ON THE GLOBALIZATION OF GEOMETRIC PARTIAL (CO)MODULES IN THE CATEGORIES OF TOPOLOGICAL SPACES AND ALGEBRAS

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Abstract. We study the globalization of partial actions on sets and topological spaces and of partial coactions on algebras by applying the general theory of globalization for geometric partial comodules, as previously developed by the authors. We show that this approach does not only allow to recover all known results in these settings, but it allows to treat new cases of interest, too.

1. Introduction

Since the very beginning of the theory of partial group actions [6], one of the main questions has been to understand if any given partial action can be obtained as a restriction of a classical (global) group action [1, 3, 9, 11, 13].

The *geometric partial (co)modules* from [10] provide a general categorical framework to study all sorts of partial actions in a unified way, subsuming partial actions of groups as well as partial (co)representations of Hopf algebras (see [15] for a detailed treatment of the globalization question in these cases). Moreover, geometric partial comodules also allow to treat cases that cannot be described by the Hopf-algebraic partial (co)actions from [4], such as genuine partial actions of algebraic groups on irreducible varieties. In a previous paper [14], we defined and studied globalizations for geometric partial comodules. In the present paper we apply the general results from [14] to discuss in more detail the globalization results for partial actions of topological monoids on topological spaces and partial comodule algebras. We show that our approach (Theorem 3.3) not only allows to recover and unify the globalization results for topological partial actions from [1] and [13] (Corollary 3.7), but it also allows to treat globalizations in new cases (Example 3.8). Next, we consider the partial comodule algebras (also called partial coactions) over bialgebras from [4] and show that these are globalizable (Theorem 4.3). Finally we explain how our globalization for partial comodule algebras differs from the enveloping coaction from [3] (Proposition 4.4).

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We denote identity on an object X by Id_X or simply by X itself.

2. The motivating example: Partial actions of monoids on sets

2.1. **Categorical formulation of partial actions.** A partial action of a monoid on a set is, intuitively, a "partially defined" action, satisfying unitality and associativity conditions whenever these make sense. Let us make this more explicit. Fix a monoid M with composition law $\Delta : M \times M \to M$ and neutral element $u : \{*\} \to M, * \mapsto e$. A *partial action datum* over M is a quadruple $(X, X \bullet M, \pi_X, \rho_X)$ consisting of two sets, X and $X \bullet M$, and of a span

$$\begin{array}{ccccc} X \times M & & & & X \\ & \overset{\pi_X}{\nwarrow} & & \overset{\rho_X}{\nearrow} & \\ & & X \bullet M & & \end{array} \tag{1}$$

in Set, where π_X is an injective map. The set $X \bullet M$ can be thought of as those "compatible pairs" for which the action is well-defined. For every $m \in M$, put $X_m \coloneqq \{x \in X \mid (x, m) \in X \bullet M\}$ and $\alpha_m : X_m \to X, x \mapsto \rho_X(x, m)$. The set X_m is the domain for the action by the element m. For the sake of simplicity, we will often write $x \cdot m \coloneqq \alpha_m(x) = \rho_X(x, m)$. We can now consider the following pullbacks:

$$\begin{array}{ccccc} & & X \times M & & \\ & \overset{\rho_X \times M}{\nearrow} & & \overset{\pi_X}{\nwarrow} & \\ (X \bullet M) \times M & & & & X \bullet M \\ & \overset{\pi_{X \bullet M}}{\nwarrow} & \wedge & \overset{\rho_X \bullet M}{\nearrow} & \\ & & (X \bullet M) \bullet M & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & X \times M & & \\ & \overset{(X \times \Delta) \circ (\pi_X \times M)}{\nearrow} & & \overset{\pi_X}{\nwarrow} & \\ (X \bullet M) \times M & & & & X \bullet M \\ & \overset{\pi_{X,\Delta}}{\nwarrow} & \wedge & \overset{X \bullet \Delta}{\nearrow} & \\ & & X \bullet (M \bullet M) & & \end{array} \tag{2}$$

Explicitly, these pullbacks can be described as the following sets:

$$\begin{aligned} (X \bullet M) \bullet M &= \{(x, m, n) \in X \bullet M \times M \mid (x \cdot m, n) \in X \bullet M\} \\ &= \{(x, m, n) \in X \times M \times M \mid x \in X_m \text{ and } x \cdot m \in X_n\}, \\ X \bullet (M \bullet M) &= \{(x, m, n) \in X \bullet M \times M \mid (x, mn) \in X \bullet M\} \\ &= \{(x, m, n) \in X \times M \times M \mid x \in X_m \text{ and } x \in X_{mn}\}. \end{aligned}$$

The quadruple $(X, X \bullet M, \pi_X, \rho_X)$ is called a *partial action* of M on X if the following two axioms are satisfied.

(PA1) Unitality: $X_e = X$ and $\alpha_e = \mathsf{Id}_X$. Equivalently, there exists a morphism $X \bullet u : X \to X \bullet M$ which makes the following diagram commutative

$$\begin{array}{ccccc} X \times M & \overset{\pi_X}{\longleftarrow} & X \bullet M & \overset{\rho_X}{\longrightarrow} & X \\ & \underset{X \times u}{\nwarrow} & \uparrow{\scriptstyle X \bullet u} & \underset{\mathsf{Id}_X}{\nearrow} & \\ & & X. & & \end{array} \tag{3}$$

(PA2) Partial associativity: $\alpha_m^{-1}(X_n) = X_{mn} \cap X_m$ and $\alpha_n \circ \alpha_m = \alpha_{mn}$ on $\alpha_m^{-1}(X_n)$ for all $m, n \in M$. Equivalently, there is an isomorphism (equality, in fact)

$$\theta : (X \bullet M) \bullet M \to X \bullet (M \bullet M)$$

such that the following diagram commutes

$$\begin{array}{ccccc} X & \xleftarrow{\rho_X} & X \bullet M & \xleftarrow{\rho_X \bullet M} & (X \bullet M) \bullet M \\ \uparrow{\scriptstyle \rho_X} & & & \swarrow{\scriptstyle \theta} & \downarrow{\scriptstyle \pi_{X \bullet M}} \\ X \bullet M & \xleftarrow[X \bullet \Delta]{} & X \bullet (M \bullet M) & \xrightarrow[\pi_{X,\Delta}]{} & (X \bullet M) \times M. \end{array} \tag{4}$$

Remark 2.1. One can easily verify that the definition of a partial action of a monoid as given above is equivalent to those considered for example in [13, Definition 2.3] and in [9, Definition 2.4]. Moreover, it was shown in [10, Section 1] that, for M a group, one recovers the definition of partial group actions as given in [7, Definition 1.2].

The above definitions of a partial action datum and of a partial action can obviously be extended to the setting of monoids in arbitrary monoidal categories with pullbacks. Indeed, if (M, Δ, u) is a monoid (or algebra) in the monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$, then we can define *partial module data* $(X, X \bullet M, \pi_X, \rho_X)$ and *geometric partial modules* by simply replacing the cartesian product by the monoidal product in diagrams (1), (2), (3) and (4) above. This is the viewpoint of [10], where the dual notions of *partial comodule datum* and *geometric partial comodule* over a comonoid (coalgebra) in an arbitrary monoidal category with pushouts $\mathcal{C}$ were defined.

If $(X, X \bullet M, \pi_X, \rho_X)$ and $(Y, Y \bullet M, \pi_Y, \rho_Y)$ are geometric partial modules, then a *morphism of geometric partial modules* is a pair $(f, f \bullet M)$ of morphisms in $\mathcal{C}$ with $f : X \to Y$ and $f \bullet M : X \bullet M \to Y \bullet M$ such that the following diagram commutes

$$\begin{array}{ccccc} X & \xleftarrow{\rho_X} & X \bullet M & \xrightarrow{\pi_X} & X \otimes M \\ {\scriptstyle f}\downarrow & & {\scriptstyle f \bullet M}\downarrow & & \downarrow{\scriptstyle f \otimes M} \\ Y & \xleftarrow[\rho_Y]{} & Y \bullet M & \xrightarrow[\pi_Y]{} & Y \otimes M. \end{array} \tag{5}$$

Observe that $f \bullet M$, when it exists, is uniquely determined by f as π_Y is a monomorphism. We will often denote a geometric partial module $(X, X \bullet M, \pi_X, \rho_X)$ simply by X and a morphism as above by f. Moreover, we will denote by gPMod_M the category of geometric partial modules over M and their morphisms and we will often omit to specify the adjective "geometric" when not strictly needed.

Note also that any usual (global) M-module (X, δ_X) is a geometric partial module with $\pi_X := \mathsf{Id}_{X \otimes M}$ and $\rho_X := \delta_X$. In fact, Mod_M is a full subcategory of gPMod_M and we denote by $\mathcal{I} : \mathsf{Mod}_M \to \mathsf{gPMod}_M$ the associated embedding functor.

2.2. **The general globalization result.** Let (M, Δ, u) be a monoid in a monoidal category $\mathcal{C}$ with pullbacks. Recall from (the dual of) [10, Example 2.5] that, for any (right) M-module (Y, δ) and any monomorphism $p : X \to Y$ in $\mathcal{C}$, the pullback

$$\begin{array}{ccccc} & \nearrow^{(p \otimes M) \circ \delta} & Y & \nwarrow^{p} & \\ X \otimes M & & \wedge & & X \\ & \nwarrow_{\pi_X} & X \bullet M & \nearrow_{\rho_X} & \end{array} \tag{6}$$

inherits a structure of geometric partial module and p becomes a morphism of partial modules. We refer to this as the *induced partial module* structure from Y to X.

Naively speaking, the globalization of a partial module X is a universal M-module "containing" X and such that the partial action is induced by the global one. The following definition is the straightforward dualization of [14, Definition 3.1].

Definition 2.2. Given a partial module $(X, X \bullet M, \pi_X, \rho_X)$, a *globalization* for X is a global module (Y, δ_Y) with a morphism $p : X \to Y$ in $\mathcal{C}$ such that

(GL1) the following diagram commutes

$$\begin{array}{ccccc} X \otimes M & \xrightarrow{p \otimes M} & Y \otimes M & \xrightarrow{\delta_Y} & Y \\ {\scriptstyle \pi_X}\uparrow & & & & \uparrow{\scriptstyle p} \\ X \bullet M & & \xrightarrow[\rho_X]{\qquad\qquad} & & X, \end{array} \tag{7}$$

that is, $p : X \to \mathcal{I}(Y)$ is a morphism of partial modules;

(GL2) diagram (7) is a pullback square in $\mathcal{C}$, that is to say, the partial module structure on X is induced by the global module structure on Y;

(GL3) the global module Y is universal, in the sense that the following map is bijective

$$\mathsf{Mod}_M(Y, Z) \to \mathsf{gPMod}_M(X, \mathcal{I}(Z)), \qquad \eta \mapsto \eta \circ p.$$

We say that X is *globalizable* if a globalization for X exists and we denote by gPMod_M^{gl} the full subcategory of gPMod_M of the globalizable partial modules.

It can be shown (see [14, Lemma 3.2]) that if (Y, p) is a globalization of a partial module X, then $p : X \to Y$ is a monomorphism. Moreover, it follows from axiom (GL3) that a globalization of a partial module is unique (up to isomorphism) whenever it exists.

The following theorem is the main results of [14], Theorem 3.5, rephrased in its dual form for the convenience of the reader.

Theorem 2.3. *Let M be a monoid in the monoidal category $\mathcal{C}$ with pullbacks. Then a geometric partial M-module $X = (X, X \bullet M, \pi_X, \rho_X)$ is globalizable if and only if*

(a) the following coequalizer exists in Mod_M:

$$(X \bullet M \otimes M, X \bullet M \otimes \Delta) \underset{(X\otimes\Delta)\circ(\pi_X \otimes M)}{\overset{\rho_X \otimes M}{\rightrightarrows}} (X \otimes M, X \otimes \Delta) \xrightarrow{\kappa} (Y_X, \delta); \tag{8}$$

(b) the following diagram is a pullback diagram in $\mathcal{C}$:

$$\begin{array}{ccccc} & & Y_X & & \\ & \overset{\kappa}{\nearrow} & & \overset{\kappa\circ(X\otimes u)}{\nwarrow} & \\ X \otimes M & & & & X \\ & \underset{\pi_X}{\nwarrow} & & \underset{\rho_X}{\nearrow} & \\ & & X \bullet M. & & \end{array} \tag{9}$$

Moreover, if these conditions hold, then the morphism $\epsilon_X := \kappa \circ (X \otimes u) \colon X \to Y_X$ is a monomorphism in $\mathcal{C}$, $\kappa = \delta \circ (\epsilon_X \otimes M)$ and (Y_X, ϵ_X) is the globalization of X.

Remark 2.4. Let (Y, δ) be a global M-module. In view of [14, Lemma 3.3], it can be easily checked that (Y, δ) fits into the coequalizer diagram (9) if and only if there exists a morphism $p : X \to Y$ in $\mathcal{C}$ satisfying (GL1) and (GL3). This suggests that one may call "pre-globalization" a global module (Y, δ) together with a morphism $p : X \to Y$ in $\mathcal{C}$ satisfying (GL1) and (GL3) (equivalently, *(a)* of Theorem 2.3) and treat (GL2) (equivalently, *(b)* of Theorem 2.3) separately, as an additional condition. Since here we are interested in globalizations in the strict sense (that is, also inducing the given partial module structure), we focus on global modules satisfying all the conditions (GL1)-(GL3), in order to stick to the point, and we leave the pre-globalization notion for a future investigation.

2.3. **Recovering the globalization of partial actions of groups and monoids.** Let us return to the situation where the monoid M (in Set) acts partially on the set X. Then the following coequalizer (in Set)

$$X \bullet M \times M \underset{(X\times\Delta)\circ(\pi_X\times M)}{\overset{\rho_X\times M}{\rightrightarrows}} X \times M \xrightarrow{\kappa} Y_X \tag{10}$$

is given by $Y_X = (X \times M)/R$, where $R \subseteq (X \times M) \times (X \times M)$ is the equivalence relation generated by $r = \Big\{\Big((x \cdot m, n), (x, mn)\Big) \mid m, n \in M, x \in X_m\Big\}$. Since the endofunctor $- \times M : \mathsf{Set} \to \mathsf{Set}$ is a left adjoint, it preserves coequalizers and hence Y_X inherits in a natural way a global action from $X \times M$. Explicitly, if $[x, m]$ denotes the class of (x, m) in Y_X, then the global action of M on Y_X is given by $[x, m] \triangleleft n := [x, mn]$. Applying our globalization Theorem 2.3, we find that Y_X will be the globalization of X if (9) is a pullback diagram. This was essentially proven in [13, Proposition 2.6]. Hence we can conclude the following result, which then shows that the globalization for monoids as described in [13, Section 2] is a special instance of the globalization for geometric partial modules discussed in the previous section.

Corollary 2.5. *For* $\mathcal{C} = \mathsf{Set}$*, we have* $\mathsf{gPMod}_M^{gl} = \mathsf{gPMod}_M$ *for every monoid* M*.*

Let us remark that if the monoid M is a group, then the globalization Y_X coincides with the globalization for partial group actions as given in [1, Theorem 1.1] or [11, Section 3.1], as already discussed in [14, Proposition 3.4].

3. Partial actions of topological monoids

Consider the category Top of topological spaces. It is a monoidal, complete and cocomplete category (see e.g. [12, Chapter V, Section 9]). Limits and colimits can be computed by endowing the corresponding limits and colimits in Set with a suitable topology. A monoid in Top is a topological monoid $((M, \tau_M), \Delta, u)$, i.e. a topological space endowed with a monoid structure whose composition is a continuous map. The notion of geometric partial module inflected in Top gives a span

$$(X \times M, \tau_X \times \tau_M) \xleftarrow{\pi_X} (X \bullet M, \tau_{X\bullet M}) \xrightarrow{\rho_X} (X, \tau_X)$$

in Top, where π_X is an injective continuous map and (PA1) and (PA2) hold.

As we know from the general globalization Theorem 2.3, the globalization of a geometric partial module in Top will exist if the coequalizer (10) exists in Mod_M and the diagram (9) is a pullback in Top. The first condition is always satisfied, since Mod_M is (complete and) cocomplete for any topological monoid M. This follows directly from (for example) [2, Corollary 1.7], since Top is co-well-powered (which means that each object has a set of quotient objects) and since the functor $- \times M : \mathsf{Top} \to \mathsf{Top}$ preserves epimorphisms (that is, surjective continuous maps). Concretely, the underlying set of a coequalizer of two parallel arrows $f, g : X \to Y$ in Mod_M is computed as the coequalizer (Q, q) in Set_M (the category of M-modules in Set), whose construction we recalled in the previous section. The coequalizer in Top would then endow Q with the quotient topology with respect to the surjective map $q : Y \to Q$. However, to obtain the coequalizer in Mod_M, one endows Q with the finest subtopology of this quotient topology for which the action $Q \times M \to Q$ becomes continuous. In general, this topology is strictly coarser than the quotient topology and hence the forgetful functor $\mathsf{Mod}_M \to \mathsf{Top}$ will not preserve coequalizers (see however Lemma 3.4 below for equivalent conditions for this to be the case).

From these observations, we can then conclude that for a geometric partial module in Top to be globalizable, the only condition is that (9) is a pullback in Top. The following example shows that, in contrast to what we saw in the previous section for Set, not every geometric partial module over a topological monoid is globalizable.

Example 3.1. Let (M, Δ, u) be a topological monoid and X a set with at least two elements, which we endow with the indiscrete topology. Consider the trivial global action $\delta_X : X \times M \to X$, given by $\delta_X(x, m) = x$ for all $x \in X$ and $m \in M$. Now we define the topological space $X \bullet M$ as the set $X \times M$ endowed with the product topology of the discrete topology $\tau_X^{\mathfrak{d}}$ on X and the given topology on M. In other words, the topology on $X \bullet M$ is generated by open sets of the form $\{x\} \times U$ where $x \in X$ and $U \in \tau_M$. Then $(X, X \bullet M, \mathsf{Id}_{X \times M}, \delta_X)$ is clearly a partial M-module datum in Top. Since the action is global, both $(X \bullet M) \bullet M$ and $X \bullet (M \bullet M)$ have $X \times M \times M$ as underlying set. On $(X \bullet M) \times M$ we have the product topology arising from the discrete topology on X and the given topology on M and since

$$\delta_X \bullet M = \left(X \times M \times M, \tau_X^{\mathfrak{d}} \times \tau_M \times \tau_M\right) \xrightarrow{\delta_X \times M} \left(X \times M, \tau_X^{\mathfrak{d}} \times \tau_M\right) \quad \text{and}$$
$$X \bullet \Delta = \left(X \times M \times M, \tau_X^{\mathfrak{d}} \times \tau_M \times \tau_M\right) \xrightarrow{X \times \Delta} \left(X \times M, \tau_X^{\mathfrak{d}} \times \tau_M\right)$$

are already continuous, we find that $(X \bullet M) \bullet M = X \bullet (M \bullet M)$ have the same topology as $(X \bullet M) \times M$ and hence X is a geometric partial M-module.

The underlying set of the coequalizer (10) in Top is X (because the original action was global) endowed with the quotient topology along $X \times M \xrightarrow{\delta_X} X$, which is the original indiscrete topology on X. Moreover, if we endow this coequalizer Y_X with the initial global action of X, one easily observes that Y_X is also coequalizer (8) in the category of (global) topological M-modules Mod_M. Then $(X, X \bullet M, \mathsf{Id}_{X \times M}, \delta_X)$ is globalizable (with globalization the global M-module (Y_X, δ_X)) if and only if (9) is a pullback in Top. By specializing (9) in our present setting, we obtain the following diagram

$$\begin{array}{ccccc} & & Y_X & & \\ & \overset{\delta_X}{\nearrow} & & \overset{\mathsf{Id}_X}{\nwarrow} & \\ X \times M & & & & X \\ & \underset{\mathsf{Id}_{X\times M}}{\nwarrow} & & \underset{\delta_X}{\nearrow} & \\ & & X \bullet M & & \end{array}$$

In order for this diagram to be a pullback in Top, $X \bullet M$ should have the coarsest topology for which $\mathsf{Id}_{X\times M}$ and δ_X are continuous, which means that the opens should be of the form $X \times U$, with $U \subset M$ open. This clearly differs from the topology we have chosen on $X \bullet M$. Therefore, the diagram above is not a pullback and hence the geometric partial module $(X, X \bullet M, \mathsf{Id}_{X\times M}, \delta_X)$ is not globalizable.

As a consequence, we can state the following result.

Proposition 3.2. *In the category* Top*, a general globalization theorem for geometric partial modules does not exist. More precisely,* $\mathsf{gPMod}_M^{gl} \subsetneq \mathsf{gPMod}_M$ *for any topological monoid* M*.*

Proof. It follows directly from Example 3.1. □

Although we know that not all geometric partial modules in Top are globalizable, Theorem 2.3 provides for us a way to describe globalizable partial modules.

Theorem 3.3. *Let* (M, τ_M) *be a topological monoid. Then the globalizable geometric partial modules over* M *in* Top *are exactly all those geometric partial modules* $(X, X \bullet M, \pi_X, \rho_X)$ *for which* $X \bullet M$ *has the coarsest topology making both* π_X *and* ρ_X *continuous.*

In particular, any geometric partial module for which π_X *is an embedding (that is,* $X \bullet M$ *has the induced topology via* π_X*), is globalizable.*

Proof. As we have observed at the beginning of the section, the sole criterion for the globalization of X to exist is that the set-theoretic pullback (9) is also a pullback in Top. Clearly, this is the case exactly if the topology on $X \bullet M$ is the coarsest topology making both π_X and ρ_X continuous.

Finally, if $(X, X \bullet M, \pi_X, \rho_X)$ is a geometric partial module for which π_X is an embedding, then this means that $X \bullet M$ has simply the induced (or subspace) topology from $X \times M$ and that ρ_X is already continuous with respect to this topology. Hence $X \bullet M$ has indeed the coarsest topology for which both π_X and ρ_X are continuous and we can conclude. □

The following Lemma shows that under some mild conditions on the topological monoid in Theorem 3.3, the coequalizers (of type (8)) in Mod_M can be more easily computed in Top.

Lemma 3.4. *If* (M, τ_M) *is a topological monoid such that one of the following conditions holds:*

(1) The endofunctor $- \times M \times M : \mathsf{Top} \to \mathsf{Top}$ *preserves coequalizers;*

(2) M *is core-compact;*

(3) (M, τ_M) *is a topological group;*

then the underlying functor from Mod_M *to* Top *creates coequalizers.*

Proof. (1). This is an application of the well-known fact that for any monoid M in a monoidal category $\mathcal{C}$, the colimit of any given diagram in the category Mod_M exists whenever the colimit of the same diagram in $\mathcal{C}$ exists and the functor $- \otimes M \otimes M : \mathcal{C} \to \mathcal{C}$ preserves it.
(2). This is a particular instance of (1), since in this case $- \times M$ is a left adjoint functor (see, for instance, [5, Theorem 5.3]).
(3). This is a particular instance of (1) too, but specialized to the coequalizers of type (10) (which is sufficient for Theorem 3.3 to hold). Recall that the endofunctor $- \times Z$ preserves coequalizers (Q, q) of open maps in Top, for every Z in Top: in fact, since the coequalizer (Q, q) of open maps is open itself, $q \times Z$ is open, surjective and continuous and hence it is a quotient map (that is, the product topology on $Q \times Z$ is equivalent to the quotient topology). Now, G being a group, the maps $\Delta : G \times G \to G$, π_X and ρ_X are all open maps. Since products and coequalizers of open maps are open again and the endofunctor $- \times G$ preserves coequalizers of open maps, $- \times G$ always preserves the coequalizer (10) in Top. □

The next definition subsumes at the same time [1, Definition 1.1] (in case $((M, \tau_M), \Delta, u)$ is a topological group) and [13, page 125] (in case M is discrete).

Definition 3.5. A *topological partial (right) action* of a topological monoid (M, τ_M) on a topological space (X, τ_X) is a pair $\left(\{X_m\}_{m\in M}, \{\alpha_m\}_{m\in M}\right)$ such that

(TP1) the set $X \bullet M = \{(x, m) \in X \times M \mid x \in X_m\}$ is an open subspace of $X \times M$ and the function $\rho_X : X \bullet M \to X, (x, m) \mapsto \alpha_m(x)$ is continuous;

(TP2) the pair forms a set-theoretic partial action of M: $X_e = X$, $\alpha_e = \mathsf{Id}_X$ and for all $m, n \in M$, $\alpha_m^{-1}(X_n) = X_{mn} \cap X_m$ and $\alpha_n \circ \alpha_m = \alpha_{mn}$ on $\alpha_m^{-1}(X_n)$.

A *morphism of topological partial actions* from (X, τ_X) with $\left(\{X_m\}_{m\in M}, \{\alpha_m\}_{m\in M}\right)$ to $(X', \tau_{X'})$ with $\left(\{X'_m\}_{m\in M}, \{\alpha'_m\}_{m\in M}\right)$ is a continuous map $f : X \to X'$ such that $f(X_m) \subseteq X'_m$ and $f \circ \alpha_m = \alpha'_m \circ f$.

Note that axiom (TP1) implies that the set $X_m = \{x \in X \mid (x, m) \in X \bullet M\}$ is an open subspace of X and $\alpha_m : X_m \to X$ is a continuous map, for all $m \in M$. This was included as an additional axiom in [1, Definition 1.1] and [13, page 125].

Any discrete partial action as in Section 2.1 is an example of a topological partial action in which every space has the discrete topology. On the other hand, not every geometric partial module in Top is a topological partial action. Indeed, the Example 3.1 is a geometric partial module in Top which is not a topological partial action. This can be seen easily from the following proposition, since in Example 3.1 $\pi_X : X \bullet M \to X \times M$ is an injective continuous map, but not an open embedding, as $X \bullet M$ does not have the induced topology along π_X.

Proposition 3.6. *Topological partial actions of a topological monoid (M, τ_M) are exactly the geometric partial M-modules in Top for which π_X is an open embedding.*

Furthermore, the category $\mathsf{TopParAct}_M$ *of topological partial actions of* M *and their morphisms is a full subcategory of the category* gPMod_M *of geometric partial* M*-modules in* Top.

Proof. As explained in Section 2.1, axiom (TP2) tells exactly that $(X, X \bullet M, \pi_X, \rho_X)$ is a geometric partial module over M in Set. Furthermore, axiom (TP1) tells that π_X and ρ_X are morphisms in Top and π_X is an open embedding. Hence we can conclude on the first assertion of the theorem if we prove that the bijection $\theta : (X \bullet M) \bullet M \to X \bullet (M \bullet M)$ (arising from the fact that $(X, X \bullet M, \pi_X, \rho_X)$ is a geometric partial module in Set) is a homeomorphism. The condition $X_m \cap X_{mn} = \alpha_m^{-1}(X_n)$ implies that the continuous map

$$X \bullet (M \bullet M) \xrightarrow{\pi_{X,\Delta}} X \bullet M \times M \xrightarrow{\rho_X \times M} X \times M$$

factors through $(X \bullet M, \pi_X)$ and since the latter has the induced topology from $X \times M$, the resulting factorization is continuous. Such a factorization is exactly the map needed to prove that the inclusion $X \bullet (M \bullet M) \to (X \bullet M) \bullet M$ is continuous by resorting to the universal property of $(X \bullet M) \bullet M$ as a pullback in Top. The other way around, the argument is similar.

Now, any morphism $f : X \to X'$ of topological partial actions induces a function $f \bullet M : X \bullet M \to X' \bullet M$ by (co)restriction of $f \times M$, which is continuous and makes (5) to commute. Moreover, if $(f, f \bullet M)$ is a morphism of partial M-modules which were induced by topological partial actions, then the condition $(f \times M) \circ \pi_X = \pi_{X'} \circ (f \bullet M)$ entails that for every $x \in X_m$, we have $f(x) \in X'_m$, and the condition $f \circ \rho_X = \rho_{X'} \circ (f \bullet M)$ entails that for every $x \in X_m$, we have $f(\alpha_m(x)) = \alpha'_m(f(x))$. Therefore, f is a morphism of topological partial actions. One easily verifies that this construction is functorial. □

By combining Proposition 3.6 with Theorem 3.3, we immediately can conclude that all topological partial actions over a topological monoid are globalizable.

Corollary 3.7. *Let* M *be a topological monoid. Then every topological partial action over* M *is globalizable and the underlying set of the globalization is given by the coequalizer* (10). *Hence* $\mathsf{TopParAct}_M$ *is a full subcategory of* gPMod_M^{gl}.

Let us remark that in the framework of topological partial actions of topological groups, Corollary 3.7 was also proven in [8, Proposition 5.5] and [1, Theorem 1.1]. In the framework of topological partial actions of a (discrete) monoid on a topological space, an analogue of this result has been established in [13, Section 3].

In general, however, the inclusion of $\mathsf{TopParAct}_M$ in gPMod_M^{gl} in Corollary 3.7 is not essentially surjective on objects, in the sense that there exist globalizable partial modules which do not come from topological partial actions.

Example 3.8. Take $M = \mathbb{R}$ acting on $Y = \mathbb{R}^2$ by vertical translation $Y \times M \to Y, \big((x, y), v\big) \mapsto (x, y + v)$ and take X to be the subspace $j : \mathbb{R} \to \mathbb{R}^2, x \mapsto (x, 0)$, (everything with Euclidean topology). Then $X \bullet M = \mathbb{R} \times \{0\}$, which is not open in $\mathbb{R} \times \mathbb{R}$. However, being the trivial partial module, $(X, X \bullet M, \pi_X, \rho_X)$ is globalizable with globalization $(X \times M, X \times \Delta)$ (see [14, Proposition 3.10]).

The next proposition further explains this phenomenon and how topological partial actions can be characterized by the way they embed in their globalization.

Proposition 3.9. *Let M be a topological monoid for which the underlying functor from Mod_M to Top preserves coequalizers (e.g. M satisfies one of the conditions from Lemma 3.4).*

If (Y,δ) is a global M-module in Top and $\epsilon : X \to Y$ is an open embedding, then the induced geometric partial M-module $(X, X \bullet M, \pi_X, \rho_X)$ in Top obtained by restricting Y along ϵ is a topological partial action of M.

Conversely, if X is a topological partial action of M with globalization Y_X, then the monomorphism $\epsilon_X : X \to Y_X$ is an open embedding.

Proof. Let (Y,δ) be a global topological M-module and $\epsilon : X \to Y$ be an open embedding. Then one can endow X with a geometric partial module structure by taking the pullback (6). Hence we can identify $X \bullet M = (\epsilon \times M)^{-1}\big(\delta^{-1}(\epsilon(X))\big)$. Since ϵ is an open embedding and δ is continuous, we can conclude that $X \bullet M$ is an open subset of $X \times M$ and ρ_X (which is the restriction of δ) is continuous with respect to the subset topology on $X \bullet M$. In other words $\pi_X : X \bullet M \to X \times M$ is an open embedding and hence X is a topological partial action by Proposition 3.6.

Suppose now that X is a globalizable topological partial action with globalization (Y_X,δ). Then the topology $\tau_{X\bullet M}$ on $X \bullet M$ has to be the limit topology and hence the coarsest topology for which π_X and ρ_X are continuous. Moreover, since $\big((Y_X,\tau_Y),\kappa\big)$ is (up to homeomorphism) the coequalizer in Top of

$$\big(X \bullet M \times M, \tau_{X\bullet M} \times \tau_M\big) \underset{(X\times\Delta)\circ(\pi_X\times M)}{\overset{\rho_X \times M}{\rightrightarrows}} \big(X \times M, \tau_X \times \tau_M\big),$$

we have that τ_Y is the quotient topology with respect to κ, that is $U \in \tau_Y$ if and only if $\kappa^{-1}(U) \in \tau_X \times \tau_M$, for every $U \subseteq Y_X$. In particular, $\epsilon_X(X) \in \tau_Y$ if and only if $X \bullet M = \kappa^{-1}\big(\epsilon_X(X)\big) \in \tau_X \times \tau_M$. Therefore, as $X \bullet M$ is open in $X \times M$ (see Proposition 3.6), also $\epsilon_X(X)$ is open in Y_X (and conversely).

We are left to check that ϵ_X is an open map. Since $X \bullet M$ is the pullback of (9) in Top, for every $V \in \tau_X$ we have that $(x,m) \in \kappa^{-1}\big(\epsilon_X(V)\big)$ if and only if there is $y \in V$ such that $\kappa(x,m) = \epsilon_X(y)$, if and only if $(x,m) \in X \bullet M$ and $\rho_X(x,m) = y$, if and only if $(x,m) \in \pi_X\big(\rho_X^{-1}(V)\big)$. Thus, $\kappa^{-1}\big(\epsilon_X(V)\big) = \pi_X\big(\rho_X^{-1}(V)\big) \in \tau_X \times \tau_M$ and so $\epsilon_X(V) \in \tau_Y$. □

Corollary 3.10. *Let M be a topological monoid as in Proposition 3.9. Then, topological partial actions over M are exactly those globalizable geometric partial modules that embed in their globalization as open subspaces.*

Let G be a topological group, X a topological partial action and Y_X the globalization of X, which we know exists from the above and which is constructed as a suitable quotient $G\times X/\sim$ (since it is the coequalizer (10)). Abadie observed in [1] that, for G a Hausdorff topological group and X a Hausdorff topological partial action, this quotient is not necessarily Hausdorff in general. Consequently, this quotient is not (isomorphic to) the coequalizer Y_X from (10) in the category Haus. However, since Haus is still complete and cocomplete (see e.g. [12, Proposition V.9.2]), one can still consider the coequalizer (Y_X, κ) of

$$X \bullet G \times G \underset{(X\times\Delta)\circ(\pi_X\times G)}{\overset{\rho_X\times G}{\rightrightarrows}} X \times G \tag{11}$$

in Haus, which is the "largest Hausdorff quotient" of the coequalizer (Y', κ') of the same pair of arrows, but computed in Top. Namely, $Y_X := Y'/\approx$ where $y \approx y'$ if and only if for every $f : Y' \to Q$ with Q in Haus we have $f(y) = f(y')$ (i.e., they cannot be distinguished by maps to Hausdorff spaces). As a matter of notation, denote by $[x, m]$ the equivalence class of (x, m) in Y_X, by $[x, m]'$ its class in Y' and by $q : Y' \to Y_X$ the canonical projection. We have $q \circ \kappa' = \kappa$.

The following result tells that the globalization for Hausdorff partial actions exists exactly when the coequalizer of (11) can be computed as in Top. This should be compared to [1, Proposition 1.2] and [8, Proposition 5.6].

Theorem 3.11. *Consider a geometric partial module $(X, X \bullet G, \pi_X, \rho_X)$ in Haus where G is a group. Then X is globalizable (as object in Haus) with globalization Y_X if and only if $Y_X = Y'$, that is, if and only if the coequalizer of (11) in Top is a Hausdorff space.*

Proof. The reverse implication holds in light of Corollary 3.7, whence let us focus on the direct one. Assume then that Y_X is the globalization of X with global action β and pick two distinct points $[x, g]', [y, h]'$ in Y' (where Y' as above denotes the coequalizer of the pair (11) in Top). Consider $[x, gh^{-1}]'$ and $[y, e]'$. In view of (GL2), if $[x, gh^{-1}] = [y, e]$, then $(x, gh^{-1}) \in X \bullet G$ and so $[x, g]' = [y, h]'$, a contradiction. Thus, $[x, gh^{-1}] \neq [y, e]$ and hence there exists Q Hausdorff and $f : Y' \to Q$ such that $f([x, gh^{-1}]') \neq f([y, e]')$. By taking the preimages of two separating open sets, we find two open subsets U, V of Y' separating $[x, gh^{-1}]'$ from $[y, e]'$. Since $\beta_h = \beta(-, h)$ is a homeomorphism, $\beta_h(U)$ and $\beta_h(V)$ are open subsets separating $[x, g]'$ from $[y, h]'$ in Y'. □

4. Partial comodule algebras

Let $\Bbbk$ be a commutative ring. Recall that the category $\mathsf{Alg}_\Bbbk$ of (unital, associative) $\Bbbk$-algebras is monoidal, the monoidal product being the tensor product of two $\Bbbk$-algebras with component-wise multiplication. In this section, we study geometric partial modules (see Section 2.1) in the monoidal category $\mathsf{Alg}_\Bbbk^{\mathrm{op}}$ or, stated otherwise, geometric partial comodules in $\mathsf{Alg}_\Bbbk$ (by using the dual terminology from [10]), which we will call *geometric partial comodule algebras.*

Firstly, recall that a coalgebra in $\mathsf{Alg}_\Bbbk$ is just a $\Bbbk$-bialgebra $(H, \mu, u, \Delta, \varepsilon)$ and a global comodule over H in $\mathsf{Alg}_\Bbbk$ is exactly an H-comodule algebra in the classical sense. A geometric partial comodule algebra is a quadruple $(A, A \bullet H, \pi_A, \rho_A)$, where A and $A \bullet H$ are algebras, $\pi_A : A \otimes H \to A \bullet H$ is an algebra epimorphism (not necessarily surjective) and $\rho_A : A \to A \bullet H$ is an algebra morphism, satisfying the counitality and coassociativity conditions dual to axioms (PA1) and (PA2). Specializing Theorem 2.3 to this setting, we find the following result.

Corollary 4.1. *Let H be a $\Bbbk$-bialgebra which is flat as left $\Bbbk$-module and consider a geometric partial H-comodule algebra $(A, A \bullet H, \pi_A, \rho_A)$. Set*

$$Y := \left\{ \sum_i a_i \otimes h_i \in A \otimes H \;\middle|\; \sum_i \rho_A(a_i) \otimes h_i = \sum_i \pi_A(a_i \otimes h_{i(1)}) \otimes h_{i(2)} \right\}.$$

Then the globalization of A exists provided that the following diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{A\otimes\varepsilon}{\swarrow} & & \searrow & \\ A & & & & A \otimes H \\ & \underset{\rho_A}{\searrow} & & \underset{\pi_A}{\swarrow} & \\ & & A \bullet H & & \end{array} \tag{12}$$

is a pushout in $\mathsf{Alg}_\Bbbk$. In this case, the globalization is given precisely by Y.

Moreover, in case $A \otimes \varepsilon : Y \to A$ is surjective, then the above condition is satisfied provided that π_A is surjective as well and $\mathsf{ker}(\pi_A)$ can be generated (as an $A \otimes H$-ideal) by elements of Y.

Proof. We know from Theorem 2.3 that the globalization of A, if it exists, should be given by the equalizer

$$Y \xrightarrow{\kappa} A \otimes H \underset{(\pi_A \otimes H)\circ(A \otimes \Delta)}{\overset{\rho_A \otimes H}{\rightrightarrows}} A \bullet H \otimes H \tag{13}$$

computed in $\mathsf{Alg}_\Bbbk^H$ (the H-comodules in $\mathsf{Alg}_\Bbbk$). Since H is flat as a left $\Bbbk$-module, this equalizer can be computed in $\mathsf{Alg}_\Bbbk$, and hence in $\mathsf{Mod}_\Bbbk$ (or even Set). Therefore, it is given by the set Y in the statement. The second condition of Theorem 2.3 states exactly that diagram (12) is a pushout square in $\mathsf{Alg}_\Bbbk$.

For the last statement, recall that the pushout of a span $R \xleftarrow{f} S \xrightarrow{g} T$ in $\mathsf{Alg}_\Bbbk$ where g is a surjective map is given by $\left(R/\langle f(\mathsf{ker}(g))\rangle, p_R, \tilde{f}\right)$ where $\langle f(\mathsf{ker}(g))\rangle$ is the two-sided ideal in R generated by $f(\mathsf{ker}(g))$, p_R is the canonical projection and $\tilde{f}$ is the unique map such that $\tilde{f} \circ g = p_R \circ f$. By applying this, we see that the diagram in the statement is indeed a pushout if and only if π_A is surjective and $\mathsf{ker}(\pi_A)$ is the ideal generated by all $\sum_i a_i \otimes h_i \in Y$ such that $\sum_i a_i \varepsilon(h_i) = 0$. □

Let us remark that, in general, π_A is not necessarily surjective and geometric partial comodule algebras are not always globalizable (see [14, Example 3.6] for an explicit example). We will now describe a particular class of geometric partial comodule algebras for which the globalization always exists.

Definition 4.2 ([4])**.** A (right) *algebraic*[(1)] *partial comodule algebra* over a bialgebra H is an algebra A with a $\Bbbk$-linear map $\delta_A : A \to A \otimes H, a \mapsto a_{[0]} \otimes a_{[1]}$, such that

[(1)] The prefix ‘algebraic’ is not standard in literature, but we use it here to distinguish these objects from geometric partial comodule algebras as introduced above.

(i) $\delta_A(ab) = \delta_A(a)\delta_A(b)$, i.e. $(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}$;
(ii) $(A \otimes \varepsilon)\delta_A(a) = a$;
(iii) $(\delta_A \otimes H)\,\delta_A(a) = (\delta_A(1_A) \otimes H) \cdot (A \otimes \Delta)\,\delta_A(a)$, i.e.

$$a_{[0][0]} \otimes a_{[0][1]} \otimes a_{[1]} = 1_{A[0]}a_{[0]} \otimes 1_{A[1]}a_{[1](1)} \otimes a_{[1](2)}.^{(2)} \tag{14}$$

It has been shown in [10, Example 4.9] that any partial comodule algebra over H in the sense of Definition 4.2 is a geometric partial comodule in the category $\mathsf{Alg}_{\Bbbk}$. Briefly, set $e' := 1_A \otimes 1_H - 1_{A[0]} \otimes 1_{A[1]}$, which is an idempotent. Consider the canonical projection

$$\pi_A : A \otimes H \to \frac{A \otimes H}{\langle e' \rangle}. \tag{15}$$

Setting $A \bullet H := A \otimes H/\langle e' \rangle$ and $\rho_A := \pi_A \circ \delta_A$ provides a geometric partial comodule structure on A in the category of $\Bbbk$-algebras.

Theorem 4.3. *Let H be a left flat $\Bbbk$-bialgebra and (A, δ_A) an algebraic partial H-comodule algebra. Then the associated geometric partial H-comodule algebra $(A, A \bullet H, \pi_A, \rho_A)$ is globalizable.*

Proof. Let Y be as in Corollary 4.1. In light of (14) and the definition (15) of π_A, the $\Bbbk$-linear map δ_A satisfies

$$(\rho_A \otimes H) \circ \delta_A = (\pi_A \otimes H) \circ (A \otimes \Delta) \circ \delta_A.$$

As a consequence, there exists a unique $\Bbbk$-linear morphism $\vartheta : A \to Y$ such that $\kappa \circ \vartheta = \delta_A$ (that is, δ_A takes values in Y) and hence, since $\mathsf{Id}_A = (A \otimes \varepsilon) \circ \delta_A$ in view of (ii), $A \otimes \varepsilon$ is still surjective when restricted to Y. Moreover, as explained above the proposition, $\mathsf{ker}(\pi_A)$ is generated as an ideal in $A \otimes H$ by the element $e' = 1_A \otimes 1_H - 1_{A[0]} \otimes 1_{A[1]}$. By denoting $1_A = 1$, we find that

$$\begin{aligned}
&(\delta \otimes H - A \otimes \Delta)(e') \\
&= 1_{[0]} \otimes 1_{[1]} \otimes 1_H - 1 \otimes 1_H \otimes 1_H - 1_{[0][0]} \otimes 1_{[0][1]} \otimes 1_{[1]} + 1_{[0]} \otimes 1_{[1](1)} \otimes 1_{[1](2)} \\
&\overset{(14)}{=} -e' \otimes 1_H - 1_{[0]}1_{[0]} \otimes 1_{[1]}1_{[1](1)} \otimes 1_{[1](2)} + 1_{[0]} \otimes 1_{[1](1)} \otimes 1_{[1](2)} \\
&= (e' \otimes 1_H)(1_{[0]} \otimes 1_{[1](1)} \otimes 1_{[1](2)} - 1 \otimes 1_H \otimes 1_H) \in \mathsf{ker}(\pi_A) \otimes H
\end{aligned}$$

Therefore, $e' \in Y$ and hence A is globalizable by Corollary 4.1. □

From now on (following [3]), let us assume that $\Bbbk$ is a field. The injective map $\vartheta : A \to Y$ from the proof of Theorem 4.3 is clearly multiplicative (as it is induced by the multiplicative map δ_A). One could consider the smallest subcomodule algebra of Y containing $\vartheta(A)$. This leads to the notion of enveloping coaction in the sense of [3]. More precisely, an enveloping coaction for an algebraic partial H-comodule algebra A is a (global) comodule algebra $(B, \delta_B : b \mapsto b^{[0]} \otimes b^{[1]})$ with an injective multiplicative map $\theta : A \to B$ such that

(2) In [4], $1_{A[0]} \otimes 1_{A[1]}$ appears on the right: $a_{[0][0]} \otimes a_{[0][1]} \otimes a_{[1]} = a_{[0]}1_{A[0]} \otimes a_{[1](1)}1_{A[1]} \otimes a_{[1](2)}$. Here we resort to the convention used in [3], for the sake of consistency with what follows. This change of side is harmless.

(a) $\theta(A)$ is a unital right ideal of B generated by $e \coloneqq \theta(1_A)$,
(b) B is generated by $\theta(A)$ as an H-comodule algebra and
(c) $(\theta \otimes H) \circ \delta_A = (\theta(1_A) \otimes H) \cdot (\delta_B \circ \theta)$ or, equivalently, for all $a \in A$

$$\theta(a_{[0]}) \otimes a_{[1]} = e\theta(a)^{[0]} \otimes \theta(a)^{[1]}. \tag{16}$$

In [3, Theorem 4], it was proven that the enveloping coaction of any algebraic partial comodule algebra exists. More precisely, B can be constructed as the H-subcomodule algebra of $A \otimes H$ (which is a right H-comodule via $A \otimes \Delta$) generated by all elements of the form $a_{[0]} \otimes a_{[1](1)} f(a_{[1](2)})$, with $a \in A$ and $f \in H^*$. The morphism $\theta : A \to B$ is then given by δ_A and $e = 1_{A[0]} \otimes 1_{A[1]}$. One may check that θ corestricts to an isomorphism $\theta : A \to eB$. Hence we can consider the projection of algebras $p : B \to A, b \mapsto \theta^{-1}(eb)$. In the realization of B as above, we find that $p(\sum_i a_i \otimes h_i) = 1_{A[0]} a_i \varepsilon(1_{A[1]} h_i)$ for any $\sum_i a_i \otimes h_i \in B$.

Proposition 4.4. *Given a partial H-comodule algebra A over a field $\Bbbk$, the enveloping coaction B of A in the sense of* [3] *is a subcomodule algebra of the globalization Y_A of A. Namely, there is a unique comodule algebra monomorphism $j : B \to Y_A$ such that one of the following (equivalent) conditions hold:*
(I) $\epsilon_A \circ j = p$;
(II) $\kappa \circ j = (p \otimes H) \circ \delta_B$.
In particular, B is co-generated by A in the sense of [14, Definition 2.10] *(i.e. $(p \otimes H) \circ \delta_B$ is a monomorphism).*

Proof. In view of the foregoing discussion, we can identify A with eB, θ with the inclusion map and p with left multiplication by e. Under this identification, we find

$$\begin{aligned}\Big((\delta_A \otimes H) \circ (p \otimes H) \circ \delta_B\Big)(a) &= (ea^{[0]})_{[0]} \otimes (ea^{[0]})_{[1]} \otimes a^{[1]} \overset{(16)}{=} a_{[0][0]} \otimes a_{[0][1]} \otimes a_{[1]} \\ &\overset{(14)}{=} 1_{A[0]} a_{[0]} \otimes 1_{A[1]} a_{[1](1)} \otimes a_{[1](2)} \qquad \text{and} \\ \Big((A \otimes \Delta) \circ (p \otimes H) \circ \delta_B\Big)(a) &= ea^{[0]} \otimes a^{[1]}_{(1)} \otimes a^{[1]}_{(2)} \overset{(16)}{=} a_{[0]} \otimes a_{[1](1)} \otimes a_{[1](2)}\end{aligned}$$

for all $a \in A$. By using the fact that $\mathsf{ker}(\pi_A)$ is generated by the element $1_A \otimes 1_H - 1_{A[0]} \otimes 1_{A[1]}$, we can conclude that $\varkappa \coloneqq (p \otimes H) \circ \delta_B : B \to A \otimes H$ satisfies

$$(\rho_A \otimes H) \circ \varkappa \circ \theta = (\pi_A \otimes H) \circ (\delta_A \otimes H) \circ \varkappa \circ \theta = (\pi_A \otimes H) \circ (A \otimes \Delta) \circ \varkappa \circ \theta.$$

Since $(\rho_A \otimes H) \circ \varkappa$ and $(\pi_A \otimes H) \circ (A \otimes \Delta) \circ \varkappa$ are H-comodule algebra maps and since B is generated by $\theta(A)$ as H-comodule algebra, we can conclude that $(\rho_A \otimes H) \circ \varkappa = (\pi_A \otimes H) \circ (A \otimes \Delta) \circ \varkappa$. Thus, since Y_A is given by the equalizer (13), we find that there is a unique morphism of H-comodule algebras $j : B \to Y_A$ such that $\kappa \circ j = \varkappa$. By composing the last identity with $A \otimes \varepsilon$, we find that $\epsilon_A \circ j = p$.

Finally remark that by the identity (II) and the fact that κ is a monomorphism, j is injective if and only if $\varkappa = (p \otimes H) \otimes \delta_B$ is so. Consider the explicit realization of B as subcomodule algebra of $A \otimes H$ from [3, proof of Theorem 4] as recalled above. For any element $a_{[0]} \otimes a_{[1]} \in \theta(A)$, we find

$$\varkappa(a_{[0]} \otimes a_{[1]}) = 1_{A[0]} a_{[0]} \varepsilon(1_{A[1]} a_{[1](1)}) \otimes a_{[1](2)} \overset{(14)}{=} a_{[0][0]} \varepsilon(a_{[0][1]}) \otimes a_{[1]} = a_{[0]} \otimes a_{[1]},$$

whence $\varkappa = (p \otimes H) \otimes \delta_B$ coincides with the inclusion $B \subset A \otimes H$ on $\theta(A)$ and since both maps are comodule algebra morphisms and B is generated by $\theta(A)$ as comodule algebra, we find that $(p \otimes H) \otimes \delta_B$ coincides with the inclusion on the whole of B. In particular, $(p \otimes H) \otimes \delta_B$ is injective. ☐

We conclude this paper by providing some examples that show how, in general, the enveloping coaction differs from the globalization.

Example 4.5 ([3, Example 1])**.** Let G be a finite group. If N is a normal subgroup of G and $\mathsf{char}(\Bbbk) \nmid |N|$, then $t = \frac{1}{|N|}\sum_{n\in N} n \in \Bbbk N$ is a central idempotent in $H := \Bbbk G$. Notice also that t is an integral in $\Bbbk N$, in the sense that $nt = t = tn$ for all $n \in N$. Let $A := t\,\Bbbk G$ be the (unital) ideal generated by t, let $p : H \to A, h \mapsto th$, be the canonical projection and let $\iota : A \to H$ be the inclusion. Consider the partial $\Bbbk G$-coaction on A given by

$$\delta_A(tg) = (t \otimes 1)\Delta(tg) = tt_1 g \otimes t_2 g = tg \otimes tg. \tag{17}$$

In this case, $A \cong \Bbbk[G/N]$ (the group algebra over G/N) with partial coaction given by the composition $A \xrightarrow{\Delta_A} A \otimes A \xrightarrow{A\otimes\iota} A \otimes H$ and $A \bullet H$ given by

$$\frac{A \otimes H}{\langle t \otimes 1 - \delta_A(t)\rangle} \overset{(17)}{=} \frac{A \otimes H}{\langle t \otimes 1 - t \otimes t\rangle} = \frac{A \otimes H}{(t \otimes 1 - t \otimes t)(A \otimes H)} = \frac{A \otimes H}{A \otimes (1-t)H},$$

which is isomorphic to $A \otimes A$ via the factorization through the quotient of the projection $\pi_A := (A \otimes p) : A \otimes H \to A \otimes A$.

To construct the globalization of A, choose a family $\{g_1, \ldots, g_r\}$ of representatives of the right cosets of N in G $\left(\text{i.e. } G = \bigsqcup_{i=1}^{r} N g_i\right)$ and observe that $\{tg_1, \ldots, tg_r\}$ forms a basis of A. Since $\rho_A = \pi_A \circ \delta_A$, we have that $z = \sum_{i=1}^{r}\sum_{g\in G} c_{i,g}(tg_i) \otimes g \in A \otimes H$ belongs to $Y_A = \mathsf{Eq}(\rho_A \otimes H, (\pi_A \otimes H) \circ (A \otimes \Delta))$ if and only if

$$\sum_{i=1}^{r}\sum_{g\in G} c_{i,g}\left(tg_i \otimes tg_i \otimes g - tg_i \otimes g \otimes g\right) \in \mathsf{ker}(\pi_A \otimes H) = A \otimes (1-t)H \otimes H,$$

if and only if $tg_i - g \in (1-t)H$, for all $g \in G$ and all $i = 1, \ldots, r$ such that $c_{i,g} \neq 0$. Thanks to the fact that $nt = t = tn$ for all $n \in N$, one may now check directly that, in fact, $z \in Y_A$ if and only if $c_{i,g} \neq 0$ only for $g \in Ng_i$, that is, $Y_A = \mathsf{span}_{\Bbbk}\{tg \otimes g \mid g \in G\}$, which is the enveloping coaction as shown in [3].

Example 4.6 ([3, Example 2])**.** Let H_4 be Sweedler's four dimensional Hopf algebra, $H_4 = \Bbbk\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx\rangle$, with g group-like and $\Delta(x) = x \otimes 1 - g \otimes x$, $\varepsilon(x) = 0$. For any $\alpha \in \Bbbk$, the element $f = \frac{1}{2}(1 + g + \alpha gx)$ is an idempotent in H_4 and, by identifying H_4 with $\Bbbk \otimes H_4$ in the canonical way, the assignment $\delta_{\Bbbk} : \Bbbk \to H_4, \lambda \mapsto \lambda f$, defines a structure of partial H_4-comodule algebra on $\Bbbk$. In this case, $f = \delta_{\Bbbk}(1_{\Bbbk})$ and $\Bbbk \bullet H_4 = \Bbbk \otimes H_4/\langle 1 - \delta(1)\rangle = H_4/\langle 1 - f\rangle$. A straightforward check reveals that $\langle 1 - f\rangle = \mathsf{ker}(\varepsilon)$ and hence $\Bbbk \bullet H_4 \cong \Bbbk$ via $\varepsilon : H_4 \to \Bbbk$. Therefore, $\Bbbk$ has the trivial partial H_4-comodule structure $(\Bbbk, H_4, \varepsilon, \mathsf{Id}_{\Bbbk})$ and so $Y_{\Bbbk} = H_4$, which strictly contains $\mathsf{span}_{\Bbbk}\{1, f\}$, that is the enveloping coaction according to [3].

In a similar way, one can check that the globalization of the partial comodule algebra from [3, Example 3] strictly contains the enveloping coaction.

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