On the integration problem for Lie algebroids

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Welcome Home 2021

22/12/2021

(Based on an ongoing joint project with A. Ardizzoni and L. El Kaoutit – arXiv:1905.10288)

The correspondence between groups, Hopf algebras and Lie algebras

Grp

From groups to Hopf algebras

Let (G, μ, e) be a (discrete) group and $\Bbbk^G := \mathrm{Fun}(G, \Bbbk)$.

$$\mathcal{R}(G) := \left\{ f : G \to \Bbbk \;\middle|\; \begin{array}{c} \exists \quad \sum_i f_{1,i} \otimes f_{2,i} \in \Bbbk^G \otimes \Bbbk^G \quad \text{s.t.} \\ f(xy) = \sum_i f_{1,i}(x) f_{2,i}(y) \quad \forall\, x, y \in G \end{array} \right\}$$

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$$\Delta : \mathcal{R}(G) \to \mathcal{R}(G) \otimes \mathcal{R}(G), \qquad f \mapsto \sum_i f_{1,i} \otimes f_{2,i},$$

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$$S : \mathcal{R}(G) \to \mathcal{R}(G), \qquad f \mapsto f\left((-)^{-1}\right)$$

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$\big(\mathcal{R}(G), \Delta, \varepsilon, S\big)$ is a *commutative Hopf algebra*.

Representative functions

Representative functions

- Any polynomial function is representative.
- exp, sin and cos from $(\mathbb{R}, +, 0)$ to $\mathbb{R}$ are representative:

$$\begin{aligned} \Delta(\exp) &= \exp \otimes \exp, & \varepsilon(\exp) &= 1, & S(\exp) &= \exp^{-1}; \\ \Delta(\sin) &= \cos \otimes \sin + \sin \otimes \cos, & \varepsilon(\sin) &= 0, & S(\sin) &= -\sin; \\ \Delta(\cos) &= \cos \otimes \cos - \sin \otimes \sin, & \varepsilon(\cos) &= 1, & S(\cos) &= \cos . \end{aligned}$$

- log from $(\mathbb{R}^{>0}, \cdot, 1)$ to $\mathbb{R}$ is representative:

$$\Delta(\log) = \log \otimes 1 + 1 \otimes \log, \qquad \varepsilon(\log) = 0, \qquad S(\log) = -\log .$$

The correspondence between groups, Hopf algebras and Lie algebras

$$\mathsf{Grp} \xrightarrow[\mathcal{R}]{} \mathsf{CommHopf}^{\mathrm{op}}$$

From Hopf algebras to groups

If $(H, \Delta, \varepsilon, S)$ is a commutative Hopf algebra, then

$$\mathcal{X}(H) := \mathsf{Alg}(H, \Bbbk)$$

is a group with

$$f * g := (f \otimes g) \circ \Delta, \qquad e := \varepsilon \qquad \text{and} \qquad f^{-1} := f \circ S.$$

$$\mathsf{CommHopf}(H, \mathcal{R}(G)) \cong \mathsf{Grp}(G, \mathcal{X}(H)).$$

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$$\mathsf{Grp} \underset{\mathcal{R}}{\overset{\mathcal{X}}{\rightleftarrows}} \mathsf{CommHopf}^{\mathrm{op}} \qquad (\mathcal{X} \dashv \mathcal{R})$$

The group-Hopf algebra dictionary

$$
\begin{aligned}
\mathsf{FinGrp} &\cong \mathsf{SsCommHopf}^{\mathsf{op}}_{\mathsf{f.d.}} \\
\mathsf{CompTopGrp} &\cong \mathsf{CommHopf}^{\mathsf{op}}_{\int, \mathcal{X}(H)\subseteq H^* \text{dense}} \\
\mathsf{CompLieGrp} &\cong \mathsf{AffHopf}^{\mathsf{op}}_{\int, \mathcal{X}(H)\subseteq H^* \text{dense}} \\
\mathsf{AffAlgGrp} &\cong \mathsf{AffHopf}^{\mathsf{op}} \\
\mathsf{AffGrpSch} &\cong \mathsf{CommHopf}^{\mathsf{op}}
\end{aligned}
$$

The correspondence between groups, Hopf algebras and Lie algebras

Lie

$$\mathsf{Grp} \underset{\mathcal{R}}{\overset{\mathcal{X}}{\rightleftarrows}} \mathsf{CommHopf}^{\mathrm{op}} \qquad (\mathcal{R} \dashv \mathcal{X})$$

From Lie algebras to Hopf algebras

Let L be a Lie algebra. Its *universal enveloping algebra*

$$\mathcal{U}(L) \coloneqq \Bbbk\big\langle X \in L \mid XY = YX + [X, Y]\big\rangle$$

becomes a *cocommutative Hopf algebra* via

$$\Delta(X) \coloneqq X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) \coloneqq 0, \qquad S(X) \coloneqq -X, \qquad \forall\, X \in L.$$

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In the other direction, if H is a Hopf algebra then the *space of primitives*

$$\mathcal{P}(H) \coloneqq \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$$

is a Lie algebra.

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Theorem

Poincaré-Birkhoff-Witt $\Rightarrow$ $L \cong \mathcal{P}\mathcal{U}(L)$.

Heyneman-Radford $\Rightarrow$ $\mathcal{U}\mathcal{P}(H) \hookrightarrow H$.

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Milnor-Moore Theorem

$H \cong \mathcal{UP}(H) \Leftrightarrow H$ primitively generated $\Leftrightarrow H$ cocommutative connected.

The correspondence between groups, Hopf algebras and Lie algebras

$$\mathsf{Lie} \underset{\mathcal{U}}{\overset{\mathcal{P}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathsf{CocommHopf}$$

$$\mathsf{Grp} \underset{\mathcal{R}}{\overset{\mathcal{X}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathsf{CommHopf}^{\mathrm{op}}$$

Sweedler's dual and Hopf algebra duality

If H is a Hopf algebra, then

$$H^\circ := \left\{ f \in H^* := \mathrm{Hom}_{\Bbbk}(H, \Bbbk) \;\middle|\; \begin{array}{cc} \exists \quad \sum_i f_{1,i} \otimes f_{2,i} \in H^* \otimes H^* & \text{s.t.} \\ f(xy) = \sum_i f_{1,i}(x) f_{2,i}(y) & \forall\, x, y \in H \end{array} \right\}$$

is a Hopf algebra again with

$$f * g := (f \otimes g) \circ \Delta, \qquad 1 := \varepsilon, \qquad \varepsilon := \mathrm{ev}_1$$
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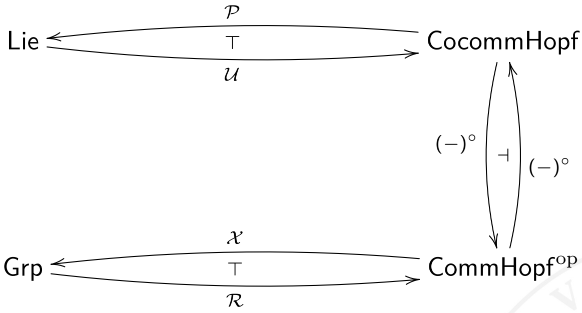
If H is commutative, then H° is cocommutative and conversely.

$$\mathrm{CommHopf}(H, K^\circ) \cong \mathrm{CocommHopf}(K, H^\circ).$$

The correspondence between groups, Hopf algebras and Lie algebras



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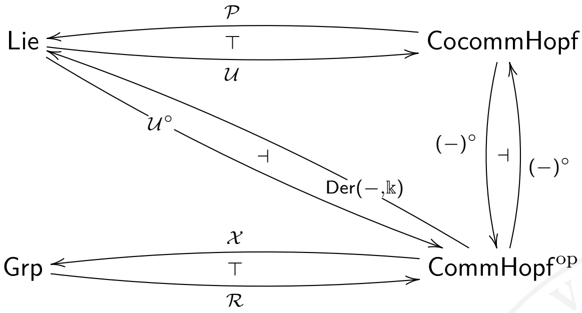
Remark

$$\mathcal{P}(H^\circ) = \{ f \in H^* \mid f(xy) = f(x)\varepsilon(y) + \varepsilon(x) f(y) \} = \mathrm{Der}(H, \Bbbk).$$

Hence $\mathcal{U}(-)^\circ \dashv \mathrm{Der}(-, \Bbbk)$.

The correspondence between groups, Hopf algebras and Lie algebras



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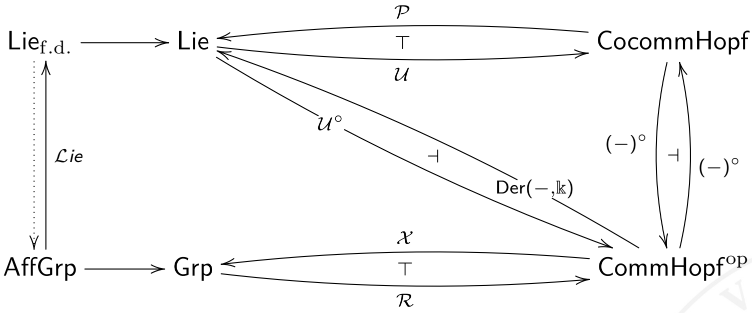

Lie-Rinehart algebras

A *Lie-Rinehart algebra* over a commutative algebra A is a Lie algebra L s.t.

- A acts on L by $\mathbb{k}$-linear endomorphisms: $\rho\colon A \to \mathrm{End}_{\mathbb{k}}(L)$,
- L acts on A by $\mathbb{k}$-linear derivations: $\omega\colon L \to \mathrm{Der}_{\mathbb{k}}(A)$,
- ω and ρ are compatible, in the sense that ω is A-linear,
- Leibniz rule: $[X, aY] = \omega(X)(a)Y + a[X, Y]$, $\forall\, X, Y \in L$, $a \in A$.

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Example

(1) $A = \mathcal{C}^\infty(M)$, $L = \mathfrak{X}(M)$, $\omega = \mathrm{Id}$.

(2) More generally, if $\mathscr{L}$ is a Lie algebroid over a smooth manifold M, then $\Gamma(\mathscr{L})$ is a Lie-Rinehart algebra over $\mathcal{C}^\infty(M)$.

Groupoids, Hopf algebroids and Lie-Rinehart algebras

LieRin_A

Universal enveloping algebroids

Let (A, L, ω) be a Lie-Rinehart algebra. Its *universal enveloping algebra*

$$\mathcal{U}(L) := \Bbbk \left\langle \begin{matrix} a \in A \\ X \in L \end{matrix} \;\middle|\; \begin{matrix} a \cdot b = ab, & XY = YX + [X, Y], \\ 1 = 1_A, & Xa = aX + \omega(X)(a) \end{matrix} \right\rangle$$

becomes a *cocommutative Hopf algebroid* via

$$\begin{aligned} \Delta(X) &:= X \otimes_A 1 + 1 \otimes_A X, & \varepsilon(X) &:= 0, \\ \gamma(X) &:= X \otimes_A 1 - 1 \otimes_A X, & \forall\, X &\in L. \end{aligned}$$

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In the other direction, if H is a cocommutative Hopf algebroid then

$$\mathcal{P}(H) := \{h \in H \mid \Delta(h) = h \otimes_A 1 + 1 \otimes_A h\}$$

is a Lie-Rinehart algebra.

Groupoids, Hopf algebroids and Lie-Rinehart algebras

$$\mathsf{LieRin}_A \underset{\mathcal{U}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{CocommHopfAlgd}_A \quad (\mathcal{U} \dashv \mathcal{P})$$

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Theorem (Moerdijk-Mrčun [MM])

$$\mathsf{LieRin}_A(L, \mathcal{P}(H)) \cong \mathsf{CocommHopfAlgd}_A(\mathcal{U}(L), H) \text{ and}$$

$$\mathsf{LieRin}_A^{\text{proj}} \cong \mathsf{CocommHopfAlgd}_A^{\text{cocompl, gr proj}}$$

[MM] I. Moerdijk, J. Mrčun, *On the universal enveloping algebra of a Lie algebroid*. Proc. Amer. Math. Soc. **138** (2010), no. 9, 3135–3145.

Groupoids and commutative Hopf algebroids

A *groupoid* is a small category in which every arrow is invertible:

$$\mathcal{G}_0 \underset{\tau}{\overset{\sigma}{\underset{\xrightarrow{e}}{\leftleftarrows}}} \mathcal{G}_1 \overset{\circ}{\longleftarrow} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1, \qquad \mathrm{inv} \colon \mathcal{G}_1 \to \mathcal{G}_1.$$

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A *commutative Hopf algebroid* is a cogroupoid in the category of commutative algebras (equivalently, an affine groupoid scheme = representable presheaf of groupoids on commutative algebras):

$$A \underset{t}{\overset{s}{\underset{\xleftarrow{\varepsilon}}{\overrightarrow{\longrightarrow}}}} H \circlearrowright^{S} \xrightarrow{\ \Delta\ } H \otimes_A H .$$

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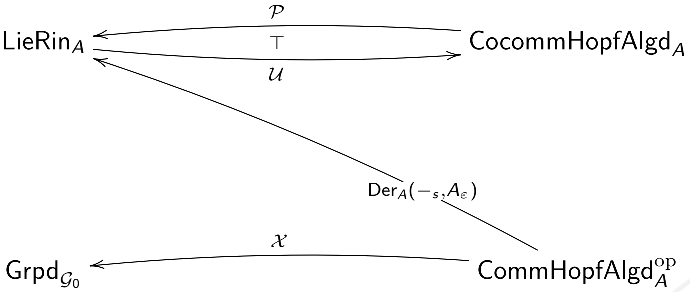
In particular, if (A, H) is a commutative Hopf algebroid, then $\mathcal{X}(A, H) := (\mathrm{Alg}(A, \Bbbk), \mathrm{Alg}_{\Bbbk}(H, \Bbbk))$ is a groupoid.

Groupoids, Hopf algebroids and Lie-Rinehart algebras

$$\mathsf{LieRin}_A \underset{\mathcal{U}}{\overset{\mathcal{P}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathsf{CocommHopfAlgd}_A$$

$$\mathsf{Grpd}_{\mathcal{G}_0} \overset{\mathcal{X}}{\longleftarrow} \mathsf{CommHopfAlgd}_A^{\mathrm{op}}$$

Groupoids, Hopf algebroids and Lie-Rinehart algebras



The dark dual

Given a $\Bbbk$-algebra extension $A \to R$, $R^\bullet$ is uniquely determined by the following *universal property*:

$R^\bullet$ admits a coassociative and counital comultiplication

$$\Delta : R^\bullet \to R^\bullet \otimes_A R^\bullet, \qquad \varepsilon : R^\bullet \to A,$$

and there exists an A-bilinear morphism $\xi : R^\bullet \to \mathrm{Hom}_A(R, A)$ such that

$$\xi(z)(rs) = \sum_i \xi\left(z_{1,i}\right)\Big(\xi\left(z_{2,i}\right)(r)s\Big), \qquad \xi(z)(1_R) = \varepsilon(z)$$

for all $z \in R^\bullet$, $r, s \in R$. Furthermore, $R^\bullet$ is universal with respect to these properties. That is, for any other $(C, \Delta, \varepsilon, \zeta)$, there exists a unique morphism $\hat{\zeta} \colon C \to R^\bullet$ (compatible with Δ and ε) such that $\xi \circ \hat{\zeta} = \zeta$.

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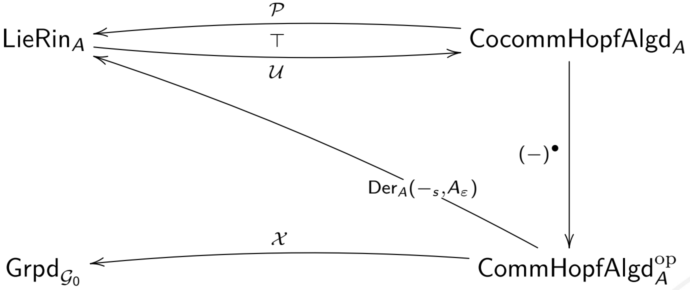
If (A, H) is a cocommutative Hopf algebroid, then $(A, H^\bullet)$ is a commutative Hopf algebroid.

Groupoids, Hopf algebroids and Lie-Rinehart algebras



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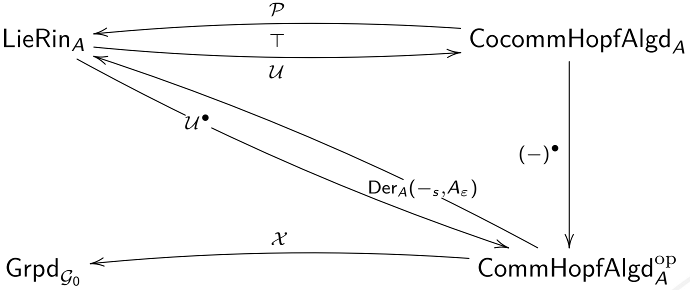


Open questions

- If A is affine, under which conditions $\mathcal{U}(L)^{\bullet}$ is affine, too?
- In such a case, do we have $L \cong \mathrm{Der}_A\big(\mathcal{U}(L)^{\bullet}, A\big)$?
- If A is affine, under which condition we can find $B \subseteq \mathcal{U}(L)^{\bullet}$ which is affine?
- Given (A, B) in $(A, \mathcal{U}(L)^{\bullet})$ affine, under which conditions $\mathrm{Der}_A\big(B, A\big) \cong L$?
- ...

Many thanks

Filling in the gaps: the group Hopf algebra

If G is a group, then $\Bbbk G = \mathrm{span}_{\Bbbk}\{g \in G\}$ is a Hopf algebra with

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}.$$

If H is a Hopf algebra, then the set of *group-like elements*

$$\mathcal{G}(H) \coloneqq \{h \in H \mid \Delta(h) = h \otimes h, \varepsilon(h) = 1\}$$

is a group with

$$\mu(h,k) \coloneqq hk, \qquad e \coloneqq 1, \qquad h^{-1} \coloneqq S(h).$$

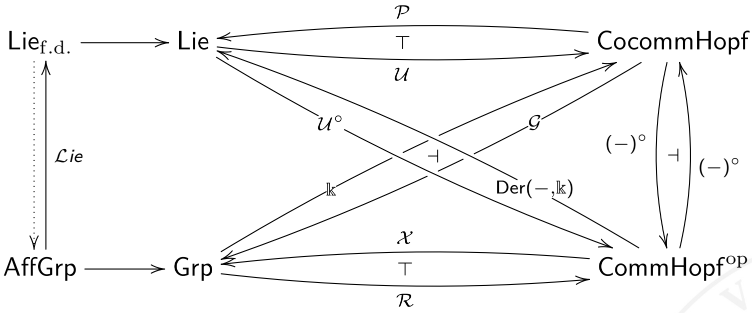
$$\mathrm{Hopf}(\Bbbk G, H) \cong \mathrm{Grp}(G, \mathcal{G}(H)).$$

Remark

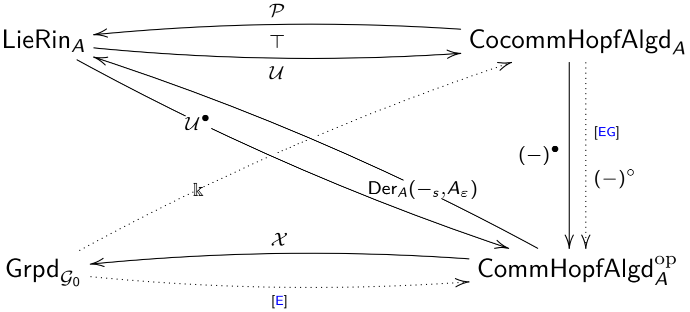
$$\mathcal{G}(H^\circ) = \{f \in H^* \mid f(xy) = f(x)f(y), f(1) = 1\} = \mathrm{Alg}(H, \Bbbk) = \mathcal{X}(H).$$

Hence $\mathcal{R}(G) = \Bbbk G^\circ$.

The full correspondence between groups, Hopf algebras and Lie algebras



Groupoids, Hopf algebroids and Lie-Rinehart algebras



[E] L. El Kaoutit, *Representative functions on discrete groupoids and duality with Hopf algebroids* (2013).

[EG] L. El Kaoutit, J. Gómez-Torrecillas, *On the finite dual of a cocommutative Hopf algebroid. Application to linear differential matrix equations and Picard-Vessiot theory*. Bull. Belg. Math. Soc. Simon Stevin **28** (2021), no. 1, 53–121.