda^\kappa$ . Its Laplace transform has the following expression, for  $\delta, x, \lambda > 0$*

$$\mathbf{E}_x^{(\delta)} \left[ e^{-\lambda Y_t} \right] = \Lambda_t^{(\delta)}(\lambda, x)$$

where

$$\Lambda_t^{(\delta)}(\lambda, x) = (1 + ct\lambda^\kappa)^{-\alpha\delta_\kappa} e^{-x\lambda(1+ct\lambda^\kappa)^{-1/\kappa}}.$$

In particular its entrance law is characterized by

$$(4.5) \quad \mathbf{E}_{0^+}^{(\delta)} \left[ e^{-\lambda Y_t} \right] = (1 + ct\lambda^\kappa)^{-\alpha\delta_\kappa}.$$

We denote this family of processes by  $(Y, \mathbb{Q}^{(\delta)})_{\delta > 0}$ .

**Proof.** The sufficient part follows readily from the definition of the cbip and by observing that for any  $a, t, x > 0$ ,  $\Lambda_{a^\kappa t}^{(\delta)}(\lambda, ax) = \Lambda_t^{(\delta)}(a\lambda, x)$ . The necessary part follows from the fact that the unique self-similar branching process has its Laplace transform given by (4.1) and thus the immigration has to satisfy the self-similarity property. Since we have for any  $a, t > 0$ ,  $a\vartheta_\lambda(a^\kappa t) = \vartheta_{a\lambda}(t)$ , we need that

$$\chi(a\vartheta_{a\lambda}(t)) = a^\kappa \chi(\vartheta_{a\lambda}(\kappa t))$$

which is only possible for  $\chi(a) = Ca^\kappa$  for some positive constant  $C$  (since  $\chi$  is the Laplace exponent of a subordinator). The claim follows.  $\square$

**Remark 4.9.** (1) We mention that  $(Y, \mathbb{Q}^{(1)})$  corresponds to the continuous state branching process  $(Y, \mathbb{Q}^{(0)})$  conditioned to never extinct in the terminology of Lambert [19]. Indeed, he showed that the latter corresponds to the cbpi with immigration  $\varphi'$ .

(2) The Laplace transform of the entrance law (4.5) appears in a paper of Pakes [31] where he studies scaled mixtures of (symmetric) stable laws. More precisely, denoting by  $Y_1^{(\delta)}$  the entrance law at time 1 of  $(Y, \mathbb{Q}^{(\delta)})$ , we have the following identity in law

$$Y_1^{(\delta)} \stackrel{(d)}{=} G(\delta)^\kappa S_\kappa,$$

where  $S_\kappa$  is a positive stable law of index  $\kappa$  and the two random variables on the right-hand side are considered to be independent.

We proceed by providing the expression of the semi-group of  $(Y, \mathbb{Q}^{(\delta)})$ .

**Proposition 4.10.** *For any  $\delta > 0$ , the semi-group of  $(Y, \mathbb{Q}^{(\delta)})$  admits a density, with respect to the Lebesgue measure, denoted by  $p_t^{(\delta)}(\cdot, \cdot)$ , which is given for any  $x, y, t > 0$  by*

$$p_{\frac{\kappa\delta}{\alpha}}^{(\delta)}(x, y) = y^{\kappa\delta-1} \sum_{n=0}^{\infty} \frac{{}_1\Psi_1\left(\begin{matrix} (1, \frac{n}{\kappa} + \delta) \\ (\kappa, \kappa\delta) \end{matrix} \middle| -y^\kappa\right)}{n! \Gamma\left(\frac{n}{\kappa} + \delta\right)} (-1)^n x^n,$$

where by self-similarity we have set  $p^{(\delta)}(xt^{-1/\kappa}, yt^{-1/\kappa}) = (ct)^{-1/\kappa} p_{\frac{t}{c}}^{(\delta)}(x, y)$ . Moreover, the semi-group of the self-similar branching process  $(Y, \mathbb{Q}^{(0)})$  admits a density, with respect to the Lebesgue measure, given, using the same notation as above, for any  $x, y > 0$  by

$$p^{(0)}(x, y) = y^{-1} \sum_{n=0}^{\infty} \frac{{}_1\Psi_1\left(\begin{matrix} (1, \frac{n}{\kappa}) \\ (\kappa, 0) \end{matrix} \middle| -y^\kappa\right)}{n! \Gamma\left(\frac{n}{\kappa}\right)} (-1)^n x^n.$$

For  $\delta, y > 0$ , the entrance law of  $(Y, \mathbb{Q}^{(\delta)})$  is given by

$$(4.6) \quad p_{\frac{\kappa\delta}{\alpha}}^{(\delta)}(0, y) = \frac{1}{\Gamma(\delta)} y^{\kappa\delta-1} {}_1\Psi_1\left(\begin{matrix} (1, \delta) \\ (\kappa, \kappa\delta) \end{matrix} \middle| -y^\kappa\right).$$

**Proof.** In what follows, we simply write  $\delta$  for  $\alpha\delta_\kappa$  and set  $c, t = 1$ . Thus, by means of the binomial formula, we get, for  $\lambda^\kappa > 1$ ,

$$(1 + \lambda^\kappa)^{-\delta} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\delta + n)}{n! \Gamma(\delta)} \lambda^{\kappa(-n-\delta)}.$$

The term-by-term inversion yields

$$p^{(\frac{\kappa\delta}{\alpha})}(0, y) = \frac{1}{\Gamma(\delta)} y^{\kappa\delta-1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + \delta)}{n! \Gamma(\kappa(n + \delta))} y^{\kappa n}.$$

Next, we have

$$\begin{aligned} e^{-\lambda x (1 + \lambda^\kappa)^{-\frac{1}{\kappa}}} (1 + \lambda^\kappa)^{-\delta} &= \sum_{n=0}^{\infty} (-1)^n (1 + \lambda^\kappa)^{-\left(\frac{n}{\kappa} + \delta\right)} \frac{1}{n!} \lambda^n x^n \\ &= \sum_{n=0}^{\infty} (-1)^n (1 + \lambda^{-\kappa})^{-\left(\frac{n}{\kappa} + \delta\right)} \frac{1}{n!} \lambda^{-\kappa\delta} x^n. \end{aligned}$$

Once again inverting term and term and using the previous result, we deduce that

$$p^{(\frac{\kappa\delta}{\alpha})}(x, y) = \sum_{n=0}^{\infty} (-1)^n F^n(y) \frac{1}{n!} x^n$$

where the term  $F^n(y)$  is given by

$$\begin{aligned} F^n(y) &= \frac{y^{\kappa\delta-1}}{\Gamma\left(\frac{n}{\kappa} + \delta\right)} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(r + \frac{n}{\kappa} + \delta)}{r! \Gamma(\kappa(r + \delta))} y^{\kappa r} \\ &= \frac{y^{\kappa\delta-1}}{\Gamma\left(\frac{n}{\kappa} + \delta\right)} {}_1\Psi_1 \left( \begin{matrix} (1, \frac{n}{\kappa} + \delta) \\ (\kappa, \kappa\delta) \end{matrix} \middle| -y^\kappa \right). \end{aligned}$$

Thus, by putting pieces together we get

$$p^{(\frac{\kappa\delta}{\alpha})}(x, y) = y^{\kappa\delta-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma\left(\frac{n}{\kappa} + \delta\right)} {}_1\Psi_1 \left( \begin{matrix} (1, \frac{n}{\kappa} + \delta) \\ (\kappa, \kappa\delta) \end{matrix} \middle| -y^\kappa \right) x^n$$

which completes the proof.  $\square$

Note that in the case  $\kappa = 1$ , we get

$$p^{(\frac{\delta}{\alpha})}(x, y) = y^{\delta-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \delta)} {}_1\Psi_1 \left( \begin{matrix} (1, n + \delta) \\ (1, \delta) \end{matrix} \middle| -y^\kappa \right) x^n.$$

To recover the well-known expression of the density of the semi-group of a Bessel squared process, see e.g. [8, p.136], we used the fact that for  $\kappa = 1$ , we have

$$\Lambda_t^{(\delta)}(\lambda, x) = (1 + ct\lambda)^{-2\delta} e^{-\frac{x}{2t}((1+2t\lambda)^{-1}-1)}$$

and used the inversion techniques, as above, to get

$$\begin{aligned} p_{\frac{t}{2}}^{(\frac{\delta}{2})}(x, y) &= \left(\frac{y}{t}\right)^{\delta-1} t^{-1} e^{-\frac{x+y}{t}} \sum_{n=0}^{\infty} \frac{\left(\frac{xy}{t^2}\right)^n}{n! \Gamma(n + \delta)} \\ &= \left(\frac{y}{xt}\right)^{\frac{\delta-1}{2}} t^{-1} e^{-\frac{x+y}{t}} I_{\delta-1} \left(\frac{2\sqrt{xy}}{t}\right), \end{aligned}$$

where we recall that

$$I_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)}$$

stands for the modified Bessel function of the first kind of index  $\nu$ , see e.g. [23].

We proceed by characterizing the Lévy processes associated to  $(Y, \mathbb{Q}^{(\delta)})$  via the Lamperti mapping.

**Proposition 4.11.** *For any  $\delta \geq 0$ , the Lévy process associated to the  $\kappa$ -sspMp  $(Y, \mathbb{Q}^{(\delta)})$  is the Lévy process  $(-\xi, \mathbb{P}^{(0, \delta)})$ .*

**Proof.** Let us first consider the case  $\delta = 1$ . Lambert [19] showed that  $(Y, \mathbb{Q}^{(1)})$  corresponds to the branching process  $(Y, \mathbb{Q}^{(0)})$  conditioned to never extinct, which is simply the  $h$ -transform in the Doob's sense, with  $h(x) = x$ , of the minimal process  $(Y, \mathbb{Q}^{(0)})$ . Let us now compute the infinitesimal generator of  $(Y, \mathbb{Q}^{(1)})$ , denoted by  $\mathbf{Q}_+^{(1)}$ . To this end, let us recall that since the process  $X$  does not have negative jumps and has a finite mean, its infinitesimal generator, denoted by  $\mathbf{Q}_+^{\dagger}$ , is given, see also [10], for a smooth function  $f$  on  $\mathbb{R}^+$  with  $f(0) = 0$  and any  $x > 0$ , by

$$\begin{aligned} \mathbf{Q}_+^{\dagger} f(x) &= \int_0^{\infty} (f(x+y) - f(x) - yf'(x)) \frac{c_-}{y^{\alpha+1}} dy \\ &= x^{-\alpha} \int_1^{\infty} (f(xu) - f(x) - xf'(x)(u-1)) \frac{c_-}{(u-1)^{\alpha+1}} du \end{aligned}$$

where we have performed the change of variable  $y = x(u-1)$ . Thus, by a formula of Volkonskii, see e.g. Rogers and Williams [36, III.21], we deduce that, for a function  $f$  as above and any  $x > 0$ ,

$$\begin{aligned} \mathbf{Q}_+^{(0)} f(x) &= x \mathbf{Q}_+^{\dagger} f(x) \\ &= x \int_0^{\infty} (f(x+y) - f(x) - yf'(x)) \frac{c_-}{y^{\alpha+1}} dy \\ &= x^{1-\alpha} \int_1^{\infty} (f(xu) - f(x) - xf'(x)(u-1)) \frac{c_-}{(u-1)^{\alpha+1}} du. \end{aligned}$$

Recalling that for any  $x > 0$ ,  $\mathbf{Q}_+^{(0)}h(x) = 0$  with  $h(x) = x$  we get, by  $h$ -transform and for a smooth function  $f$  on  $\mathbb{R}^+$ , that

$$\begin{aligned}\mathbf{Q}_+^{(1)}f(x) &= \frac{1}{h(x)}\mathbf{Q}_+^{(0)}(hf)(x) \\ &= x^{1-\alpha} \left( \mathbf{Q}_+^{(0)}f(x) + \int_0^\infty (f(x+y) - f(x)) \frac{c_-}{y^\alpha} dy \right) \\ &= x^{1-\alpha} \left( \mathbf{Q}_+^{(0)}f(x) + \int_1^\infty (f(ux) - f(x)) \frac{c_-}{(u-1)^\alpha} du \right) \\ &= x^{1-\alpha} \left( \mathbf{Q}_+^{(0)}f(x) + \int_0^1 \left( f\left(\frac{x}{u}\right) - f(x) \right) \frac{c_- u^{\alpha-2}}{(1-u)^\alpha} du \right).\end{aligned}$$

We have already shown, see Lemma 4.3, that the Lévy process associated via the Lamperti mapping to  $(Y, \mathbb{Q}_+^{(0)})$  is  $(-\xi, \mathbb{P}^{(0)})$ . Next, consider the function  $p_\lambda(x) = x^\lambda$ , with  $\lambda < 0$  and  $x > 0$ , and note that  $\mathbf{Q}_+^{(1)}p_\lambda(x) = x^{\lambda-\alpha}\psi(-\lambda)$ . Thus, using the integral (2.3) we obtain that

$$\begin{aligned}\int_0^1 \left( u^{-\lambda} - 1 \right) \frac{c_\alpha u^{\alpha-2}}{(1-u)^\alpha} du &= (-\lambda)_\alpha \frac{c_\alpha \Gamma(1-\alpha)}{-\lambda + \kappa} \\ &= -c(-\lambda)_\alpha \frac{1}{-\lambda + \kappa}\end{aligned}$$

Using the recurrence formula of the Gamma function, we deduce that the image, via the Lamperti mapping, of  $(Y, \mathbb{Q}_+^{(1)})$  is  $(-\xi, \mathbb{P}^{(0,1)})$ . The general case is deduced from the previous one by recalling that for any  $\delta, x, \lambda > 0$ , and writing  $e_\lambda(x) = e^{-\lambda x}$ ,  $\lambda \geq 0$ , we have, see e.g. [24], for any  $x > 0$ ,

$$\mathbf{Q}_+^{(\delta)}e_\lambda(x) = -e_\lambda(x)(x\varphi(\lambda) + \delta\chi(\lambda))$$

where  $\mathbf{Q}_+^{(\delta)}$  stands for the infinitesimal generator of  $(Y, \mathbb{Q}_+^{(\delta)})$ . □

**Remark 4.12.** We observe that the process  $(-\xi, \mathbb{P}^{(0,1)})$  is equivalent to  $(-\xi, \mathbb{P}^{(-1)})$ . This is not really surprising since as mentioned in the proof, the process  $(Y, \mathbb{Q}_+^{(1)})$  is obtained from  $(Y, \mathbb{Q}_+^{(0)})$  by  $h$ -transform with  $h(x) = x$ . The corresponding Lévy process is thus the  $\theta = 1$ -Esscher transform of  $(-\xi, \mathbb{P}^{(0)})$  which is  $(-\xi, \mathbb{P}^{(-1)})$ .

The Theorem is proved by putting pieces together and using the following easy result.

**Lemma 4.13.** *For any  $v > 0$  and  $\delta > \frac{\kappa}{\alpha}$ , we have*

$$f_\infty^{(\delta)}(v) = |\mathbb{E}^{(0,\delta)}[\xi_1]| v^{-\frac{1}{\kappa}} p_1^{(\delta)}(0, v^{-\frac{1}{\kappa}}).$$

**Proof.** In [32, Lemma 3.2], the following identity is proved

$$\mathbb{E}^{(0,\delta)} [e^{-qy^\kappa \Sigma_\infty}] = |\mathbb{E}^{(0,\delta)}[\xi_1]| \int_0^\infty e^{-qt} p_1^{(\delta)}(0, yt^{-\frac{1}{\kappa}}) y^{1-\kappa} dy.$$

Performing the change of variable  $t = y^\alpha v$ , the proof is completed by invoking the injectivity of the Laplace transform. □

## 5. Some concluding remarks

5.1. **On the family**  $(Y, \mathbb{Q}^\delta)_{\delta \geq 0}$ . Recalling that for  $\delta > \frac{\kappa}{\alpha}$  we have  $\psi^{(0, \delta)}(\theta) = 0$  where  $\theta = \alpha\delta - \kappa$ . We deduce readily, from Rivero [35], the behavior of  $(Y, \mathbb{Q}^\delta)$  at the boundary point 0.

**Proposition 5.1.** (1) For  $\delta \geq \frac{\kappa}{\alpha}$ , 0 is unattainable a.s.  
(2) For  $\delta < \frac{\kappa}{\alpha}$ , 0 is reached a.s.. Moreover, if  $0 < \delta < \frac{\kappa}{\alpha}$ , the boundary 0 is recurrent and reflecting, i.e. there exists an unique recurrent extension of the minimal process which hits and leaves 0 continuously a.s. and which is  $\kappa$ -self-similar on  $[0, \infty)$ .  
(3) For  $\delta = 0$ , the point 0 is a trap.

Next, we provide the Laplace transform of the first passage time below for the continuous state branching processes  $(Y, \mathbb{Q}^\delta)$ ,  $\delta = 0, 1$ . That is for the stopping time

$$T_a^Y = \inf\{s \geq 0; Y_s = a\}.$$

**Proposition 5.2.** Let  $x \geq a \geq 0$ , we have, for any  $q \geq 0$ ,

$$\begin{aligned} \mathbf{E}_x^{(0)} \left[ e^{-qT_a^Y} \right] &= \frac{\widehat{\mathcal{I}}(q^{\frac{1}{\kappa}} c \kappa x)}{\widehat{\mathcal{I}}(q^{\frac{1}{\kappa}} c \kappa a)}, \\ \mathbf{E}_x^{(1)} \left[ e^{-qT_a^Y} \mathbb{I}_{\{T_a^Y < \infty\}} \right] &= \frac{a \widehat{\mathcal{I}}(q^{\frac{1}{\kappa}} c \kappa x)}{x \widehat{\mathcal{I}}(q^{\frac{1}{\kappa}} c \kappa a)} \end{aligned}$$

where

$$\begin{aligned} \widehat{\mathcal{I}}(x) &= x \int_0^\infty e^{-t - xt^{-\frac{1}{\kappa}}} t^{-\frac{1}{\kappa} - 1} dt \\ &= \sum_{n=0}^\infty (-1)^n \frac{\Gamma(-\frac{n}{\kappa} + 1)}{n!} x^n + \kappa x^{-\kappa} {}_1\Psi_0 \left( (\kappa, \kappa) \mid -x^{-\kappa} \right). \end{aligned}$$

**Proof.** First, from (4.2), we deduce readily the identity

$$\mathbf{E}_x^{(0)} \left[ e^{-qT_0^Y} \right] = \widehat{\mathcal{I}}(q^{\frac{1}{\kappa}} c \kappa x), \quad q \geq 0.$$

Thus, the first claim is obtained by an application of the strong Markov property and using the fact that  $(Y, \mathbb{Q}^{(0)})$  has no negative jumps. The second identity follows readily from the first one by means of  $h$ -transform and the optional stopping theorem. To get the expression of the integral as a power series we follow a line of reasoning similar to Neretin [30]. First, consider the space  $L^2(\mathbb{R}^+, \frac{dx}{x})$  and denote by  $I_{\varrho, h}(x, v)$  for  $\varrho < 0$  and  $\Re(x), \Re(v), \Re(h) > 0$  the inner product of the functions  $e_x^\varrho(t) = e^{-xt^\varrho}$  and  $e_{v, h}(t) = t^h e^{-vt}$ , i.e.

$$I_{\varrho, h}(x, v) = \int_0^\infty e^{-vt - xt^\varrho} t^{h-1} dt.$$

Then, the Mellin transform of  $e_x^\varrho$  is

$$\begin{aligned}\tilde{e}_x^\varrho(\lambda) &= \int_0^\infty e^{-xt^\varrho} t^{\lambda-1} dt \\ &= \frac{\operatorname{sgn}(\varrho)}{\varrho} \int_0^\infty e^{-xu} u^{\lambda/\varrho-1} du \\ &= \frac{\operatorname{sgn}(\varrho)\Gamma(\lambda/\varrho)}{\varrho x^{\lambda/\varrho}}.\end{aligned}$$

While the Mellin transform of  $e_x^{v,h}$  is

$$\tilde{e}_x^{v,h}(\lambda) = v^{-h+\lambda}\Gamma(h).$$

By the Plancherel formula for the Mellin transform, we have

$$\int_0^\infty e_x^\varrho \overline{e_x^{v,h}} \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{e}_x^{v,h}(i\lambda) \overline{\tilde{e}_x^\varrho(i\lambda)} d\lambda$$

that is

$$I_{\varrho,h}(x,v) = \frac{\operatorname{sgn}(\varrho)}{2\pi\varrho v^h} \int_{-\infty}^\infty \Gamma(i\lambda/\varrho)\Gamma(h+i\lambda) u^{-i\lambda/\varrho} v^{i\lambda} d\lambda.$$

Then we perform the change of variable  $z = i\lambda/\varrho$  and consider an arbitrary contour  $\mathfrak{C}$  coinciding with the imaginary axis near  $\pm\infty$  and leaving all the poles of the integrand of the left side. We get the series expansions by summing the residues and by choosing  $h = v = 1$  and  $\varrho = -\frac{1}{\kappa}$ .  $\square$

We proceed by characterizing the class of sspMp which enjoy the infinite decomposability property introduced by Shiga and Watanabe [39]. More specifically, let  $\mathfrak{D}_M^+$  the Skorohod space of nonnegative valued homogeneous Markov process with càdlàg paths. Let  $(\mathbb{Q}_x)_{x \geq 0}$ ,  $(\mathbb{Q}_x^1)_{x \geq 0}$  and  $(\mathbb{Q}_x^2)_{x > 0}$  be three systems of probability measures defined on  $(\mathfrak{D}_M^+, \mathfrak{B}(\mathfrak{D}_M^+))$ , where  $\mathfrak{B}(\mathfrak{D}_M^+)$  be the  $\sigma$ -field on  $\mathfrak{D}_M^+$  generated by the Borel cylinder sets. Then, define

$$\mathbb{Q} = \mathbb{Q}^1 * \mathbb{Q}^2$$

if and only if for every  $x, y \geq 0$ ,  $\mathbb{Q}_{x+y} = \Phi(\mathbb{Q}_x^1 \times \mathbb{Q}_x^2)$  where  $\Phi$  is the mapping  $\mathfrak{D}_M^+ \times \mathfrak{D}_M^+ \rightarrow \mathfrak{D}_M^+$  defined by

$$\Phi(x_1, x_2) = x_1 + x_2.$$

A positive valued Markov process with law  $\mathbb{Q}$  is infinitely decomposable if for every  $n \geq 1$  there exists a  $\mathbb{Q}^n$  such that

$$(5.1) \quad \mathbb{Q} = \underbrace{\mathbb{Q}^n * \dots * \mathbb{Q}^n}_n.$$

They showed that there is one to one mapping between cbip and conservative Markov processes having the property (5.1). Thus, from the propositions 4.11 and 5.1 we get the following result.

**Corollary 5.3.** *There is one to one mapping between the family of sspMp satisfying the Shiga-Watanabe infinite decomposability property (5.1) and the family  $(Y, \mathbb{Q}^{(\delta)})_{\delta>0}$ , i.e. the family of spectrally positive sspMp associated, via the Lamperti mapping, to the family of Lévy processes  $(-\xi, \mathbb{P}^{(0,\delta)})_{\delta>0}$ .*

Finally, we provide an extension of the previous result the  $\kappa$ -sspMp Ornstein-Uhlenbeck processes. More specifically, for any  $\eta \in \mathbb{R}$ , let us introduce the family of laws  $(\mathbb{Q}^{\eta,(\delta)})_{\delta \geq 0}$  of self-similar Ornstein-Uhlenbeck processes associated to  $(Y, \mathbb{Q}^{(\delta)})_{\delta \geq 0}$  by the following time-space transform, for any  $t \geq 0$  and  $\delta \geq 0$ ,

$$\left( Y_t, \mathbb{Q}^{\eta,(\delta)} \right) = \left( e^{-\eta t} Y_{\tau_\eta(t)}, \mathbb{Q}^{(\delta)} \right),$$

where

$$\tau_\eta(t) = \frac{1 - e^{-\eta \kappa t}}{\eta \kappa}.$$

For any  $\eta \in \mathbb{R}$ ,  $(Y, \mathbb{Q}^{\eta,(\delta)})$  is an homogenous Markov process and for  $\eta, \delta > 0$ , it has an unique stationary measure which is the entrance law of  $(Y, \mathbb{Q}^{(\delta)})$ , see e.g. [33]. Moreover, its semi-group is characterized by its Laplace transform as follows

$$\mathbf{E}_x^{\eta,(\delta)} \left[ e^{-\lambda U_t} \right] = (1 + c\tau_\eta(t)\lambda^\kappa)^{-\delta/\kappa} e^{-x e^{-\eta t} \lambda(1 + c\tau_\eta(t)\lambda^\kappa)^{-1/\kappa}}.$$

It is easily shown that the infinitesimal generator of  $(Y, \mathbb{Q}^{\eta,(\delta)})$  has the following form, for a smooth function  $f$  on  $\mathbb{R}^+$ ,

$$\mathbf{Q}_+^{\eta,(\delta)} f(x) = \mathbf{Q}_+^{(\delta)} f(x) - \eta x f'(x), \quad x > 0.$$

Hence we deduce from the identity

$$\mathbf{Q}_+^{(\delta)} e_\lambda(x) = -e_\lambda(x) (x\varphi(\lambda) + \delta\chi(\lambda)),$$

that  $(Y, \mathbb{Q}^{\eta,(\delta)})$  is a cbip with branching mechanism  $\varphi_\eta(\lambda) = \varphi\lambda - \eta\lambda$  and immigration  $\chi(\lambda)$ . Its semi-group is absolutely continuous with a density denoted  $p_t^{(\delta,\eta)}(x, y)$  and given, for any  $x, y, t > 0$ , by

$$p_t^{(\delta,\eta)}(x, y) = e^{-\kappa\eta t} p_{\tau_\eta(t)}^{(\delta)}(x, e^{-\kappa\eta t} y).$$

**5.2. Representations of some  ${}_p\Psi_q$  functions.** Let us recall that, for  $\frac{\kappa}{\alpha} < \delta < \frac{2\kappa}{\alpha}$ , the Laplace transform of  $(\Sigma_\infty, \mathbb{P}^{(0,\delta)})$  has been computed by Patie [32] as follows, for any  $x \geq 0$ ,

$$(5.2) \quad \mathbb{E}^{(0,\delta)} \left[ e^{-x\Sigma_\infty} \right] = \mathcal{N}_{\kappa,\delta}(x)$$

where, by setting  $0 < m_\kappa = 2 - \alpha\delta_\kappa < 1$ ,

$$\mathcal{N}_{\kappa,\delta}(x) = \mathcal{I}_{\kappa,\delta}(x) - C_{m_\kappa} x^{\frac{\alpha\delta}{\kappa}-1} \mathcal{I}_{\kappa,\delta,m_\kappa}(x), \quad x \geq 0,$$

and

$$\begin{aligned}
\mathcal{I}_{\kappa,\delta}(c\kappa x) &= \Gamma(m_\kappa)\Gamma(\kappa) \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+m_\kappa)\Gamma(\kappa(n+1))} \\
&= \Gamma(m_\kappa)\Gamma(\kappa) {}_1\Psi_2 \left( \begin{matrix} (1,1) \\ (1,m_\kappa)(\kappa,\kappa) \end{matrix} \middle| x \right), \\
\mathcal{I}_{\kappa,\delta,m_\kappa}(c\kappa x) &= \Gamma(\alpha\delta) \sum_{n=0}^{\infty} \frac{x^n}{n!\Gamma(\kappa n + \alpha\delta)} \\
&= \Gamma(\alpha\delta) {}_0\Psi_1 \left( (\kappa, \alpha\delta) \middle| x \right),
\end{aligned}$$

and where  $C_{m_\kappa}$  is determined by

$$(5.3) \quad \mathcal{I}_{\kappa,\delta}(x) \sim C_{m_\kappa} x^{\frac{\alpha\delta}{\kappa}-1} \mathcal{I}_{\kappa,\delta,m_\kappa}(x) \quad \text{as } x \rightarrow \infty.$$

Using the exponentially infinite asymptotic expansions (6.2) and (6.3), we deduce that

$$C_{m_\kappa} = \frac{\Gamma(m_\kappa)\Gamma(\kappa)}{\Gamma(\alpha\delta)}.$$

As a consequence of Theorem 4.6 we have these representations of the function  $\mathcal{N}_{\kappa,\delta}$ .

**Corollary 5.4.** *For  $\frac{\kappa}{\alpha} < \delta < \frac{2\kappa}{\alpha}$  and  $x \geq 0$ , the following representation*

$$\mathcal{N}_{\kappa,\delta}(x) = \frac{M_\delta}{\Gamma(\alpha\delta_\kappa)} \int_0^\infty \frac{u^{\alpha\delta_\kappa-1}}{x+u} {}_0\Psi_1 \left( (\kappa, \alpha\delta) \middle| -u \right) du$$

*holds. Finally, we have the following asymptotic expansion for large  $x$ ,*

$$\mathcal{N}_{\kappa,\delta}(x) \sim \exp\left(-(\kappa^{-\kappa}\alpha^\alpha x)^{\frac{1}{\alpha}}\right) x^{-\frac{\delta}{\kappa}-\frac{1}{\alpha}+1}.$$

**Proof.** Note that, for any  $\delta > \frac{\kappa}{\alpha}$  and  $y > 0$ , we have

$$\begin{aligned}
f_\infty^{(\delta)}(y) &= \frac{M_\delta}{\Gamma(\alpha\delta_\kappa)} \int_0^\infty e^{-uy} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\kappa n + \alpha\delta)} u^{n+\alpha\delta_\kappa-1} du \\
&= \frac{M_\delta}{\Gamma(\alpha\delta_\kappa)} \int_0^\infty e^{-uy} u^{\alpha\delta_\kappa-1} {}_0\Psi_1 \left( (\kappa, \alpha\delta) \middle| -u \right) du
\end{aligned}$$

where we have used the analyticity of the mapping  $f_\infty^{(\delta)}$  and an argument of dominated convergence. The first representation is obtained by taking the Laplace transform on both sides on the previous equation and using (5.2). Finally, the asymptotic behavior of the function  $\mathcal{N}_{\kappa,\delta}$  is deduced from the exponentially infinite asymptotic expansions (6.2) and (6.3).  $\square$

## 6. Asymptotic expansions of the Wright hypergeometric functions

Special classes of the Wright hypergeometric functions have been considered among others by Mittag-Leffler [29], Barnes [1], Fox [14] while the general case  ${}_p\Psi_q$  has been considered by Wright [40]. We refer to Braaksma [9, Chap. 12] for a detailed account of

this function and its relation to the  $G$ -function. In the sequel, we simply indicate special properties, which can be found in [9, Chap. 12].

We proceed by recalling that the Wright hypergeometric function is defined as

$${}_p\Psi_q \left( \begin{matrix} (A_1, a_1), \dots, (A_p, a_p) \\ (B_1, b_1), \dots, (B_q, b_q) \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(A_i n + a_i)}{\prod_{i=1}^q \Gamma(B_i n + b_i)} \frac{z^n}{n!}$$

where  $p, q$  are nonnegative integers,  $a_i \in \mathbb{C}$  ( $i = 1 \dots p$ ),  $b_j \in \mathbb{C}$  ( $j = 1 \dots q$ ), the coefficients  $A_i \in \mathbb{R}^+$  ( $i = 1 \dots p$ ) and  $B_j \in \mathbb{R}^+$  ( $j = 1 \dots q$ ) are such that

$$A_i n + a_i \neq 0, -1, -2, \dots \quad (i = 0, 1, \dots, p; n = 0, 1, \dots).$$

In what follows, we will also use the number  $S$  and  $T$  defined respectively by

$$\begin{aligned} S &= 1 + \sum_{i=1}^q B_i - \sum_{i=1}^p A_i \\ T &= \prod_{i=1}^p A_i^{A_i} \prod_{i=1}^q B_i^{-B_i}. \end{aligned}$$

Throughout this part we assume that  $S$  is positive. In such a case, the series is convergent for all values of  $z$  and it defines an integral function of  $z$  (the case  $S = 0$  is also treated in [9]). Next, for  $S$  positive, the function  ${}_p\Psi_q$  admits a contour integral representation. More precisely, we have

$${}_p\Psi_q \left( \begin{matrix} (A_1, a_1), \dots, (A_p, a_p) \\ (B_1, b_1), \dots, (B_q, b_q) \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\prod_{i=1}^p \Gamma(A_i n + a_i)}{\prod_{i=1}^q \Gamma(B_i n + b_i)} \Gamma(-s) (-s)^z ds$$

where  $\mathfrak{C}$  is a contour in the complex  $s$ -plane which runs from  $s = a - i\infty$  to  $s = a + i\infty$  ( $a$  an arbitrary real number) so that the points  $s = 0, 1, 2 \dots$  and  $s = -\frac{a_i + n}{A_i}$ , ( $i = 0, 1, \dots, p; n = 0, 1, \dots$ ) lie to the right left of  $\mathfrak{C}$ . Next we introduce the following functions

$$\begin{aligned} P(z) &= \sum_{s \in R_p} z^s \Gamma(-s) \operatorname{Res} \left( \frac{\prod_{i=1}^p \Gamma(A_i s + a_i)}{\prod_{i=1}^q \Gamma(B_i s + b_i)} \right) \\ E(z) &= \frac{\exp\left((TS^S z)^{\frac{1}{S}}\right)}{S} \sum_{k=0}^{\infty} H_k (TS^S z)^{\frac{1-G-k}{S}} \end{aligned}$$

where  $\operatorname{Res}$  stands for residuum, we set  $R_p = \{r_{i,n} = -\frac{a_i + n}{A_i}, i = 0, 1, \dots, p; n = 0, 1, \dots\}$  and the constant  $G$  is given by

$$G = \sum_{i=1}^q b_i - \sum_{i=1}^p a_i + \frac{p-q}{2} + 1.$$

The coefficients  $(H_k)_{k \geq 0}$  are determined by

$$\frac{\prod_{i=1}^p \Gamma(A_i s + a_i)}{\prod_{i=1}^q \Gamma(B_i s + b_i)} (TS^S)^{-s} \sim \sum_{k=0}^{\infty} \frac{H_k}{\Gamma(k + Ss + G)}.$$

In particular,

$$H_0 = (2\pi)^{\frac{p-q}{2}} S^{G-\frac{1}{2}} \prod_{i=1}^p A_i^{\alpha_i-\frac{1}{2}} \prod_{i=1}^q B_i^{\frac{1}{2}-b_i}.$$

We have the following asymptotic expansions.

(1) Suppose  $S > 0$  and  $p > 0$ . Then, the following algebraic asymptotic expansion

$${}_p\Psi_q(z) \sim P(-z)$$

holds for  $|z| \rightarrow \infty$  uniformly on every closed subsector of

$$|\arg(-z)| < \left(1 - \frac{S}{2}\right) \pi$$

(2) Suppose  $S > 0$ . Then, the following exponentially infinite asymptotic expansion

$${}_p\Psi_q(z) \sim E(z)$$

holds for  $|z| \rightarrow \infty$  uniformly on every closed sector (vertex in 0) contained in  $\arg(z) < \min(S, 2)\frac{\pi}{2}$ .

For the convenience of the reader, we list below the asymptotic expansion corresponding to  ${}_p\Psi_q$  functions which appear in this paper. For  $y \rightarrow \infty$ , recalling that  $0 < m_\kappa = 2 - \alpha\delta_\kappa < 1$ , we have

$$(6.1) \quad {}_1\Psi_1 \left( \begin{matrix} (1, \alpha\delta_\kappa) \\ (\kappa, \alpha\delta) \end{matrix} \middle| -y \right) \sim \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + \alpha\delta_\kappa)}{\Gamma(-\kappa n)} \frac{y^{-\alpha\delta_\kappa - n}}{n!},$$

$$(6.2) \quad {}_1\Psi_2 \left( \begin{matrix} (1, 1) \\ (1, m_\kappa)(\kappa, \kappa) \end{matrix} \middle| y \right) \sim (2\pi\alpha)^{-\frac{1}{2}} \kappa^{\frac{1}{2} - \delta - \frac{\kappa}{2\alpha}} \exp\left(\left(\kappa^{-\kappa} \alpha^\alpha y\right)^{\frac{1}{\alpha}}\right) y^{\frac{\delta}{\kappa} + \frac{1}{\alpha} - 1}$$

$$(6.3) \quad {}_0\Psi_1 \left( (\kappa, \alpha\delta) \middle| y \right) \sim (2\pi\alpha)^{-\frac{1}{2}} \kappa^{\frac{1}{2} - \delta - \frac{\kappa}{2\alpha}} \exp\left(\left(\kappa^{-\kappa} \alpha^\alpha y\right)^{\frac{1}{\alpha}}\right) y^{\frac{1}{2\alpha} - \delta}.$$

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