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A pair of Frobenius pairs for Hopf modules

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Rings, modules, and Hopf algebras, 15 May 2019

Report on

Hopf modules, Frobenius functors and (one-sided) Hopf algebras - arXiv:1904.13065

Antipodes, preantipodes and Frobenius functors - to appear

$\mathbb{k}$ is a commutative ring (from time to time a field). B a $\mathbb{k}$-bialgebra.

Definition ([GNT, 1980])

A left (resp. right) convolution inverse of Id_B is called a left (resp. right) antipode and B a left (resp. right) Hopf algebra.

Definition

A $\mathbb{k}$-algebra A is Frobenius if $\exists\, \psi \in A^*$ and $e \in A \otimes A$ such that

$$(\psi \otimes A)(e) = 1 = (A \otimes \psi)(e) \qquad \text{and} \qquad ae = ea \qquad (\forall\, a \in A).$$

Equivalently, if A is fgp and ${}_AA \cong {}_AA^*$ with regular structures.

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Definition ([CMZ, 1997],[CGN, 1999])

- A pair of functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ is called a Frobenius pair if $\mathcal{G} \dashv \mathcal{F} \dashv \mathcal{G}$ (equivalently, $\mathcal{F} \dashv \mathcal{G} \dashv \mathcal{F}$).
- A functor $\mathcal{F}$ is Frobenius if $\exists\, \mathcal{G}$ such that $(\mathcal{F}, \mathcal{G})$ is a Frobenius pair.

Theorem ([M, 1965])

A $\Bbbk$-algebra A is Frobenius iff $U : {}_A\mathfrak{M} \to {}_\Bbbk\mathfrak{M}$ is Frobenius.

- Frobenius functors are a natural extension of Frobenius algebras to category theory.

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Theorem ([LS, 1969])

Any fgp Hopf algebra over a PID is Frobenius.

Theorem ([P, 1971])

A bialgebra B is an fgp Hopf algebra with $\int_r B^* \cong \Bbbk$ iff it is Frobenius with Frobenius homomorphism $\psi \in \int_r B^*$.

Argument

$$\eta_M : M \to B \otimes {}^{B}\overline{M} \qquad \epsilon_V : {}^{B}\overline{B \otimes V} \xrightarrow{\cong} V$$

$$\Bbbk \otimes_B - : {}^B_B\mathfrak{M} \to {}_\Bbbk\mathfrak{M}, \qquad B \otimes - : {}_\Bbbk\mathfrak{M} \to {}^B_B\mathfrak{M}, \qquad {}^{\mathrm{co}B}(-) : {}^B_B\mathfrak{M} \to {}_\Bbbk\mathfrak{M}$$

$$\gamma_V : V \xrightarrow{\cong} {}^{\mathrm{co}B}(B \otimes V) \qquad \theta_M : B \otimes {}^{\mathrm{co}B}M \to M$$

- B fgp Hopf $\Rightarrow B^* \in {}^B_B\mathfrak{M}$ and $\theta_{B^*} : B \otimes \int_r B^* \cong B^* \Rightarrow {}_BB \cong {}_BB^*$.

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Lemma

- *There is a canonical morphism*

$$\sigma_M : \left({}^{\mathrm{co}B}M \xrightarrow{\epsilon_{\mathrm{co}B}^{-1}} {}^{B}\overline{B \otimes {}^{\mathrm{co}B}M} \xrightarrow{{}^{B}\overline{\theta_M}} {}^{B}\overline{M} \right),$$

natural in $M \in {}_{B}^{B}\mathfrak{M}$, given by $\sigma_M(m) = \overline{m}$ for all $m \in M$.

- *$B \otimes -$ is Frobenius iff σ is a natural iso, iff $M \cong {}^{\mathrm{co}B}M \oplus B^{+}M$ ($\forall\, M$).*

- What can we say about B when $B \otimes - : {}_{\Bbbk}\mathfrak{M} \to {}_{B}^{B}\mathfrak{M}$ is Frobenius?

Consider $B \widehat{\otimes} B := {}^{\bullet}_{\bullet}B \otimes {}_{\bullet}B \in {}_{B}^{B}\mathfrak{M}$ and $\sigma_{B \hat{\otimes} B} : {}^{\mathrm{co}B}(B \widehat{\otimes} B) \to {}^{B}\overline{B \widehat{\otimes} B}$.

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The first main results: one-sided Hopf algebras

Theorem

The endomorphism $S := (\varepsilon \otimes B)\left(\sigma^{-1}_{B\hat{\otimes}B}(- \otimes 1)\right)$ *of* B *satisfies*

- $S(1) = 1$, $\varepsilon \circ S = \varepsilon$;
- $S(a_1 b)a_2 = \varepsilon(a)S(b)$, $\forall\, a, b \in B$.

In particular, it is a left antipode and B *is a left Hopf algebra. Moreover,* S *is anti-multiplicative and anti-comultiplicative.*

Theorem

TFAE for a bialgebra B

- B *is a left Hopf algebra with anti-(co)multiplicative left antipode;*
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Example ([GNT, 1980])

Consider $T := \Bbbk\left\langle e_{i,j}^{(k)} \mid 1 \leq i,j \leq n, k \geq 0\right\rangle$ with

$$\Delta\left(e_{i,j}^{(k)}\right) := \sum_{h=1}^{n} e_{i,h}^{(k)} \otimes e_{h,j}^{(k)}, \quad \varepsilon\left(e_{i,j}^{(k)}\right) := \delta_{i,j} \quad \text{and} \quad s\left(e_{i,j}^{(k)}\right) := e_{j,i}^{(k+1)}.$$

The ideal I generated by

$$\left\{ \sum_{h=1}^{n} e_{h,i}^{(k+1)} e_{h,j}^{(k)} - \delta_{i,j}1, \ \sum_{h=1}^{n} e_{i,h}^{(l)} e_{j,h}^{(l+1)} - \delta_{i,j}1 \ \middle| \ 1 \leq i,j \leq n, k \geq 0, l \geq 1 \right\}$$

is an s-stable bi-ideal, whence T/I is a left Hopf algebra with anti-(co)multiplicative left antipode which is not an antipode.

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Example ([RT, 2005])

Let $q \in \Bbbk^\times$ and consider the algebra $\widetilde{\mathrm{SL}}_q(2)$ generated by $X_{i,j}$, $1 \leq i, j \leq 2$, and subject to the relations

$$X_{2,1}X_{1,1} = qX_{1,1}X_{2,1}, \quad X_{2,2}X_{1,2} = qX_{1,2}X_{2,2},$$
$$X_{2,2}X_{1,1} = qX_{1,2}X_{2,1} + 1, \quad X_{2,1}X_{1,2} = qX_{1,1}X_{2,2} - q.$$

This is a left Hopf algebra which is not Hopf and no left antipode is anti-multiplicative.

[RT] Rodríguez-Romo, Taft, *A left quantum group*. J. Algebra **286** (2005).

Theorem

TFAE for a fgp $\Bbbk$-bialgebra B.

(1) *The functor* $-\otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ *is Frobenius and* $\int_r B^* \cong \Bbbk$.

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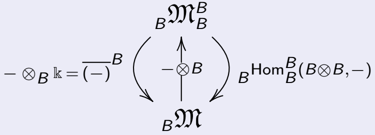

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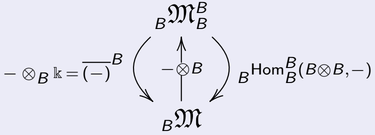

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Heartfelt wishes
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