

# From left ideals two-sided coideals to normal Hopf ideals in Hopf algebroids, and groupoids

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(based on an ongoing joint work with A. Ghobadi, L. El Kaoutit, J. Vercruysse)

## A soft (?!) introduction

▶ Let *G* be a (discrete, linear, Lie) group.

• Let  $\mathcal{R}_{\Bbbk}(G)$  be the algebra of "representative" functions on G

| $Fun(G,\Bbbk)$     | $\Bbbk[G]$ | $\mathcal{R}_{\Bbbk}(G)$ |
|--------------------|------------|--------------------------|
| (finite, discrete) | (linear)   | (Lie)                    |

•  $\mathcal{R}_{\Bbbk}(G)$  is an algebra together with  $\Delta \colon \mathcal{R}_{\Bbbk}(G) \to \mathcal{R}_{\Bbbk}(G) \otimes \mathcal{R}_{\Bbbk}(G)$ uniquely determined by

$$\Delta(f) = \sum f_1 \otimes f_2 \quad \iff \quad \sum f_1(x)f_2(y) = f(xy), \quad \forall x, y \in G,$$

ε: R<sub>k</sub>(G) → k, f ↦ f(1<sub>G</sub>), and S: R<sub>k</sub>(G) → R<sub>k</sub>(G), f ↦ f ∘ (-)<sup>-1</sup>.
If I ⊆ R<sub>k</sub>(G) is an ideal, then R<sub>k</sub>(G)/I "is a subspace of" G.
If I is also a coideal, R<sub>k</sub>(G)/I "is a submonoid of" G.
If B ⊆ R<sub>k</sub>(G) is a subalgebra, then B "is a quotient space of" G.
If B is also a subcoalgebra, B "is a quotient monoid of" G.
If B is also a sub-Hopf algebra, then B "is a quotient group of" G.

## Where do we want to go?

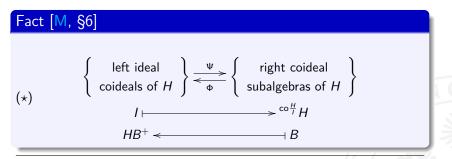
Let H be a Hopf algebra over a field  $\Bbbk$  (a bialgebra would suffice).

▶ If  $I \subseteq H$  is a coideal, then  $\pi: H \to \frac{H}{I}$  is a coalgebra map and

$${}^{\operatorname{co}rac{H}{T}}H\coloneqq\{h\in H\mid \pi(h_1)\otimes h_2=1\otimes h\}.$$

▶ If  $B \subseteq H$  is a subalgebra,

$$B^+ \coloneqq B \cap \ker(\varepsilon).$$



[M] S. Montgomery, Hopf Galois theory: a survey. New topological contexts for Galois theory and algebraic geometry, 367–400, Geom. Topol. Monogr., 16, Geom. Topol. Publ., Coventry, 2009.

## Why do we want to go there?

- [T] Suppose that H is commutative. The correspondence (★) is a bijection from the set of all sub-Hopf algebras of H onto the set of all normal Hopf ideals of H. This gives purely algebraic proofs of a number of results in the theory of affine k-groups, among which the fact that affine abelian groups form an abelian category.
- ► [N] Suppose that H is cocommutative (i.e., a formal group scheme). The correspondence (★) is a bijection from the set of all sub-Hopf algebras of H onto the set of all left ideal coideals of H.
- [S] The correspondence (\*) is a bijection from the set of all normal Hopf subalgebras B such that H<sub>B</sub> is faithfully flat onto the set of all normal Hopf ideals I such that H<sup>H/I</sup> is faithfully coflat. This suggests the "correct" definition of s.e.s. of quantum groups.
- (\*) can be extended to Hopf-Galois extensions of k different from H.
   For a classical Galois extension K/k with Galois group G and for H := Fun(G, k), it is the well-known Galois correspondence.
- K. Newman, A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras. J. Algebra 36 (1975), no. 1, 1–15.
- [S] H.-J. Schneider, Some remarks on exact sequences of quantum groups. Comm. Algebra 21 (1993), no. 9, 3337–3357.
- [T] M. Takeuchi, A correspondence between Hopf ideals and sub-Hopf algebras. Manuscripta Math. 7 (1972), 251–270.

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## Why (and what are) Hopf algebroids?

#### What

A (left) Hopf algebroid is a pair  $(A, \mathcal{H})$  of  $\Bbbk$ -algebras such that

- ►  $\mathcal{H}$  is an  $A^{e} = A \otimes A^{\mathrm{op}}$ -ring via an algebra map  $s \otimes t = \eta \colon A^{e} \to \mathcal{H}$ ;
- ▶  $_{\eta}\mathcal{H}$  admits an *A*-coring structure  $(\mathcal{H}, \Delta : \mathcal{H} \to \mathcal{H} \otimes_{A} \mathcal{H}, \varepsilon : \mathcal{H} \to A)$ ;
- $\Delta$  is multiplicative, i.e.  $\Delta(u)\Delta(v)$  makes sense and equals  $\Delta(uv)$ ;
- ▶  $\varepsilon$  is a left character, i.e.  $\varepsilon(u \, s \varepsilon(v)) = \varepsilon(u \, t \varepsilon(v));$
- $\blacktriangleright \ \beta \colon \mathcal{H} \otimes_{\mathcal{A}^{\mathrm{op}}} \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}, \ u \otimes_{\mathcal{A}^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_{\mathcal{A}} u_{(2)} v \text{ is bijective.}$

## Why (sloppily)

- Commutative Hopf algebroids are affine groupoid schemes.
- Cocommutative Hopf algebroids are formal groupoid schemes.
- Hopf algebroids are quantum groupoids.

## Examples

#### Commutative

►  $(A, A \otimes A)$  is a Hopf algebroid w.r.t.  $s(a) = a \otimes 1$ ,  $t(a) = 1 \otimes a$ ,  $\Delta(a \otimes b) = (a \otimes 1) \otimes_A (1 \otimes b)$ ,  $\varepsilon(a \otimes b) = ab$ ,  $S(a \otimes b) = b \otimes a$ . It corresponds to the groupoid of pairs  $(X, X \times X)$  on the set X.

▶  $(A, (A \otimes A)[T])$  is a Hopf algebroid w.r.t. the structure above and  $\Delta(T) = T \otimes_A 1 + 1 \otimes_A T$ ,  $\varepsilon(T) = 0$ , S(T) = -T. It corresponds to the additive groupoid  $(X, X \times G \times X)$  where X is a set, G an abelian group, and composition is given by  $(x, a, y) \circ (y, b, z) = (x, a + b, z)$ .

#### Cocommutative

Let  $(A, L, \omega)$  be a Lie-Rinehart algebra. Its universal enveloping algebra  $\mathcal{U}_A(L)$  is a Hopf algebroid with  $\Delta(X) = X \otimes_A 1 + 1 \otimes_A X$ ,  $\varepsilon(X) = 0$  and  $\beta^{-1}(X \otimes_A 1) = X \otimes_A 1 - 1 \otimes_A X$  for all  $X \in L$ .

#### Quantum

• The pair  $(A, A \otimes A^{op})$  is a Hopf algebroid as above.

▶ If *U* is a Hopf algebra and *A* is a left *U*-module algebra, then the Connes-Moscovici bialgebroid  $A \odot U \odot A$  is a Hopf algebroid over *A*.

Coideal subrings and left ideal two-sided coideals

Let  $(A, \mathcal{H})$  be a bialgebroid.

▶ If  $I \subseteq \mathcal{H}$  is a 2-sided coideal, then  $\pi \colon \mathcal{H} \to \frac{\mathcal{H}}{I}$  is an *A*-coring map and

$${}^{\operatorname{co}rac{\mathcal{H}}{\mathcal{T}}}\mathcal{H}\coloneqq\{h\in\mathcal{H}\mid \pi(h_1)\otimes_{\mathcal{A}}h_2=\pi(1)\otimes_{\mathcal{A}}h\}.$$

▶ If  $B \subseteq \mathcal{H}$  is a subalgebra,

$$B^+ := B \cap \ker(\varepsilon).$$

#### Proposition

Under the assumption that  ${}_{s}\mathcal{H}$  is A-flat, we have well-defined inclusion-preserving correspondences

## The canonical inclusions and the Galois connection

#### Theorem

Let  $(A, \mathcal{H})$  be a bialgebroid such that  ${}_{s}\mathcal{H}$  is A-flat.

- If B is a right H-comodule A<sup>op</sup>-subring via t of H, then we have an inclusion η<sub>B</sub>: B ⊆ <sup>co H</sup><sub>HB+</sub> H = ΨΦ(B). Moreover, ΦΨΦ(B) = Φ(B).
- ▶ If *I* is a left ideal 2-sided coideal in  $\mathcal{H}$ , then we have an inclusion  $\epsilon_I : \Psi \Phi(I) = \mathcal{H} \left( {}^{\operatorname{co} \frac{\mathcal{H}}{I}} \mathcal{H} \right)^+ \subseteq I$ . Moreover,  $\Psi \Phi \Psi(I) = \Psi(I)$ .

In other words,  $\Phi$  and  $\Psi$  form a monotone Galois connection (or, equivalently, an adjunction) between the two lattices and we have that

$$\mathcal{H}B^+\subseteq I\quad\iff\quad B\subseteq {}^{\operatorname{co}\frac{\mathcal{H}}{I}}\mathcal{H}.$$

## The Hopf algebroid case

Let us fix a Hopf algebroid  $(A, \mathcal{H})$  such that  ${}_{s}\mathcal{H}$  is A-flat.

#### Proposition

Let *B* be a right  $\mathcal{H}$ -comodule  $A^{\mathrm{op}}$ -subring via *t* of  $\mathcal{H}$  such that  $\beta^{-1}(B \otimes_A 1) \subseteq B \otimes_{A^{\mathrm{op}}} \mathcal{H}$  and such that  $\mathcal{H}$  is faithfully flat over *B* on the right. Then  $B = {}^{\mathrm{co}}\frac{\mathcal{H}}{\mathcal{H}B^+}\mathcal{H}$ , that is  $\Psi\Phi(B) = B$ .

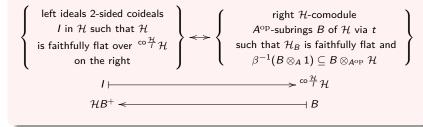
#### Proposition

Let  $I \subseteq \mathcal{H}$  be a left ideal 2-sided coideal such that  $\mathcal{H}$  is faithfully flat on  $B := {}^{\operatorname{co} \frac{\mathcal{H}}{I}}\mathcal{H}$  on the right. Then  $I = \mathcal{H}B^+$ , that is to say,  $\Phi\Psi(I) = I$ .

## The main result

#### Theorem

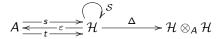
We have a well-defined inclusion-preserving bijective correspondence



## The commutative case

A commutative Hopf algebroid is a cogroupoid object in the category of commutative algebras or, equivalently, an affine groupoid scheme (i.e. a representable presheaf of groupoids on  $Aff_{k}$ ).

▶ It consists of a pair of commutative  $\Bbbk$ -algebras  $(A, \mathcal{H})$  together with a diagram of algebra maps



satisfying the duals of the groupoid conditions.

▶ the inverse of the Hopf-Galois map  $\beta$ :  $u \otimes_A v \mapsto u_1 \otimes_A u_2 v$  is

$$\beta^{-1} \colon \mathcal{H} \otimes_A \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}, \qquad u \otimes_A v \mapsto u_1 \otimes_A \mathcal{S}(u_2)v.$$

▶ If  $(A, \mathcal{H})$  is a commutative Hopf algebroid then  $\mathscr{G} := (\mathscr{G}_A, \mathscr{G}_{\mathcal{H}})$  is the associated groupoid scheme:

$$\mathscr{G}(R) = \left(\mathsf{CAlg}_{\Bbbk}(A, R), \mathsf{CAlg}_{\Bbbk}(\mathcal{H}, R)\right)$$

## Hopf ideals and subgroupoids

▶ An ideal  $I \subseteq \mathcal{H}$  in a commutative Hopf algebroid  $(A, \mathcal{H})$  is called a *(wide) Hopf ideal* if

 $\varepsilon(I) = 0, \qquad \Delta(I) \subseteq \operatorname{im}(\mathcal{H} \otimes_A I + I \otimes_A \mathcal{H}), \qquad \mathcal{S}(I) \subseteq I.$ 

▶  $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I})$  is a (wide closed) subgroupoid of  $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}})$  if  $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I})$  is a groupoid itself and  $\pi^* : \mathscr{G}_{\mathcal{H}/I} \hookrightarrow \mathscr{G}_{\mathcal{H}}$  induces a morphism of groupoids  $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}/I}) \to (\mathscr{G}_A, \mathscr{G}_{\mathcal{H}})$ , that is, if and only if *I* is a Hopf ideal.

#### Example (The isotropy Hopf algebroid)

- The ideal  $\langle s(a) t(a) | a \in A \rangle$  is a Hopf ideal in  $\mathcal{H}$ .
- $\mathcal{H}_{(i)} \coloneqq \mathcal{H}/\langle s t \rangle$  is a commutative Hopf *A*-algebra.

▶ The presheaf of groupoids  $(\mathscr{G}_A, \mathscr{G}_{\mathcal{H}}^{(i)})$  obtained by taking the isotropy groupoid of  $(\mathscr{G}_A(R), \mathscr{G}_{\mathcal{H}}(R))$  for every R in CAlg<sub>k</sub> is represented by  $\mathcal{H}_{(i)}$ :

$$\left(\mathscr{G}_{\mathcal{A}},\mathscr{G}_{\mathcal{H}}^{(i)}
ight)\cong \left(\mathscr{G}_{\mathcal{A}},\mathscr{G}_{\mathcal{H}_{(i)}}
ight).$$

## Normal Hopf ideals and normal subgroupoids

 $\blacktriangleright$   $\mathcal{H}_{(i)}$  is a right  $\mathcal{H}$ -comodule algebra with coaction defined by

$$\delta_{\mathcal{H}_{(i)}} \colon \mathcal{H}_{(i)} o \mathcal{H}_{(i)} \otimes_A \mathcal{H}, \qquad \overline{h} \mapsto \overline{h_2} \otimes_A \mathcal{S}(h_1)h_3.$$

A Hopf ideal *I* of  $\mathcal{H}$  is said to be *normal* if  $\langle s - t \rangle \subseteq I$  and for all  $\overline{x}$  in  $I_{(i)} := I/\langle s - t \rangle$ , we have

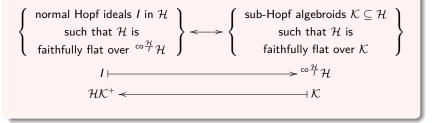
$$\delta_{\mathcal{H}_{(i)}}(\overline{x}) = \overline{x_2} \otimes_{\mathcal{A}} \mathcal{S}(x_1) x_3 \in \operatorname{im} \left( I_{(i)} \otimes_{\mathcal{A}} \mathcal{H} \right).$$

▶ *I* is a normal Hopf ideal of  $(A, \mathcal{H})$  if and only if  $(\mathcal{G}_A, \mathcal{G}_{\mathcal{H}/I})$  is a normal subgroupoid of  $(\mathcal{G}_A, \mathcal{G}_{\mathcal{H}})$ .

## The main result in the commutative setting

#### Theorem

If  $(A, \mathcal{H})$  is a commutative Hopf algebroid such that  ${}_{s}\mathcal{H}$  is flat, then we have a well-defined inclusion-preserving bijective correspondence



#### Proposition

Let  $\Bbbk = \overline{\Bbbk}$  and let  $(A, \mathcal{K})$  be a sub-Hopf algebroid of  $(A, \mathcal{H})$  such that  $\mathcal{H}$  is faithfully flat over  $\mathcal{K}$ . Then  $\Theta \colon \mathsf{CAlg}_{\Bbbk}(\mathcal{H}, \Bbbk) \to \mathsf{CAlg}_{\Bbbk}(\mathcal{K}, \Bbbk)$  is surjective. In particular, in the setting of the theorem there are canonical isos

 $\mathscr{G}_{\mathcal{H}}(\Bbbk)/\mathscr{G}_{\mathcal{H}/\mathcal{H}\mathcal{K}^+}(\Bbbk) \;\cong\; \mathscr{G}_{\mathcal{K}}(\Bbbk) \quad \text{and} \quad \mathscr{G}_{\mathcal{H}}(\Bbbk)/\mathscr{G}_{\mathcal{H}/I}(\Bbbk) \cong \mathscr{G}_{^{\mathrm{co}\mathcal{H}/I}\mathcal{H}}(\Bbbk) \,.$ 

## The end

## Thank you

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