Globalization for geometric partial comodules

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ItaCa Fest 2021

15/06/2021

Based on an ongoing project with J. Vercruysse (ULB) - arXiv:2001.07669

Outline

1. Partial actions of groups

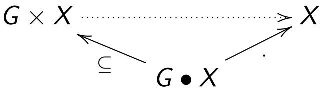
2. Geometric partial comodules

3. Globalization of geometric partial comodules

4. Applications

Partial actions of groups

Definition [Exel, 1998]
A partial action of a group G on a set X is a partial function

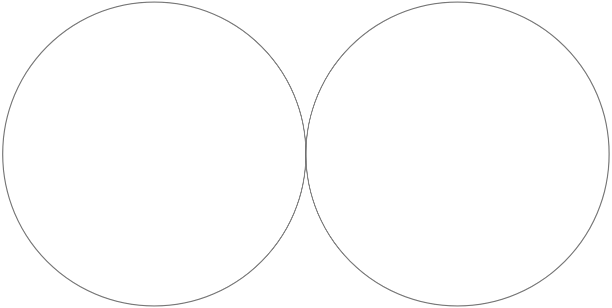


satisfying

- $\exists\ 1_G \cdot x$ for every x and $1_G \cdot x = x$;
- if $\exists\ g \cdot x$, then $\exists\ g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$;
- if $\exists\ g \cdot (h \cdot x)$, then $\exists\ gh \cdot x$ and $g \cdot (h \cdot x) = gh \cdot x$.

Example
The Möbius group acts globally on the Riemann sphere but only partially on the complex plane.

Partial actions of groups

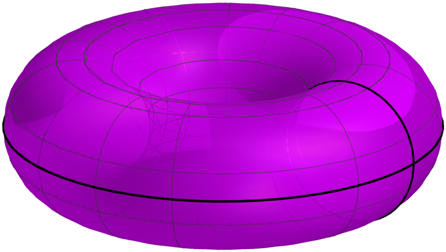


Definition

Let $\beta : G \times Y \to Y$ be a global action of G on Y and let $X \subseteq Y$.

Define $G \bullet X \dot{\to} X$ by declaring that $\exists\ g \cdot x$ if $\beta(g, x) \in X$.

This gives a partial action of G on X, called the induced partial action.



Globalization of partial actions of groups

Definition
A globalization for $G \bullet X \dot{\to} X$ is a G-set Y with an injection $\epsilon : X \rightarrowtail Y$ s.t. the partial action on X is induced by the global one on Y and Y is universal (initial) among the G-sets satisfying this property.

Theorem [Abadie, 2003]
Every partial action of a group G on a set X admits a globalization (unique up to iso) which can be realized as

$$G \times X / \sim$$

where $(g, x) \sim (h, y)$ iff $\exists\ h^{-1}g \cdot x$ and $y = h^{-1}g \cdot x$.

Example

- The Riemann sphere is the globalization of the partial action of the Möbius group on the complex plane.
- The torus is the globalization of the partial action of $S^1 \times S^1$ on the tangent circumferences.

Round up of partial (co)actions

- Partial actions of groups on sets
- Partial actions of (topological) groups on topological spaces
- Partial actions of (C^*-quantum) groups on C^*-algebras
- Partial representations of groups in algebras
- ...

- Partial modules over Hopf algebras
- Partial comodules over Hopf algebras
- Partial comodule algebras over Hopf algebras
- Partial representations of Hopf algebras in algebras
- Partial actions of Hopf algebras on $\mathbb{k}$-linear categories
- Partial actions of multiplier Hopf algebras
- Partial actions of groupoids on rings
- ...

Outline

1. Partial actions of groups

2. Geometric partial comodules

3. Globalization of geometric partial comodules

4. Applications

Setting the stage

Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal cat with pushouts and (H, Δ, ε) a coalgebra.

Monoidal category

A category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbb{I}$ such that

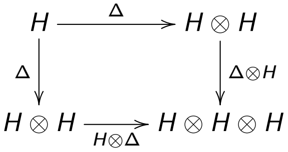
$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \qquad \mathbb{I} \otimes X \cong X \cong X \otimes \mathbb{I}$$

and the pentagon and triangle axioms hold.

Coalgebra

An object H in $(\mathcal{C}, \otimes, \mathbb{I})$ together with

$$\Delta : H \to H \otimes H \qquad \text{and} \qquad \varepsilon : H \to \mathbb{I} \qquad \text{such that}$$



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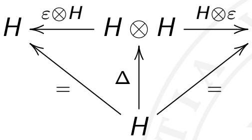
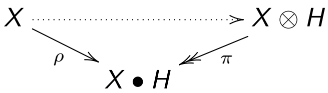


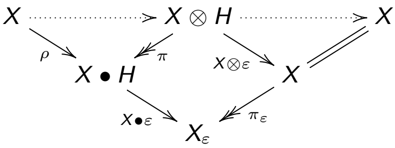
Geometric partial comodules

Definition [Hu-Vercruysse, 2018]
A geometric partial comodule (gpc_H) is a cospan



such that

- the following pushouts



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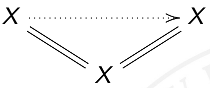
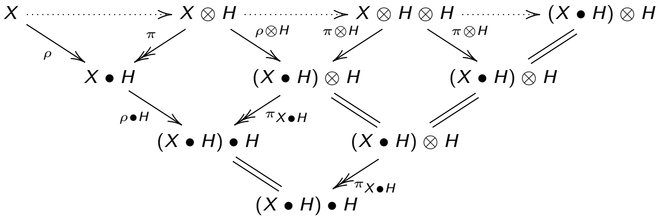


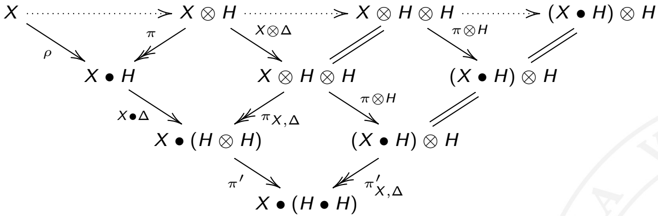
coincide (up to isomorphism);

- ...

Geometric partial comodules

- the following pushouts





coincide (up to isomorphism).

The category of geometric partial comodules

Definition

If $(X, X\bullet H, \pi_X, \rho_X)$ and $(Y, Y\bullet H, \pi_Y, \rho_Y)$ are gpcs_H, then a morphism of geometric partial comodules is a pair $(f, f\bullet H)$ of morphisms in $\mathcal{C}$ such that

$$\begin{array}{ccccc} X & \xrightarrow{\rho_X} & X\bullet H & \xleftarrow{\pi_X} & X\otimes H \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\bullet H} & & \downarrow{\scriptstyle f\otimes H} \\ Y & \xrightarrow{\rho_Y} & Y\bullet H & \xleftarrow{\pi_Y} & Y\otimes H \end{array}$$

commutes. We denote by gPCom^H the category of geometric partial comodules over H and their morphisms.

Examples of geometric partial comodules

Example

A partial action of a group G on a set X

$$\begin{array}{ccc} G \times X & \dashrightarrow & X \\ & \underset{\subseteq}{\nwarrow} \quad G \bullet X \quad \overset{\cdot}{\nearrow} & \end{array}$$

is a geometric partial G-comodule in $\mathsf{Set}^{\mathrm{op}}$.

- Partial actions of groups/monoids on sets ($\mathcal{C} = \mathsf{Set}^{\mathrm{op}}$)
- Partial actions of (topological) groups/monoids on topological spaces ($\mathcal{C} = \mathsf{Top}^{\mathrm{op}}$)
- Partial modules over Hopf algebras ($\mathcal{C} = \mathsf{Vect}_{\Bbbk}^{\mathrm{op}}$)
- Partial comodules over Hopf algebras ($\mathcal{C} = \mathsf{Vect}_{\Bbbk}$)
- Partial comodule algebras over Hopf algebras ($\mathcal{C} = \mathsf{Alg}_{\Bbbk}$)
- . . .

Induced geometric partial comodules

Definition [Hu-Vercruysse, 2018]
Let (Y, δ) be an H-comodule and $p : Y \to X$ an epi in $\mathcal{C}$. The pushout

$$
\begin{array}{ccccc}
 & & Y & & \\
 & \overset{p}{\swarrow} & & \overset{(p\otimes H)\circ\delta}{\searrow} & \\
X & & & & X \otimes H \\
 & \underset{\rho}{\searrow} & & \underset{\pi}{\swarrow} & \\
 & & X \bullet H & &
\end{array}
$$

makes of X a gpc_H and of p a morphism of gpcs_H. We refer to this as the induced partial comodule structure from Y to X.

Example
If (Y, β) is a G-set and $j : X \subseteq Y$ is any subset, then the pullback

$$
\begin{array}{ccccc}
 & & Y & & \\
 & \overset{\beta\circ(G\times j)}{\nearrow} & & \overset{j}{\nwarrow} & \\
G \times X & & & & X \\
 & \underset{\iota}{\nwarrow} & & \underset{\alpha}{\nearrow} & \\
 & & G \bullet X & &
\end{array}
$$

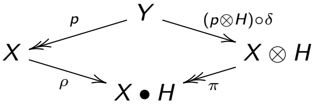
gives the induced partial action of G on X.

Globalization for geometric partial comodules

Definition

Given a gpc_H $(X, X \bullet H, \pi, \rho)$, a globalization for X is an H-comodule (Y, δ) with an epimorphism $p : Y \to X$ in $\mathcal{C}$ such that

- the diagram



  commutes and it is a pushout square in $\mathcal{C}$;

- Y is universal among all comodules admitting a map of gpcs_H to X:
  if (Z, δ') is global and $p' : Z \to X$ is of gpcs_H, then there exists a unique morphism of coms_H $\eta : Z \to Y$ such that $p \circ \eta = p'$.

X is globalizable if a globalization for X exists and we denote by gPCom^H_{gl} the full subcategory of the globalizable partial comodules.

When are geometric partial comodules globalizable?

Theorem [S.-Vercruysse]
Let H be a coalgebra in the monoidal category $\mathcal{C}$. Then a geometric partial comodule $(X, X \bullet H, \pi, \rho)$ is globalizable iff

- the equalizer

$$(\dagger) \qquad (Y_X, \delta) \xrightarrow{\kappa} (X \otimes H, X \otimes \Delta) \underset{(\pi \otimes H)\circ(X \otimes \Delta)}{\overset{\rho \otimes H}{\rightrightarrows}} (X \bullet H \otimes H, X \bullet H \otimes \Delta)$$

  exists in Com^H;

- the commuting diagram

$$\begin{array}{ccccc} & & Y_X & & \\ & \swarrow^{\epsilon := (X \otimes \varepsilon)\circ\kappa} & & \searrow^{\kappa} & \\ X & & & & X \otimes H \\ & \searrow_{\rho} & & \swarrow_{\pi} & \\ & & X \bullet H & & \end{array}$$

  is a pushout diagram in $\mathcal{C}$.

Under these equivalent conditions Y_X is the globalization of X.

When are geometric partial comodules globalizable?

$\mathcal{I}(Y,\delta) := (Y, Y\otimes H, id, \delta)$ induces a functor $\mathcal{I} : \mathrm{Com}^H \to \mathrm{gPCom}^H$.

Lemma

For a gpc_H $(X, X\bullet H, \pi, \rho)$ and a com_H (Y,δ), there is a bijective correspondence

$$\mathrm{gPCom}^H(\mathcal{I}Y, X) \cong \{f \in \mathrm{Com}^H(Y, X\otimes H) \mid f \text{ equalizes } (\dagger)\}$$
$$g \mapsto (g\otimes H)\circ\delta, \qquad (X\otimes\varepsilon)\circ f \leftarrow f.$$

Moreover, this correspondence is natural in both arguments Y and X.

Theorem [S.-Vercruysse]

Then the assignment $X \mapsto Y_X$ induces a functor

$$\mathcal{G} : \mathrm{gPCom}^H_{gl} \to \mathrm{Com}^H$$

which is right adjoint to the fully faithful functor $\mathcal{I} : \mathrm{Com}^H \to \mathrm{gPCom}^H_{gl}$.

Partial actions of monoids and groups

Set $\mathcal{C} = \mathrm{Set}^{\mathrm{op}}$. A gpc_H is a partial action of a monoid H.

Corollary [Megrelishvili-Schröder, 2004]
If X is a partial action of H, then $Y_X = (X \times M)/R$ is the globalization of X, where $R \subseteq (X \times M) \times (X \times M)$ is the equivalence relation generated by

$$\left\{ ((x \cdot m, n), (x, mn)) \mid m, n \in M, x \in X_m \right\}.$$

In particular, we have $\mathrm{gPCom}^H_{gl} = \mathrm{gPCom}^H$ for every monoid H.

Y_X is the coequalizer of $X \bullet H \times H \underset{(\pi\times H)\circ(X\times\mu)}{\overset{\rho\times H}{\rightrightarrows}} X \times H.$

Corollary [Abadie, 2003]
For every group G, $\mathrm{gPCom}^G_{gl} = \mathrm{gPCom}^G$.

Partial actions of topological monoids and groups

Set $\mathcal{C} = \mathsf{Top}^{\mathrm{op}}$.

Caveat

There exists an example of a geometric partial comodule (X, τ_X) over a topological group (H, τ_H) which does not admit a globalization.

Endow a group H acting on a set X with the indiscrete topology. We get a global action of (H, τ_H^{i}) on (X, τ_X^{i}). Set $X \bullet H := X \times H$ with the discrete topology $\tau^{\mathfrak{d}}_{X \bullet H}$. Then $(X, X \bullet H, \mathsf{Id}, \delta)$ is a gpc_H in $\mathcal{C}$, but

$$\begin{array}{ccccc} & & Y_X = (X, \tau_X^{\mathrm{i}}) & & \\ & \nearrow^{\delta} & & \nwarrow^{\mathsf{Id}} & \\ (X \times H, \tau_X^{\mathrm{i}} \times \tau_G^{\mathrm{i}}) & & & & (X, \tau_X^{\mathrm{i}}) \\ & \nwarrow_{\mathsf{Id}} & & \nearrow_{\delta} & \\ & & (X \bullet H, \tau^{\mathfrak{d}}_{X \bullet H}) & & \end{array}$$

cannot be a pullback square.

Therefore, in general, $\mathsf{gPCom}^H_{gl} \subsetneq \mathsf{gPCom}^H$ in $\mathsf{Top}^{\mathrm{op}}$. Nevertheless

Corollary [Abadie, 2003]

For a topological group H, $\mathsf{TopParAct}_H$ is a full subcat of gPCom^H_{gl}.

Geometric partial comodules in abelian categories

Let $\mathcal{C}$ be any abelian monoidal category.

Proposition [S.-Vercruysse]
Assume that $(X, X \bullet H, \pi, \rho)$ is a counital pcd_H. Consider the pullback

$$\begin{array}{ccccc} & & T & & \\ & \overset{\varpi}{\swarrow} & & \overset{\lambda}{\searrow} & \\ X & & & & X \otimes H \\ & \underset{\rho}{\searrow} & & \underset{\pi}{\twoheadleftarrow} & \\ & & X \bullet H & & \end{array}$$

in $\mathcal{C}$. Then $(X, X \bullet H, \pi, \rho)$ is a gpc_H iff

$$T \xrightarrow{\lambda} X \otimes H \underset{(\pi \otimes H)\circ(X \otimes \Delta)}{\overset{\rho \otimes H}{\rightrightarrows}} X \bullet H \otimes H$$

is an equalizer in $\mathcal{C}$.

Theorem [S.-Vercruysse]
If H is a coalgebra in $\mathcal{C}$ s.t. Com^H admits equalizers and $\mathrm{Com}^H \to \mathcal{C}$ preserves them, then $\mathrm{gPCom}^H_{gl} = \mathrm{gPCom}^H$.

Geometric partial (co)modules in vector spaces

Let $\Bbbk$ be a field.

Corollary

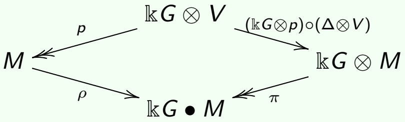
- If H is a $\Bbbk$-algebra, then gpcs_H can be identified with H-modules together with a chosen generating subspace.
- If H is a $\Bbbk$-coalgebra, then gpcs_H can be identified with H-comodules together with a chosen co-generating quotient space.

- Partial modules over Hopf algebras are globalizable and the globalization coincides with their standard dilation [Alves-Batista-Vercruysse]
- Partial representations of finite groups are globalizable [D'Adderio-Hautekiet-S.-Vercruysse]
- Partial comodules over Hopf algebras are globalizable
- Partial comodule algebras over Hopf algebras are globalizable, but their globalization is not their enveloping coaction [Alves-Batista]

Partially graded representations

Let G be a group. $\mathcal{C} = \mathrm{Rep}_G$ is an abelian monoidal category and $\Bbbk G$ is a coalgebra therein. We call partially graded G-representation a partial comodule over $\Bbbk G$ in $\mathcal{C}$.

Partially graded G-representations are all and only of the following form. For a vector space V, consider $\Bbbk G \otimes V$ with regular action and coaction. Pick a $\Bbbk G$-submodule $N \subseteq \Bbbk G \otimes V$ and define $M := (\Bbbk G \otimes V)/N$. Then M with the induced structure



is a partially graded G-representation.

In particular, this construction induces a bijective correspondence between structures of partially graded G-representation on the base field $\Bbbk$ (up to isomorphism) and linear characters of the group G.

Many thanks