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An Hopf Algebroid Approach to Jet Spaces and Lie Algebroid Integration

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Outline

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Toward Hopf algebroids: Groups and Hopf Algebras

Fix an algebraically closed field $\mathbb{k}$ of characteristic 0.

Example (Coordinate ring of a group)

Let (G, μ, e, ι) be an (affine) algebraic group and $\mathbb{k}[G]$ its coordinate ring:

- μ induces a map $\Delta : \mathbb{k}[G] \to \mathbb{k}[G \times G] \cong \mathbb{k}[G] \otimes \mathbb{k}[G]$ such that $\Delta(f) = \sum_i g_i \otimes h_i$ iff for all $x, y \in G$;
- e induces $\varepsilon : \mathbb{k}[G] \to \mathbb{k}$ such that $\varepsilon(f) = f(e)$ and
- ι induces $S : \mathbb{k}[G] \to \mathbb{k}[G]$ via pre-composition: $S(f) = f \circ \iota$.

What is a Hopf algebra?

A Hopf algebra is a $\mathbb{k}$-algebra H endowed with 3 additional structure maps

$$H$$

satisfying
$$(\Delta \otimes H)\Delta = (H \otimes \Delta)\Delta \qquad (\varepsilon \otimes H)\Delta = H = (H \otimes \varepsilon)\Delta$$
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Example (Coordinate ring of a group - continued)

For an (affine) algebraic group G, $\mathbb{k}[G]$ is a Hopf algebra. Conversely, if H is a Hopf algebra then $\mathrm{Alg}_{\mathbb{k}}(H, \mathbb{k})$ is a group. The pair of functors

$$\mathsf{AffGrp}_{\mathbb{k}} \underset{\mathsf{Alg}_{\mathbb{k}}(-,\mathbb{k})}{\overset{\mathbb{k}[-]}{\rightleftarrows}} \mathsf{CHopf}_{\mathbb{k}}(+\cdots)$$

defines an anti-equivalence of categories.

Actions of groups on varieties $G \times X \to X$ correspond to coactions of the Hopf algebras $\mathbb{k}[G]$ on $\mathbb{k}[X]$, i.e. $\mathbb{k}[X] \to \mathbb{k}[G] \otimes \mathbb{k}[X]$.

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Example (Hopf algebras of representative functions)

Let G be a group and $\mathcal{R}_{\Bbbk}(G)$ be the algebra of representative functions on G (i.e. the algebra generated by those $f : G \to \Bbbk$ that appear as coefficients of finite-dimensional reprs $\rho : G \to \mathrm{GL}_n(\Bbbk)$). It has the same Hopf algebra structure of the previous example:

- $\Delta : \mathcal{R}_{\Bbbk}(G) \to \mathcal{R}_{\Bbbk}(G) \otimes \mathcal{R}_{\Bbbk}(G)$ induced by μ,
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$$\mathsf{Grp} \underset{\mathsf{Alg}_{\Bbbk}(-,\Bbbk)}{\overset{\mathcal{R}_{\Bbbk}(-)}{\rightleftarrows}} \mathsf{CHopf}_{\Bbbk}$$

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- finite groups and f.d. semisimple commutative Hopf algebras;
- compact real Lie groups and f.g. commutative $\mathbb{R}$-Hopf algebras (plus other properties).

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Example (Universal enveloping Hopf algebras)

Let $\mathfrak{g}$ be a Lie algebra. Its universal enveloping algebra $U(\mathfrak{g})$ is a co-commutative Hopf algebra where, for every $X \in \mathfrak{g}$,

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad \varepsilon(X) = 0 \qquad S(X) = -X.$$

If H is a Hopf algebra, $\mathcal{P}(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$ is the Lie algebra of primitive elements. The pair of functors

$$\mathrm{Lie}_{\Bbbk} \underset{\mathcal{P}(-)}{\overset{U(-)}{\rightleftarrows}} \mathrm{CCHopf}_{\Bbbk}(+\cdots)$$

defines an equivalence of categories.

The finite (or Sweedler) dual

If H is a Hopf algebra, then H^* is not a Hopf algebra in general. Instead,

$$H^\circ := \{f \in H^* \mid \ker(f) \text{ contains a finite-codimensional ideal}\}$$

is a Hopf algebra. The image of $\Bbbk G^\circ \subseteq \Bbbk G^* \cong \mathrm{Fun}(G, \Bbbk)$ is $\mathcal{R}_{\Bbbk}(G)$.

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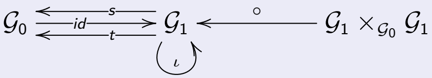
A (abstract) groupoid is a (small) category in which every arrow is invertible, i.e. it is the datum of two sets $\mathcal{G}_0, \mathcal{G}_1$ together with functions



satisfying some reasonable properties.

Example (The fundamental groupoid)

For a topological space X, the sets $\mathcal{G}_0 = X$, $\mathcal{G}_1 = \{\text{paths on } X\}/_{\langle\text{homotopy}\rangle}$ with obvious source, target and composition maps give the fundamental groupoid of X.

Example (The general linear groupoid)

The set $\mathrm{GL}_*(\Bbbk)$ of all invertible matrices of any order over $\Bbbk$ form a groupoid $(\mathbb{N}, \mathrm{GL}_*(\Bbbk))$ with the ordinary matrix multiplication.

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Example (Groupoids and partial actions)

A partial action of a group G on a set X consists of a family $\{X_g \mid g \in G\}$ of subsets of X and a family $\{\alpha_g : X_{g^{-1}} \to X_g \mid g \in G\}$ of bijections such that

- $X_e = X$, $\alpha_e = \mathrm{Id}_X$,
- $\alpha_h^{-1}(X_{g^{-1}} \cap X_h) = X_{(gh)^{-1}} \cap X_{h^{-1}}$,
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Taking $\mathcal{G}_0 = X$, $\mathcal{G}_1 = \{(g, x) \in G \times X \mid x \in X_{g^{-1}}\}$ and

$$(g, x) \cdot (h, y) = \begin{cases} (gh, y) & \text{if } x = \alpha_h(y) \\ \text{---} & \text{otherwise} \end{cases}$$

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Commutative Hopf algebroids

A commutative Hopf algebroid is a groupoid object in the category $\mathsf{Aff}_{\Bbbk}$ or, equivalently, a cogroupoid object in the category $\mathsf{CAlg}_{\Bbbk}$. Thus, it consists of a pair of commutative algebras (A, H) together with structural algebra morphisms

$$A \underset{t}{\overset{s}{\rightrightarrows}} H \xrightarrow{\ \Delta\ } H \otimes_A H, \qquad H \xrightarrow{\varepsilon} A, \qquad \mathcal{S} : H \to H$$

such that

$$(\Delta \otimes_A H)\Delta = (H \otimes_A \Delta)\Delta \qquad (\varepsilon \otimes_A H)\Delta = H = (H \otimes_A \varepsilon)\Delta$$
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Example (Hopf algebroid of pairs)

Let A be a commutative algebra and $H = A \otimes A$. Then

$$s(a) = a \otimes 1 \qquad \Delta(a \otimes b) = (a \otimes 1) \otimes_A (1 \otimes b)$$
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make of $(A, A \otimes A)$ a commutative Hopf algebroid.

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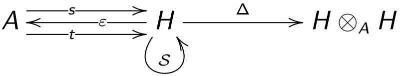

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First application: the integration problem

Lie Groups $\xrightarrow{\text{Differentiation at the identity}}$ Lie Algebras

Lie Groups $\xleftarrow{\text{Lie 3rd Theorem}}$ Lie Algebras

Lie Groupoids $\xrightarrow{\text{Differentiation at the identities}}$ Lie Algebroids

Kapranov [K]: for a (suitable) Lie algebroid $\mathcal{L}$, $\mathcal{V}_A(\mathcal{L})^*$ is a topological bialgebroid whose formal spectrum is a formal groupoid integrating $\mathcal{L}$.

$\mathcal{V}_A(\mathcal{L})$ co-commutative Hopf algd $\rightsquigarrow$ $\mathcal{V}_A(\mathcal{L})^\circ$ commutative Hopf algd

$$\mathcal{V}_A(\mathcal{L})^\circ \qquad ? \qquad \mathcal{V}_A(\mathcal{L})^*$$

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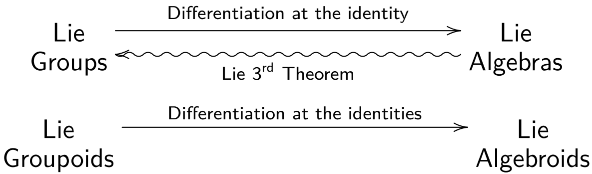
[M] K. Mackenzie, *General theory of Lie groupoids and Lie algebroids*. London Mathematical Society Lecture Note Series, **213**. Cambridge University Press, Cambridge, 2005.



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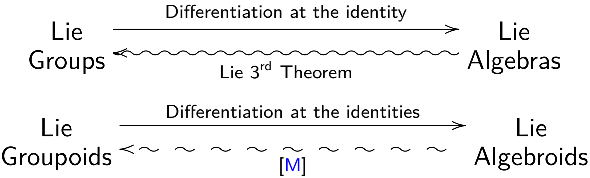
First application: the integration problem



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Lie Groups $\leftsquigarrow$ Lie Algebras (Lie 3rd Theorem)

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Lie Groupoids $\leftsquigarrow$ Lie Algebroids ([M])

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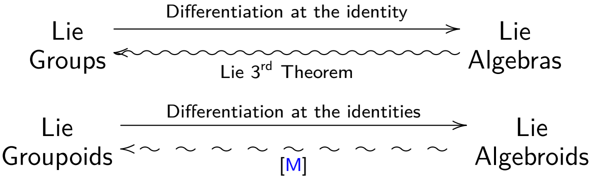

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Henceforth, assume that A is a commutative $\Bbbk$-algebra and denote by $\mathrm{Der}(A)$ the Lie algebra of $\Bbbk$-derivations of A.

A Lie-Rinehart algebra over A is a triple (A, L, ω) where L is a Lie algebra which is also an A-module and $\omega : L \to \mathrm{Der}(A)$ is a Lie algebra map (the anchor) such that for all $a \in A$ and $X, Y \in L$

$$\omega(a \cdot X) = a \cdot X \qquad \text{and} \qquad [X, a \cdot Y] = \omega(X)(a) \cdot Y + a \cdot [X, Y].$$

Example (Lie algebroids)

A Lie algebroid is a vector bundle $\mathcal{L} \to \mathcal{M}$ over a smooth manifold $\mathcal{M}$ with a structure of Lie algebra in the space $\Gamma(\mathcal{L})$ of sections of $\mathcal{L}$ and a morphism of vector bundles $\omega : \mathcal{L} \to T\mathcal{M}$ such that $\Gamma(\omega) : \Gamma(\mathcal{L}) \to \Gamma(T\mathcal{M})$ is a Lie algebra map and

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Universal enveloping algebra and filtered Hopf algebroids

The universal enveloping algebra of a Lie-Rinehart algebra (A, L, ω) is a triple $\big(\mathcal{V}_A(L), \iota_A, \iota_L\big)$ composed by a $\Bbbk$-algebra $\mathcal{V}_A(L)$, an algebra map $\iota_A : A \to \mathcal{V}_A(L)$ and a Lie algebra map $\iota_L : L \to \mathcal{V}_A(L)$ satisfying

$$\iota_L(a \cdot X) = \iota_L(X)\iota_A(a) \quad \text{and} \quad \big[\iota_L(X), \iota_A(a)\big] = \iota_A(\omega(X)(a)), \qquad (\dagger)$$

that enjoys the following universal property:

For any triple (U, ϕ_A, ϕ_L) as above satisfying $(\dagger)$ there exists a unique algebra map $\Phi : \mathcal{V}_A(L) \to U$ such that $\Phi \circ \iota_A = \phi_A$ and $\Phi \circ \iota_L = \phi_L$.

Explicitly, $$\mathcal{V}_A(L) := \frac{T_A(A \otimes L)}{\big\langle\, [\eta(X), \eta(Y)] - \eta([X, Y])\,,\ [\eta(X), a] - \omega(X)(a) \,\big\rangle}$$

with $\iota_A : A \to \mathcal{V}_A(L);\ a \mapsto a$ and $\iota_L : L \to \mathcal{V}_A(L);\ X \mapsto \eta(X) := 1 \otimes X$.

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The $\Bbbk$-algebra $\mathcal{V}_A(L)$ comes endowed with

(HA1) an (injective) $\Bbbk$-algebra map $\iota_A : A \to \mathcal{V}_A(L)$;

(HA2) a (co-commutative) comultiplication Δ and a counit ε given by

$$\varepsilon(\iota_A(a)) = a, \qquad \Delta(\iota_A(a)) = \iota_A(a) \otimes_A 1 = 1 \otimes_A \iota_A(a),$$
$$\varepsilon(\iota_L(X)) = 0, \qquad \Delta(\iota_L(X)) = \iota_L(X) \otimes_A 1 + 1 \otimes_A \iota_L(X),$$

such that $\varepsilon(uv) = \varepsilon(\varepsilon(u)v)$ for all $u, v \in \mathcal{V}_A(L)$ and Δ factors through an A-ring map $\Delta : \mathcal{V}_A(L) \to \mathcal{V}_A(L) \times_A \mathcal{V}_A(L)$;

(HA3) an inverse for the map $\mathfrak{can} : \mathcal{V}_A(L) \otimes_A \mathcal{V}_A(L) \to \mathcal{V}_A(L) \otimes_A \mathcal{V}_A(L)$, $\mathfrak{can}(u \otimes_A v) = (u \otimes_A 1)\Delta(v)$, which is uniquely determined by

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A pair of $\Bbbk$-algebras (A, U) satisfying (HA1) - (HA3) is called a co-commutative (right) Hopf algebroid (Schauenburg [S]).

[S] P. Schauenburg, *Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras)*, New trends in Hopf algebra theory, Contemp. Math., vol. **267**, Amer. Math. Soc., Providence, RI, 2000.

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Even more:

(FHA1) The algebra $\mathcal{V}_A(L)$ carries an exhaustive ascending filtration

$$0 \subset F^0(\mathcal{V}_A(L)) \subset F^1(\mathcal{V}_A(L)) \subset F^2(\mathcal{V}_A(L)) \subset \cdots$$

where $F^0(\mathcal{V}_A(L)) = A$ and $F^p(\mathcal{V}_A(L))$ is the right A-submodule of $\mathcal{V}_A(L)$ generated by products of at most p elements of $\iota_L(L)$. If we assume A to be filtered with the discrete filtration $F^nA = 0$ for all $n < 0$ and $F^nA = A$ for all $n \geq 0$, then the structure maps of $\mathcal{V}_A(L)$ turn out to be filtered. In particular, it does so the translation map

$$\delta : \mathcal{V}_A(L) \to \mathcal{V}_A(L) \otimes_A \mathcal{V}_A(L),\ u \mapsto \mathfrak{can}^{-1}(1 \otimes_A u) := u_- \otimes_A u_+.$$

(FHA2) If L is a finitely generated and projective A-module, then the quotient modules $F^n(\mathcal{V}_A(L))/F^{n-1}(\mathcal{V}_A(L))$ are finitely generated and projective right A-modules as well (e.g. $L = \Gamma(\mathcal{L})$, $\mathcal{L}$ a Lie algebroid).

A co-commutative Hopf algebroid (A, U) satisfying (FHA1) is said to be filtered. If it satisfies (FHA2) as well, then it is said to have an admissible filtration.

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Let (A, U) be a co-commutative Hopf algd with an admissible filtration. Consider its full (right) linear dual $U^* = \mathrm{Hom}_{-,A}(U, A) \cong \varprojlim (F^n(U)^*)$.
Recall that a filtered $\Bbbk$-algebra (R, F_nR) is complete iff $R \cong \varprojlim (R/F_nR)$.

Kapranov [Ka]: U^* inherits a natural decreasing filtration

$$G_0(U^*) = U^* \quad \text{and} \quad G_{n+1}(U^*) = \mathfrak{Ann}(F^n(U)),\ n \geq 0.$$

such that U^* becomes a complete commutative $\Bbbk$-algebra w.r.t. the convolution product. The counit induces

$$\eta = s \otimes t : A \otimes A \to U^*, \quad (a \otimes b \mapsto [u \mapsto \varepsilon(bu)a]).$$

Unit and multiplication of U induce $\varepsilon_* : U^* \to A$ and

$$\Delta_* : U^* \to U^* \widehat{\otimes}_A U^* = \varprojlim \left(\frac{U^* \otimes_A U^*}{\sum_{i+j=n} G_i(U^*) \otimes_A G_j(U^*)} \right).$$

[Ka] M. Kapranov, *Free Lie algebroids and the space of paths*. Sel. Math., New ser. **13** (2007).

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Even more: the translation map δ induces a complete $\Bbbk$-algebra map

$$\mathcal{S} : U^* \to U^*, \quad f \mapsto [u \mapsto \varepsilon(f(u_-)u_+)] \,.$$

Summing up, U^* is a complete commutative algebra with a diagram

$$A \underset{t}{\overset{s}{\rightrightarrows}} U^* \xrightarrow{\Delta_*} U^* \mathbin{\widehat{\otimes}}_A U^*, \qquad \varepsilon_* : U^* \to A, \qquad \mathcal{S} : U^* \to U^* \tag{$\dagger$}$$

of complete algebra maps such that

(CHA1) $(U^*, \Delta_*, \varepsilon_*)$ is a coalgebra in ${}_A\mathrm{Bim}^{\mathsf{c}}_A$;

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A complete Hopf algebroid is of a pair of complete comm algebras (A, H) together with a diagram of algebra maps ($\dagger$) satisfying (CHA1) - (CHA3).

Equivalently, a complete Hopf algebroid is a cogroupoid object in the category of complete commutative algebras (see e.g. [De]).

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Even more: the translation map δ induces a complete $\Bbbk$-algebra map

$$\mathcal{S} : U^* \to U^*, \quad f \mapsto [u \mapsto \varepsilon(f(u_-)u_+)] .$$

Summing up, U^* is a complete commutative algebra with a diagram

$$A \underset{t}{\overset{s}{\rightrightarrows}} U^* \xrightarrow{\Delta_*} U^* \widehat{\otimes}_A U^*, \qquad \varepsilon_* : U^* \to A, \qquad \mathcal{S} : U^* \to U^* \tag{$\dagger$}$$

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Let (A, U) be a co-commutative Hopf algebroid.

El Kaoutit, Gómez-Torrecillas [EG]: The category $\mathcal{A}_U$ of those right U-modules whose underlying A-module is finitely generated and projective is a symmetric rigid monoidal $\Bbbk$-linear category and the forgetful functor $\omega : \mathcal{A}_U \to \mathfrak{proj}(A)$ is a strict monoidal additive faithful functor. As a consequence, the Tannaka reconstruction process provides us for a commutative Hopf algebroid (A, U°) (the finite dual) and a strict monoidal functor $\chi : \mathcal{A}_U \to \mathcal{A}^{U^\circ}$.

Namely, $U^\circ := \dfrac{\bigoplus_{M \in \mathcal{A}_U} M^* \otimes_{T_M} M}{\langle \varphi \otimes_{T_N} f(m) - \varphi \circ f \otimes_{T_M} m \mid \varphi \in N^*, m \in M, f \in T_{M,N} \rangle}$

where $T_{M,N} = \mathrm{Hom}_{\mathcal{A}_U}(M, N)$ and $T_M = T_{M,M}$. Furthermore, there is a canonical $A \otimes A$-algebra map

$$\zeta : U^\circ \to U^*, \ \overline{\varphi \otimes_{T_M} m} \mapsto [u \mapsto \varphi(m \cdot u)]$$

whose injectivity implies that χ is an isomorphism.

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The main morphism of complete Hopf algds

Assume that (A, U) is endowed with an admissible filtration $\{F^n U\}_{n\geq 0}$.

The commutative Hopf algebroid (A, U°) can be filtered with the augmentation filtration $G_0(U^\circ) = U^\circ$ and $G_n(U^\circ) = \ker(\varepsilon_\circ)^n$ and its completion $\left(A, \widehat{U^\circ}\right)$ is a complete Hopf algebroid (A discretely filtered).

Theorem

The canonical map $\zeta : U^\circ \to U^*$ is filtered and hence it can be lifted to a morphism $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ of complete Hopf algebroids such that

$$\begin{array}{ccc} U^\circ & \xrightarrow{\quad\zeta\quad} & U^* \\ & {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle\widehat{\zeta}} & \\ & \widehat{U^\circ} & \end{array}$$

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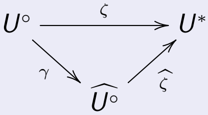

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Idea

If $\mathcal{V}_A(L)^\circ$ is separated and $\widehat{\zeta}$ is an isomorphism, ζ is injective. It follows then that $\widehat{\mathcal{V}_A(L)^\circ}$ can be seen as a formal groupoid which integrates L and that is "canonically" associated with a groupoid whose category of representations is equivalent to the category of modules of L.

Theorem

Let (A, U) be a co-commutative Hopf algebroid with an admissible filtration and assume that $\zeta : U^\circ \to U^*$ is injective. TFAE

(a) $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ *is a filtered isomorphism,*

(b) $\widehat{\zeta}$ *is surjective and the augmentation filtration on U° coincides with the induced one,*

Moreover, the following assertions are equivalent as well

(c) $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ *is an homeomorphism,*

(d) $\widehat{\zeta} : \widehat{U^\circ} \to U^*$ *is open and injective and U° is dense in U^*,*

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Even when ζ is injective and $A = \mathbb{k}$, $\widehat{\zeta}$ may not be an isomorphism.

Example (from [ES])

Let $L = \mathbb{C}X$ be the one dimensional (abelian) complex Lie algebra.

- It is trivially a Lie-Rinehart algebra over $\mathbb{C}$
- Its universal enveloping algebra is the Hopf algebra $\mathbb{C}[X]$
- The finite dual of $\mathbb{C}[X]$ coincides with the usual Sweedler dual $\mathbb{C}[X]^\circ$
- The morphism ζ is the inclusion $\mathbb{C}[X]^\circ \subseteq \mathbb{C}[X]^*$
- Let $\xi \in \mathbb{C}[X]^\circ$ be given by $\xi(X^n) = \delta_{n,1}$ (Kronecker delta). Either the augmentation filtration on $\mathbb{C}[X]^\circ$ and the filtration on $\mathbb{C}[X]^*$ are the $\langle\, \xi \,\rangle$-adic ones

In this case, it turns out that $\widehat{\zeta}$ is surjective but the $\langle \xi \rangle$-adic filtration on $\mathbb{C}[X]^\circ$ is strictly finer then the one induced by $\mathbb{C}[X]^*$, whence $\widehat{\zeta}$ cannot be a filtered isomorphism (in fact, not even an homeomorphism).

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Second application: jet spaces and differential operators

The question

Set $K = \ker(m : A \otimes A \to A)$. The quotients $\mathcal{J}^k(A) = (A \otimes A)/K^{k+1}$ are the modules of k-jets over A and the limit $\mathcal{J}(A) = \varprojlim(\mathcal{J}^k(A))$ is the algebra of infinite jets of A.

The duality $\mathrm{Diff}_k(A) \cong {}^*\mathcal{J}^k(A)$ [K] gives a morphism $\mathcal{J}(A) \to \mathrm{Diff}(A)^*$. Notice that $\mathcal{J}(A)$ is the K-adic completion of $A \otimes A$ and it turns out to be also a complete Hopf algebroid. In some favourable cases, $\mathrm{Diff}(A)$ is a cocommutative Hopf algebroid, so we may consider

$$\begin{array}{ccc} \mathrm{Diff}(A)^\circ & \xrightarrow{\zeta} & \mathrm{Diff}(A)^* \\ \uparrow & & \uparrow \\ A \otimes A & \longrightarrow & \mathcal{J}(A) \end{array}$$

Question: Is $\mathcal{J}(A)$ some kind of completion of $\mathrm{Diff}(A)^\circ$?

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Consequences of the existence of $\widehat{\zeta}$

Idea: In some very favourable cases (e.g. $A = \Bbbk[X_1, \ldots, X_n]$):

- $\mathrm{Diff}(A)$ is a cocommutative Hopf algebroid with an admissible filtration and hence $\mathrm{Diff}(A)^*$ is a complete Hopf algebroid.
- If we endow $A \otimes A$ with the K-adic filtration, then the diagram

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Thank you