On the Structure Theorem for quasi-Hopf bimodules

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New trends in Hopf algebras and tensor categories

Royal Flemish Academy of Belgium for Science and the Arts - Brussels

June 2-5, 2015

Report of: P. Saracco, *On the Structure Theorem for Quasi-Hopf Bimodules* (arXiv:1501.06061)

The main aim

Fix $\Bbbk$ a field. We assume to work in the category $\mathfrak{M} := \mathrm{Vect}_{\Bbbk}$ of $\Bbbk$-vector spaces.

Fact

H bialgebra $\Rightarrow$ the category $\mathfrak{M}_H$ of H-modules is monoidal. The category $\mathfrak{M}_H^H$ of Hopf modules is the category of comodules on the H-module coalgebra H: $(\mathfrak{M}_H)^H$.

Our aim is to extend the following result to the framework of quasi-bialgebras.

Theorem

T.F.A.E. for a bialgebra H:

1. *the functor $(-) \otimes H : \mathfrak{M} \to \mathfrak{M}_H^H$ is an equivalence of categories with quasi-inverse $(-)^{\mathrm{co}H} : \mathfrak{M}_H^H \to \mathfrak{M}$, where $M^{\mathrm{co}H} := \{m \in M \mid \rho(m) = m \otimes 1\}$;*
2. *H is a Hopf algebra, i.e. it admits an antipode $s : H \to H$.*

Sketch of proof.

The assignment $[m \mapsto \tau_M(m_0) \otimes m_1]$, where $\tau_M : M \to M^{\mathrm{co}H}, [m \mapsto m_0 \cdot s(m_1)]$, defines the inverse for the counit $\vartheta_M : M^{\mathrm{co}H} \otimes H \to M, [m \otimes h \mapsto m \cdot h]$. The unit is always invertible. □

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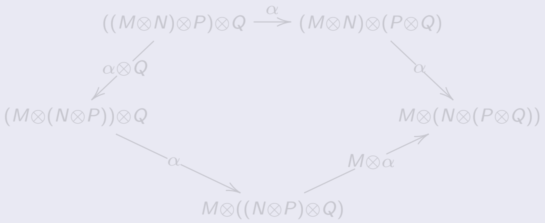
Definition (Benabou/Mac Lane, 1963)

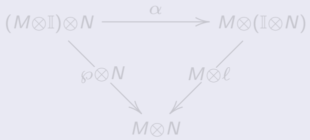
A monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \alpha, \ell, \wp)$ is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (tensor product), an object $\mathbb{I}$ (unit) and 3 natural isomorphisms:

$$\alpha_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P) \quad \text{(associativity constraint)}$$

$$\ell_M : \mathbb{I} \otimes M \to M, \qquad \wp_N : N \otimes \mathbb{I} \to N \qquad \text{(unit constraints)}$$

such that the following diagrams commute (pentagon and triangle axioms):





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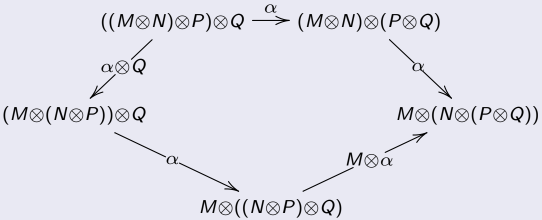

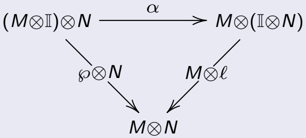

Quasi-bialgebras

Definition (Drinfel'd, [Dr, 1989])

A quasi-bialgebra is a datum $(A, m, u, \Delta, \varepsilon, \Phi)$ where:

1. (A, m, u) is an associative and unital algebra;
2. $\Delta : A \to A \otimes A$ (comultiplication) and $\varepsilon : A \to \Bbbk$ (counit) are algebra maps;
3. $\Phi \in A \otimes A \otimes A$ is an invertible element (reassociator) that satisfies:

$$(A \otimes A \otimes \Delta)(\Phi) \cdot (\Delta \otimes A \otimes A)(\Phi) = (1 \otimes \Phi) \cdot (A \otimes \Delta \otimes A)(\Phi) \cdot (\Phi \otimes 1),$$
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Moreover, ε is a counit for Δ and Δ is quasi-coassociative, i.e.

$$\Phi \cdot ((\Delta \otimes A) \circ \Delta) = ((A \otimes \Delta) \circ \Delta) \cdot \Phi.$$

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$((A, m, m), \Delta, \varepsilon)$ is a coassociative A-bimodule coalgebra. Its category of (right) quasi-Hopf bimodules is the category of A-comodules in ${}_A\mathfrak{M}_A$: ${}_A\mathfrak{M}_A^A := \left({}_A\mathfrak{M}_A\right)^A$.

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An adjunction between ${}_A\mathfrak{M}$ and ${}_A\mathfrak{M}_A^A$

Henceforth, let us fix a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ and denote by $A^+ := \ker(\varepsilon)$ its augmentation ideal.
The subsequent result is contained in the proof of Theorem 3.1 in

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Theorem

Set $\overline{M} := \frac{M}{MA^+} \in {}_A\mathfrak{M}$. We have that the functor $R := (-) \otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ is right adjoint to the functor $L := \overline{(-)} : {}_A\mathfrak{M}_A^A \to {}_A\mathfrak{M}$. Unit and counit are given by:

$$\eta_M : M \to \overline{M} \otimes A, \left[m \mapsto \overline{m_0} \otimes m_1\right] \quad \text{and} \quad \epsilon_N : \overline{N \otimes A} \to N, \left[\overline{n \otimes a} \mapsto n\,\varepsilon(a)\right]$$

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Main question: When is R an equivalence of categories?

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Answering the main question (I)

Consider the quasi-Hopf bimodule $A \widehat{\otimes} A$ with underling vector space $A \otimes A$ and structures given explicitly by:

$$a \cdot (x \otimes y) = x \otimes ay, \qquad (x \otimes y) \cdot a = xa_1 \otimes ya_2,$$
$$\rho(x \otimes y) = \big((x \otimes y_1) \otimes y_2\big) \cdot \Phi$$

The component of the unit associated to $A \widehat{\otimes} A$ satisfies:

$$\widehat{\eta}_A := \eta_{A \widehat{\otimes} A} \colon A \widehat{\otimes} A \to \overline{A \widehat{\otimes} A} \otimes A, \left[a \otimes b \mapsto \overline{a\Phi^1 \otimes b_1 \Phi^2} \otimes b_2 \Phi^3\right]$$

Definition

A preantipode for a quasi-bialgebra (A, Φ) is a linear map $S \colon A \to A$ that satisfies:

(P1) $b_1 S(ab_2) = S(a)\varepsilon(b)$, $\forall\, a, b \in A$; $\overset{a=1}{\rightsquigarrow}$ $b_1 S(b_2) = S(1)\varepsilon(b)$

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Answering the main question (I)

Consider the quasi-Hopf bimodule $A \,\widehat{\otimes}\, A$ with underling vector space $A \otimes A$ and structures given explicitly by:

$$a \cdot (x \otimes y) = x \otimes ay, \qquad (x \otimes y) \cdot a = xa_1 \otimes ya_2,$$
$$\rho(x \otimes y) = \big((x \otimes y_1) \otimes y_2\big) \cdot \Phi$$

The component of the unit associated to $A \,\widehat{\otimes}\, A$ satisfies:

$$\widehat{\eta}_A := \eta_{A \widehat{\otimes} A} \colon A \,\widehat{\otimes}\, A \to \overline{A \,\widehat{\otimes}\, A} \otimes A, \left[a \otimes b \mapsto \overline{a\Phi^1 \otimes b_1\Phi^2} \otimes b_2\Phi^3\right]$$

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Answering the main question (II)

Theorem (Structure Theorem for quasi-Hopf bimodules)

Let $(A, m, u, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. T.F.A.E.:

(i) (L, R, η, ϵ) *is an equivalence of categories;*

(ii) $\widehat{\eta}_A \colon A \widehat{\otimes} A \to \overline{A \widehat{\otimes} A} \otimes A$ *is an isomorphism;*

(iii) *A admits a preantipode;*

(iv) *for every $M \in {}_A\mathfrak{M}_A^A$ there exists a linear map $\widetilde{\tau}_M \colon \overline{M} \to M$ such that*

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Proof.

(i) $\Rightarrow$ (ii) Trivial.

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Coinvariants for quasi-Hopf bimodules

Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode and $M \in {}_A\mathfrak{M}_A^A$.

Definition

The space of coinvariant elements of a $M \in {}_A\mathfrak{M}_A^A$ is $M^{\mathrm{co}A} := \tau_M(M)$.

Proposition

- $M^{\mathrm{co}A}$ is a left A-module via $a \triangleright m := \tau_M(a \cdot m)$, $\forall\, a \in A, m \in M$.
- The map $\widetilde{\tau}_M \colon \overline{M} \xrightarrow{\sim} M^{\mathrm{co}A}$ is an isomorphism in ${}_A\mathfrak{M}$ with inverse map $\sigma_M \colon M^{\mathrm{co}A} \to \overline{M}, \left[m \mapsto \overline{m}\right]$.

Corollary

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Revisiting classical results (I)

Hopf case

Let $(H, m, u, \Delta, \varepsilon)$ be an ordinary bialgebra.

- (H, s) is a Hopf algebra with antipode s if and only if $(H, m, u, \Delta, \varepsilon, \Phi, s)$ is a quasi-bialgebra with preantipode s and reassociator $\Phi = 1 \otimes 1 \otimes 1$.
  One checks that the two maps τ_M coincide for all $M \in \mathfrak{M}_H^H$ and then the inverse to the original counit is given by:

$$\vartheta_M^{-1} \colon m \mapsto \left(\widetilde{\tau}_M \otimes H\right)\left(\eta_M(m)\right) = \tau_M(m_0) \otimes m_1.$$

- If every H-Hopf module satisfies the Fundamental Theorem, then one can verify that for every $M \in {}_H\mathfrak{M}_H^H$

$$\tau_M(m) := \left(M^{\mathrm{co}H} \otimes \varepsilon\right)\left(\vartheta_M^{-1}(m)\right)$$

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In this context, the Structure Theorem for quasi-Hopf bimodules reduces to the classical Fundamental Theorem of Hopf modules.

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Let $(H, m, u, \Delta, \varepsilon)$ be an ordinary bialgebra.

- (H, s) is a Hopf algebra with antipode s if and only if $(H, m, u, \Delta, \varepsilon, \Phi, s)$ is a quasi-bialgebra with preantipode s and reassociator $\Phi = 1 \otimes 1 \otimes 1$.
  One checks that the two maps τ_M coincide for all $M \in \mathfrak{M}_H^H$ and then the inverse to the original counit is given by:

$$\vartheta_M^{-1} \colon m \mapsto (\widetilde{\tau}_M \otimes H)\left(\eta_M(m)\right) = \tau_M(m_0) \otimes m_1.$$

- If every H-Hopf module satisfies the Fundamental Theorem, then one can verify that for every $M \in {}_H\mathfrak{M}_H^H$

$$\tau_M(m) := \left(M^{\mathrm{co}H} \otimes \varepsilon\right)\left(\vartheta_M^{-1}(m)\right)$$

  factors through $\widetilde{\tau}_M : \overline{M} \to M^{\mathrm{co}H}$ and that it satisfies condition (iv) of the Structure Theorem.

In this context, the Structure Theorem for quasi-Hopf bimodules reduces to the classical Fundamental Theorem of Hopf modules.

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In this context, the Structure Theorem for quasi-Hopf bimodules reduces to the classical Fundamental Theorem of Hopf modules.

Revisiting classical results (II)

Definition (Drinfel'd, 1989)

We say that a quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra if it is endowed with an algebra anti-homomorphism $s\colon A \to A$ and two distinguished elements α and β such that:

$$s(a_1)\alpha a_2 = \alpha\,\varepsilon(a) \qquad a_1\beta s(a_2) = \beta\,\varepsilon(a)$$

$$\Phi^1\beta s(\Phi^2)\alpha\Phi^3 = 1 \qquad s(\phi^1)\alpha\phi^2\beta s(\phi^3) = 1$$

The triple (s, α, β) is called quasi-antipode.

Quasi-Hopf case

1. Every quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ admits a preantipode:
$$S(\cdot) := \beta s(\cdot)\alpha.$$
2. If s is invertible, then τ_M coincides with the projection E of Hausser and Nill:
$$\tau(m) = \Phi^1 \cdot m_0 \cdot \beta s(\Phi^2 m_1)\alpha\Phi^3 = \Phi^1 \cdot m_0 \cdot \beta s(s^{-1}(\alpha\Phi^3)\Phi^2 m_1) = E(m).$$

It is then possible to obtain Hausser and Nill's result from our Structure Theorem.

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From preantipodes to quasi-antipodes (I)

It is sometimes possible to produce a quasi-antipode given a preantipode. E.g. we have implicitly seen the case of ordinary bialgebras.

Proposition

If $(A, m, u, \Delta, \varepsilon, \Phi, S)$ is a commutative quasi-bialgebra with preantipode, then A is an Hopf algebra with antipode $s(a) = \Phi^1 S(a\Phi^2)\Phi^3$ and $(A, m, u, \Delta, \varepsilon, \Phi, s, 1, S(1))$ is a quasi-Hopf algebra.

Theorem (Theorem 3.1 in [Sc])

For a finite dimensional quasi-bialgebra $(A, m, u, \Delta, \varepsilon, \Phi)$, T.F.A.E.:

1. *A is a quasi-Hopf algebra.*
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$$\xi\left(\overline{a \otimes b}\right) = aS(b) \quad \text{and} \quad a_1 S(a_2) = \varepsilon(a)S(1).$$

However, *a posteriori*, $\widetilde{\gamma}(\overline{a \otimes b}) = a\beta s(b)$ while $\xi(\overline{a \otimes b}) = a\beta s(b)\alpha$ and α cannot be expected to be invertible in general.

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If ξ is invertible then $\left(\left(a \xrightarrow{s} 1^1 S(a1^2)\right), 1, S(1)\right)$, where $\overline{1^1 \otimes 1^2} = \xi^{-1}(1)$, defines a quasi-antipode (without any hypothesis on the dimension of A).

Corollary

If $(A, m, u, \Delta, \varepsilon, \Phi, s, \alpha, \beta)$ is a finite dimensional quasi-Hopf algebra and α is invertible, then we can recover explicitly the quasi-antipode from the preantipode.

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Concluding example

Example (Preliminaries 2.3 in [EG])

Let $C_2 = \langle g \rangle$ be the cyclic group of order 2 and let $H(2) := \Bbbk C_2$ be its group algebra ($\mathrm{char}(\Bbbk) \neq 2$):

$$m(p \otimes q) = p \cdot q, \quad u(1_\Bbbk) = 1_{C_2}, \quad \Delta(p) = p \otimes p, \quad \varepsilon(p) = 1_\Bbbk \quad (\forall\, p, q \in C_2\,).$$

Let us consider the non trivial reassociator:

$$\Phi := (1 \otimes 1 \otimes 1) - 2(\lambda \otimes \lambda \otimes \lambda) \quad \text{where} \quad \lambda := \frac{1}{2}(1 - g).$$

One can verify that $(H(2), m, u, \Delta, \varepsilon, \Phi, \mathrm{Id}_{H(2)}, g, 1)$ is a quasi-Hopf algebra. Therefore $S\colon H(2) \to H(2), [\, z \mapsto z \cdot g \,]$ provides a preantipode for $H(2)$ and

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Work in progress

Missing example

A quasi-bialgebra with preantipode that is not a quasi-Hopf algebra.

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