ULB UNIVERSITÉ LIBRE DE BRUXELLES

ANTIPODES VS PREANTIPODES

PAOLO SARACCO - UNIVERSITÉ LIBRE DE BRUXELLES

INTRODUCTION

Hopf algebras are nowadays recognized as an important field of study, also because of their extraordinary ubiquity. For example, in (non)commutative geometry they are the natural analogue of groups: the **quantum groups**.

Roughly speaking, they represent spaces with additional structure: a commutative Hopf algebra is an algebra (the space) with an additional comultiplication (the composition law), a counit (the neutral element) and an antipode (the inverses).

Quasi and coquasi-bialgebras (informally, **quasi-quantum monoids**) have been introduced in the 90s by Drinfeld and Majid in connection with Quantum Field Theory. A natural question is:

What could be a "good" analogue of the antipode in this context?

CHARACTERIZING ANTIPODES

Hopf algebras over a field $\Bbbk$ are:

(♗) Bialgebras H whose Hopf modules satisfy the **Structure Theorem**

$$M \cong M^{\mathrm{co}H} \otimes H;$$

(♘) Coalgebras whose finite-dimensional comodules form a **rigid monoidal category** with monoidal underlying functor;

(♖) Algebras whose modules form a (right) **closed monoidal category** with monoidal underlying functor preserving internal homs.

REFERENCES

[1] A. Ardizzoni and A. Pavarin. Preantipodes for dual quasi-bialgebras. *Israel J. Math.*, 2012.

[2] P. Saracco. Coquasi-bialgebras with preantipode and rigid monoidal categories. *arXiv:1611.06819*, 2016.

[3] P. Saracco. On the structure theorem for quasi-hopf bimodules. *Appl. Categ. Structures*, 2017.

(CO)QUASI-BIALGEBRAS AND (CO)QUASI-HOPF ALGEBRAS

A **quasi-bialgebra** is an algebra A with $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \Bbbk$ algebra maps and $\Phi \in A \otimes A \otimes A$ invertible 3-cocycle such that $\Phi \cdot (\Delta \otimes A)(\Delta(a)) = (A \otimes \Delta)(\Delta(a)) \cdot \Phi$ for all $a \in A$.
A **quasi-Hopf algebra** is a quasi-bialgebra A with $s : A \to A$ anti-algebra map and $\alpha, \beta \in A$ such that

$$a_1 \beta s(a_2) = \beta\varepsilon(a), \quad s(a_1)\alpha a_2 = \alpha\varepsilon(a), \quad \Phi^1 \beta s(\Phi^2)\alpha\Phi^3 = 1 \quad \text{for all } a \in A.$$

Dualizing these definitions, one gets those of **coquasi-bialgebras** and **coquasi-Hopf algebras**.

PREANTIPODES

A **preantipode** [3] for a quasi-bialgebra A is a linear endomorphism S such that

$$a_1 S(ba_2) = \varepsilon(a)S(b) = S(a_1 b)a_2, \quad \Phi^1 S(\Phi^2)\Phi^3 = 1.$$

A **preantipode** [1] for a coquasi-bialgebra H is a linear endomorphism S such that

$$S(h_1)_1 h_2 \otimes S(h_1)_2 = 1 \otimes S(h), \quad S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1, \quad \omega(h_1 \otimes S(h_2) \otimes h_3) = \varepsilon(h).$$

WHAT WAS KNOWN

If A is a (co)quasi-Hopf algebra then

(♗) There exists a notion of $M^{\mathrm{co}A}$ for every $M \in {}_A\mathfrak{M}_A^A$ (resp. ${}^A\mathfrak{M}_A^A$) such that

$$M \cong M^{\mathrm{co}A} \otimes A$$

(♘) The category ${}_f\mathfrak{M}_A$ (resp. ${}^A\mathfrak{M}_f$) is rigid:

$$M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk).$$

(♖) The category $\mathfrak{M}_A$ (resp. ${}^A\mathfrak{M}$) is closed:

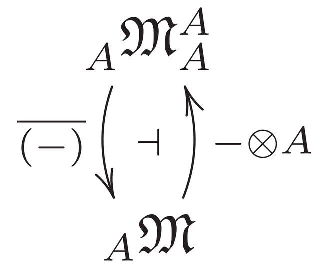
$$\underline{\mathrm{hom}}(M,N) = \mathrm{Hom}_{\Bbbk}(M,N).$$

What about the converses?

THE MAIN RESULTS

Structure theorem for quasi-Hopf modules [3]

Let A be a quasi-bialgebra and ${}_A\mathfrak{M}_A^A := \left({}_A\mathfrak{M}_A\right)^A$.
For M in ${}_A\mathfrak{M}_A^A$, $\overline{M} := \dfrac{M}{MA^+} \in {}_A\mathfrak{M}$.



The following are equivalent

(α) $M \cong \overline{M} \otimes A$ for all M;

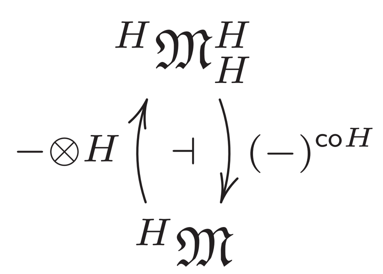
(β) $A \otimes A \cong \dfrac{A \otimes A}{(A \otimes A)A^+} \otimes A$;

(γ) A admits a preantipode.

(♗) holds for quasi-bialgebras with preantipode.

Structure thm for coquasi-Hopf bicomodules [1]

Let H be a coquasi-bialgebra, ${}^H\mathfrak{M}_H^H := \left({}^H\mathfrak{M}^H\right)_H$.
For M in ${}^H\mathfrak{M}_H^H$, $M^{\mathrm{co}H} = \{m \in M \mid \delta(m) = m \otimes 1\}$ is a left comodule.



The following are equivalent

(α') $M \cong M^{\mathrm{co}H} \otimes H$ for all M;

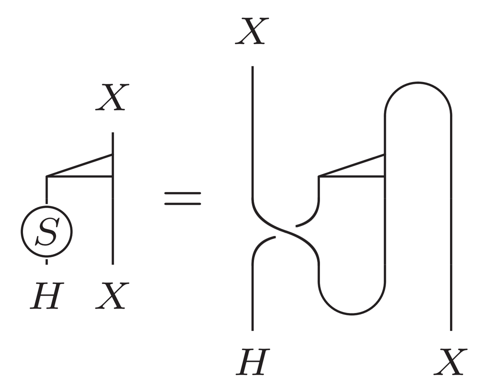
(β') $(H \otimes H)^{\mathrm{co}H} \otimes H \cong H \otimes H$;

(γ') H admits a preantipode.

(♗) holds for coquasi-bialgebras with preantipode.

Reconstruction theorem for coquasi-bialgebras with preantipode [2]

Let $\mathcal{C}$ be a rigid monoidal category with a quasi-monoidal functor $\boldsymbol{\omega} : \mathcal{C} \to \mathfrak{M}_f$. The representing object H of the functor $\mathrm{Nat}(\boldsymbol{\omega}, - \otimes \boldsymbol{\omega})$ is a (universal) coquasi-bialgebra with preantipode:



If $\boldsymbol{\omega}(X^*) \cong \boldsymbol{\omega}(X)^*$ then H is coquasi-Hopf. The reverse constructions hold as well.

(♘) holds for coquasi-bialgebras with preantipode.

OPEN QUESTIONS

1) ${}_A\mathfrak{M}$ is right closed with internal hom ${}_A\mathrm{Hom}(A \otimes N, -)$. Can quasi-bialgebras with preantipode be characterized via an analogue of (♖)?

2) The functor $- \otimes A$ is part of an adjoint triple with right adjoint ${}_A\mathrm{Hom}_A^A(A \otimes A, -)$. What can we say if $- \otimes A$ is just Frobenius?

CONTACT INFORMATION

Web sites.google.com/view/paolo-saracco
Email paolo.saracco@ulb.ac.be
Address Département de Mathématique, ULB, Bd du Triomphe, B-1050 Brussels, Belgium.