INTRODUCTION TO DEFINITE CONNECTIONS

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1. Outline

The aim of this talk is to give some gentle introduction to the subject of definite connections based upon the work of Fine–Krasnov–Panov [**1**], [**2**] and [**3**]. I will mainly focus on the first few pages of the recent preprint [**3**]

A gauge theoretic approach to Einstein 4-manifolds

which is also the starting point for the problems I study in my PhD thesis.

As the framework of definite connections makes abundant use of the language of gauge theory, I first want to illustrate by means of electromagnetism and Yang–Mills theory some rudimental aspects of gauge theory. Then I explain how to apply the same ideas to Einstein's equations, thus entering the core of the subject.

2. Electromagnetism

When studying electromagnetism one is interested in the *electromagnetic field*. Indeed, the field-strength tensor is the "observable quantity" which allows us to describe the motion of a charged particle. Nevertheless, it turns out to be a good idea to first look at an auxiliary quantity, the *electromagnetic potential*.

2.1. Construction of the electromagnetic potential. Denote the electric field strength by $\vec{E}(t,\vec{x})$ and the magnetic field strength by $\vec{B}(t,\vec{x})$. The dimension of the total electromagnetic field $(\vec{E},\vec{B})$ is $3+3=6=\binom{4}{2}$. The two homogeneous Maxwell equations (Maxwell–Faraday and Gauss's Law of Magnetism) can be written, assuming $c=1$, as

$$\begin{cases} \vec{\nabla}\times\vec{E}+\frac{\partial}{\partial t}\vec{B} & = & 0 \\ \vec{\nabla}\cdot\vec{B} & = & 0. \end{cases}$$

Poincaré's Lemma roughly states that (on a simply-connected spacetime) curl-free vector fields are gradients and divergence-free vector fields are curls. The second equation then gives existence of a vector field $\vec{A}(t,\vec{x})$, called *vector potential field* (or magnetic potential) such that $\vec{B}=\vec{\nabla}\times\vec{A}$. If we inject this into the first equation we get

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial}{\partial t}\vec{A}\right)=0.$$

Applying Poincaré once more we get that there exists a scalar field $\Phi(t,\vec{x})$, called *scalar potential field* (or electric potential), such that $\vec{E}+\frac{\partial}{\partial t}\vec{A}=\vec{\nabla}\Phi(t,\vec{x})$.

We conclude that for a given field configuration $(\vec{E}, \vec{B})$ we can find an associated potential configuration $(\Phi, \vec{A})$.

This potential is not unique. Indeed, for any scalar function $\chi(t, \vec{x})$ we can apply the transformation

$$\begin{cases} \Phi' &= \Phi + \frac{\partial}{\partial t}\chi \\ \vec{A}' &= \vec{A} + \vec{\nabla}\chi. \end{cases}$$

This clearly changes the potential, but it does not affect the electromagnetic field:

$$\begin{cases} \vec{E}' &= \vec{\nabla}\Phi + \vec{\nabla}\frac{\partial}{\partial t}\chi - \frac{\partial}{\partial t}\vec{A} - \frac{\partial}{\partial t}\vec{\nabla}\chi &= \vec{E} \\ \vec{B}' &= \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{\nabla}\chi &= \vec{B}. \end{cases}$$

We call such a transformation a *gauge transformation*. As χ can be an arbitrary function in space-time (think for example of a bump function), we say that the gauge transformation is *local*.

2.2. The mathematical point of view. If we pass from physics to mathematics, we make no more distinction between space and time or between electro and magnetism. So we take as space-time a 4-dimensional (Riemannian) manifold.

Define the electromagnetic 4-potential as the 1-form $A := \Phi dt + A_i dx_i$. In terms of differential forms, the two homogeneous Maxwell equations can be expressed as

$$\begin{cases} E &= (\partial_i\Phi - \partial_t A_i) \quad dx_i \wedge dt \\ B &= (\partial_i A_j - \partial_j A_i) \quad dx_i \wedge dx_j. \end{cases}$$

We define a 2-form $F := \frac{1}{2}F_{\mu\nu}\ dx_\mu \wedge dx_\nu$ with the skew-symmetric electromagnetic field tensor $F_{\mu\nu}$ completely determined by the 2-forms E and B. Moreover, a quick calculation shows that $F = dA$.

On the other hand, we can consider the 1-form A to be the connection 1-form of a $U(1)$-connection. The curvature of a connection on a principal bundle is given by the second structure equation as $dA + A \wedge A$. As the Lie algebra of $U(1)$ is abelian, the curvature of the connection is simply the exterior derivative of A and thus coincides with the above defined 2-form F.

We deduce that the two homogeneous Maxwell equations can be written under the compact form

$$dF = 0.$$

In this setup a gauge transformation writes as

$$A_\mu \to A'_\mu := A_\mu + \partial_\mu\chi$$

for some smooth scalar function χ on the space-time 4-manifold.

2.3. Conclusion. In a differential geometric language, the electromagnetic field F can be interpreted as the curvature of the 4-potential A, which is a $U(1)$-connection. The second structure equation tells us that the curvature 2-form (also called *Faraday 2-form*) is

$$F = dA + A \wedge A.$$

We have seen that the two homogeneous Maxwell equations reduce to the Bianchi identity

$$dF = d^2 A = 0.$$

Similarly, one could prove that the two inhomogeneous Maxwell equations reduce to the source equation

$$d^* F = J$$

where J is a current 3-form built out of the electric charge and the magnetic current. Note that the conservation of current is expressed here through the continuity equation $dJ = 0$ which is implied through $d^2 = 0$.

It is worth pointing out that the Bianchi identity only depends on the topology of the space-time 4-manifold, whereas the two inhomogeneous Maxwell equations involve the Hodge star operator and thus depend on the metric, or more precisely on the conformal class of the metric.

Finally, note that if there is no charge in the game, i.e. $J = 0$, then the Maxwell equations become

$$dF = d^* F = 0.$$

In this case, the Faraday 2-form is harmonic.

3. Yang–Mills theory

Instead of playing with a $U(1)$-bundle, we now consider a $U(n)$-bundle E over a Riemannian manifold (M, g). Take A a compatible connection in E i.e. a unitary connection in the Hermitian vector bundle E. Then its curvature F_A is given by the second structure equation

$$F_A = dA + A \wedge A.$$

If $n > 1$, the term $A \wedge A$ no longer disappears. The gauge theory is said to be *non-abelian* and the curvature becomes non-linear in the connection.

We can build an interesting functional out of the curvature of the connection. Define the *Yang–Mills energy* (or *Yang–Mills action*) to be the L_2-norm of the curvature

$$S_{YM}(A) := \int_M |F_A|^2 \; dvol(g)$$

which is well-defined for M compact. The critical points of this action are called *Yang–Mills connections* and play an important role in physics.

Proposition 1. *The Yang–Mills connections are exactly the solutions of the Yang–Mills equation*

$$d_A^* F_A = 0.$$

The proof is standard. Nevertheless I include it because we will stumble upon some similar calculation when talking about definite connections.

Proof. Consider the directional derivative of S_{YM} in the direction τ. Then we need

$$\frac{d}{dt} \left(S_{YM}(A + t\tau) \right) |_{t=0} = 0$$

for all τ in order to have a critical point. A quick calculation gives

$$F_{A+t\tau} = F_A + t d_A \tau + \frac{1}{2} t^2 [\tau, \tau]$$

where $d_A \tau = d\tau + [A, \tau]$ and so we obtain

$$|F_{A+t\tau}|^2 = |F_A|^2 + 2t \langle d_A \tau, F_A \rangle + o(t^2).$$

After derivation and evaluation in $t = 0$ only the linear term in t remains and we are left with

$$2 \int_M \langle d_A \tau, F_A \rangle \ dvol(g) = 0$$

for all τ, which is equivalent to $d_A^* F_A = 0$. q.e.d.

Remark that the equation $d_A F_A = 0$ is automatically satisfied by the Bianchi identity. Consequently, we may interpret the Yang–Mills equation as some "twisted version" of Maxwell's equations, saying that A has "harmonic curvature". Note moreover that, as the curvature is first order in the connection, the Yang–Mills equation is a second order PDE in A which is non-linear for any $n > 1$.

Up to now, the reasoning did not depend on the dimension of the base manifold M. However, if the dimension of M happens to be four, then we can apply some special linear algebraic features of dimension 4 to obtain more refined results. Suppose (M^4, g) is an oriented Riemannian 4-manifold. Then the Hodge star operator sends 2-forms to 2-forms

$$\star : \Lambda^2 \to \Lambda^2 \text{ with } \star^2 = 1.$$

This implies that we can split a 2-form in a *self-dual* part (the $(+1)$-eigenspace of the Hodge star operator) and an *anti-self-dual* part (the (-1)-eigenspace of $\star$). In particular, as the curvature is a 2-form, F_A splits in two pieces.

$$\begin{aligned} S_{YM}(A) &= \textstyle\int_M |F_A|^2 \ dvol(g) \\ &= \textstyle\int_M tr(F_A \wedge \star F_A) \\ &= \textstyle\int_M \left(|F_A^+|^2 + |F_A^-|^2 \right) \ dvol(g) \\ &\geq \textstyle\int_M \left(|F_A^+|^2 - |F_A^-|^2 \right) \ dvol(g) \\ &= \textstyle\int_M tr(F_A \wedge F_A) \\ &= k \cdot p_1(E) \end{aligned}$$

The last equality is true by Chern–Weil theory for some constant k. Roughly speaking, $F_A \wedge F_A$ is a 4-form with values in endomorphisms of the bundle E. The trace then applies to the endomorphism part, spitting out a number which itself does not depend on the connection but only on the bundle. It turns out that this number is a multiple of the first Pontrjagin class of the bundle E.

We conclude that in dimension 4, there is a topological a priori lower bound on the Yang–Mills action

$$S_{YM}(A) \geq k \cdot p_1(E)$$

and that equality is attained precisely when $F_A^- = 0$. A connection A such that $F_A^- = 0$ is called an *instanton*. The equation $F_A^- = 0$ is a first order PDE in A. It is interesting to note that any instanton is automatically a

Yang–Mills connection as any minimum is a critical point. In other words, the first order equation $F_A^- = 0$ implies the second order equation $d_A^* F_A = 0$. Of course, this can also be checked explicitly: if $F_A^- = 0$, then $\star F_A = F_A$ and so $d_A^* F_A = -\star d_A \star F_A = -\star d_A F_A = 0$ by the Bianchi identity.

4. Definite connections

There are some formal similarities between Yang–Mills theory and the theory of definite connections. I find it instructive to aboard the subject by means of the following table which gives the equivalent notions of the main objects in both theories.

	Yang–Mills theory	Theory of definite connections
auxiliary object	$U(n)$-connections	definite $SO(3)$-connections
object of interest	curvature F_A	metric $g_A = (\bar{g_A}, \nu_A)$
functional	Yang–Mills action	Krasnov action
critical points	Yang–Mills connections	Einstein metrics
minima/ maxima	instantons	anti-self-dual Einstein metrics
topological bound	lower bound $p_1(E)$	upper bound $p_1(E)$
geometric flow	Yang–Mills flow	Levi–Civita flow / Krasnov flow

In Yang–Mills theory, passing from the auxiliary object to the object of interest is easy because the curvature of the connection A is given by the second structure equation. However, passing from a connection to a Riemannian metric is more delicate. This leads to a first question which we treat after reviewing some properties of 4-dimensional geometry.

Question 2. *How to associate a Riemannian metric g_A to a (definite) $SO(3)$-connection A?*

4.1. An excursion in 4-dimensional geometry. Let (M^4, g) be an oriented Riemannian 4-manifold. Then there is a unique torsion-free metric connection on the tangent bundle, the Levi–Civita connection ∇.

The conformal class $\bar{g}$ of the metric g induces through the Hodge star operator a splitting of 2-forms

$$\Lambda^2 = \Lambda^+(\bar{g}) \oplus \Lambda^-(\bar{g}).$$

In other words, it determines a 3-dimensional subbundle $\Lambda^+(\bar{g})$ of the 6-dimensional bundle Λ^2. Moreover, by construction, the wedge product is positive definite on Λ^+ and negative definite on Λ^-, so $\alpha \wedge \alpha > 0$ for all $\alpha \in \Lambda^+$. I call a 3-dimensional subbundle $V \subset \Lambda^2$ a *definite bundle* if it satisfies the algebraic property $\alpha \wedge \alpha > 0$ for all $\alpha \in V$. To resume, the choice of a conformal class $\bar{g}$ fixes a definite bundle $\Lambda^+(\bar{g}) \subset \Lambda^2$.

The Levi–Civita connection ∇ on the tangent bundle induces another metric connection, also denoted by ∇, on $\Lambda^+(\bar{g})$. To any vector in $\mathbb{R}^3$,we can associate a rotation around the vector. So we deduce that ∇ on Λ^+ is an $SO(3)$-connection.

For future use, we would like to single out the induced Levi–Civita connection on Λ^+ from other metric connections. For a connection B on Λ^+ we can consider the following sequence of maps

$$\Gamma(\Lambda^+) \xrightarrow{B} \Gamma(\Lambda^1 \otimes \Lambda^+) \xrightarrow{\sigma} \Gamma(\Lambda^3)$$

where σ stands for skew-symmetrization. The torsion $\tau \in \Lambda^3$ of B is defined as

$$\tau(B) := \sigma \circ B - d.$$

It can be checked ([**2**], Proposition 2.3) that the induced Levi–Civita connection on Λ^+ is the unique torsion-free metric connection on Λ^+.

4.2. Associate a conformal class to a definite connection.

Proposition 3. *Let V be a definite bundle. Then there exists a unique conformal class of metric $\bar{g}$ such that $V = \Lambda^+(\bar{g})$.*

Proof. The space of all conformal structures is

$$\frac{GL(4,\mathbb{R})}{\mathbb{R}_+ \times SO(4)} \cong \frac{SL(4,\mathbb{R})}{SO(4)}.$$

We would like this to be isomorphic to the set of definite bundles. A nice proof of this can be found in J. Distexhe's master thesis ([**4**], Chapter 1.3). We present here the main ideas.

Define $SO(3,3) := \{g \in SL(4,\mathbb{R}) \mid g^t Q g = Q\}$ for Q a non-degenerate quadratic form of signature $(3,3)$ and denote by $SO_0(3,3)$ its connected component of the identity. Then it can be shown that $SL(4,\mathbb{R})$ double covers $SO_0(3,3)$ and that $SO(4)$ double covers $SO(3) \times SO(3)$. Moreover, both $SO(4)$ and $SO(3) \times SO(3)$ are maximal compact subgroups of $SL(4,\mathbb{R})$ and $SO_0(3,3)$ respectively. We deduce that we get another isomorphism

$$\frac{SL(4,\mathbb{R})}{SO(4)} \cong \frac{SO_0(3,3)}{SO(3) \times SO(3)}.$$

As $SO_0(3,3)$ leaves Q invariant, the space $\frac{SO_0(3,3)}{SO(3)\times SO(3)}$ corresponds precisely to the set of all 3-dimensional subspaces of Λ^2 on which Q is positive definite, i.e. to the set of definite bundles. q.e.d.

Let E be an $SO(3)$-bundle over the 4-manifold M and let A be an $SO(3)$-connection on E. Locally we can write

$$F_A = \sum_i F_i \otimes e_i$$

where $\{F_i\}$ is a triple of 2-forms and $\{e_i\}$ a local frame of the Lie algebra of E. Clearly, $\langle F_i \rangle$ is 3-dimensional subbundle of Λ^2 and we can ask whether it is definite.

Definition 4. An $SO(3)$-connection A over M is definite if the span generated by the triple of curvature 2-forms $\langle F_i \rangle \subset \Lambda^2$ is a definite bundle.

In other words, if we write

$$F_i \wedge F_J = M_{ij}\nu$$

for some arbitrary volume form ν, then A is definite if and only if M_{ij} is a positive definite 3 by 3 matrix (hence the name).

We conclude by Proposition 3 that we can associate to any definite connection A a unique conformal class $\bar{g_A}$ such that $\langle F_i \rangle = \Lambda^+(\bar{g_A})$, i.e. such that A is an instanton for $\bar{g_A}$.

4.3. Associate a volume form to a definite connection. There are different possibilities depending on what we are interested in (Einstein metrics, anti-self-dual Einstein metrics, ...). I will explain the choice of volume form one should make if one wants to hit Einstein metrics.

Suppose g is an Einstein metric, i.e. that $Ric(g) = \Lambda g$. If we take the trace on both sides, we see immediately that the scalar curvature is constant and equals four times the cosmological constant Λ.

The first column of the Riemann curvature tensor gives the curvature of the Levi–Civita connection ∇ on $\Lambda^+(g)$. Moreover, as it is an instanton the part $\Lambda^+ \to \Lambda^-$ vanishes and we are left with the upper left block which can be shown to be $W^+ + \frac{s}{4}$. Consequently, the trace of the curvature is constant and equals $\frac{s}{4} = \Lambda$.

On the other hand, $F_i \wedge F_j = M_{ij}\nu$ so $|F_\nabla|^2 = M$. We conclude that, with the correct sign choice,

$$tr(\sqrt{M}) = |\Lambda|.$$

The scale of M fixes the volume form and conversely. This leads to the following choice: the volume form ν_A of a definite connection A is defined such that $tr(\sqrt{M}) = |\Lambda|$.

4.4. Einstein metrics.

Question 5. *What condition do we need to impose on a definite connection A to assure that the associated metric g_A is Einstein?*

By definition, a definite connection A is an instanton for the associated metric g_A. On the other hand, if the Levi–Civita connection on $\Lambda^+(g_A)$ is an instanton, then the metric g_A is Einstein. We deduce that, naïvely, we would like to have A to be the Levi–Civita connection of $\Lambda^+(g_A)$. The problem is that A is a connection on the $SO(3)$-bundle E, and not on $\Lambda^+(g_A)$. So we first need to identify the bundles E and $\Lambda^+(g_A)$.

Recall that for a definite connection A, the curvature $F_A = \sum F_i \otimes e_i$ is such that the triple of 2-forms is self-dual, $F_i \in \Lambda^+(g_A)$. Also the local frame $\{e_i\}$ is a frame for the Lie algebra of E which can be identified with E. In other words, F_A gives an identification (isomorphism) between E and $\Lambda^+(g_A)$. However, in general, we can not use this to push the connection A to a metric connection on $\Lambda^+(g_A)$ because F_A is not an isometry. Indeed, the F_i are not orthonormal because $F_i \wedge F_j = M_{ij}\nu$ and M_{ij} need not be a multiple of the identity. So in order to get an isometric identification, we need to twist the curvature F_A.

Define

$$\Phi_A := \sum_i \varphi_i \otimes e_i \quad \text{with} \quad \varphi_i := \sum_j (M^{-\frac{1}{2}})_{ij} F_j.$$

Then $\Phi_A = M^{-\frac{1}{2}} F_A$ gives an isometric identification of E and $\Lambda^+(g_A)$. This implies that $\Phi_{A*}A$ is a metric connection on $\Lambda^+(g_A)$.

We have already seen that the Levi–Civita connection is the unique metric connection on $\Lambda^+(g_A)$ without torsion. Consequently we need to solve the equation $\tau(\Phi_{A*}A) = 0$.

Intuitively, the torsion belongs to $\Lambda^3 \otimes \Lambda^+(g_A)$. On the other hand, the coupled exterior derivative $d_A\Phi_A$ belongs to $\Lambda^3 \otimes E$ and the bundles E and $\Lambda^+(g_A)$ are isometrically identified via Φ_A. One may hope to replace the vanishing torsion condition by the easier to manipulate condition $d_A\Phi_A = 0$. The proof of this statement can be found in [**3**], Theorem 2.6. Accepting this, we get the first part of the following Proposition.

Proposition 6. *Let (M^4, g) be an oriented Riemannian 4-manifold with a definite connection A.*

1) *If $d_A\Phi_A = 0$, then g_A is an Einstein metric.*
2) *If $M = k \cdot Id$ then g_A is an anti-self-dual Einstein metric.*

For the second point, note that, if the endomorphism M is a multiple of the identity, then F_A gives an isometric identification between E and $\Lambda^+(g_A)$. So $\Phi_A = F_A$ and the equation $d_A\Phi_A = 0$ is automatically satisfied by Bianchi. But in addition to g_A being Einstein, in this case, the trace-free part of the curvature also vanishes. So $W_+ = 0$ and the metric g_A is anti-self-dual.

We conclude that Einstein metrics are obtained as solutions to the second order PDE $d_A\Phi_A = 0$. Moreover Φ_A is self-dual, so this is equivalent to $d_A^*\Phi_A = 0$ and thus we face once more some "twisted Bianchi identity" just like in Yang–Mills theory. Finally, there is a first order PDE $M = k \cdot Id$ which implies the second order PDE. Its solutions are anti-self-dual Einstein metrics, also called *gravitational instantons* in physics.

4.5. A volume functional. Does the theory of definite connections admit some action with similar features than the Yang–Mills action? It turns out that such an action, called *Krasnov action*, is provided by the volume functional

$$S(A) := \int_M \nu_A.$$

The following Proposition (Theorem 2.9 and Proposition 2.10 in [**3**]) then becomes a routine calculation which we include for the sake of completeness.

Proposition 7. *Let (M^4, g) be an oriented Riemannian 4-manifold with a definite connection A on the $SO(3)$-bundle E and the volume functional S defined as above.*

1) *A is a critical point of S if and only if* $d_A\Phi_A = 0$.
2) $S(A) \leq k \cdot p_1(E)$ *with equality if and only if* M_A *is a multiple of the identity.*

Proof. For the first part, consider the change in the connection $A \to A+ta$. To get the Euler–Lagrange equation, we need to solve

$$\frac{d}{dt}\left(\nu_{A+ta}\right)_{t=0} = 0$$

for all a, i.e. $\nu_a = 0$.

For this we start with the identity

$$M_{A+ta}\ \nu_{A+ta} = F_{A+ta} \wedge F_{A+ta}.$$

Only the linear term in t survives the differentiation and evaluation in $t = 0$:

$$M_A\nu_a + M_a\nu_A = F_A \wedge F_a + F_a \wedge F_A.$$

If we multiply all the terms by $M_A^{-\frac{1}{2}}$ on the left and take the trace we get

$$tr(M_A^{\frac{1}{2}})\nu_a + tr(M_A^{-\frac{1}{2}}M_a)\nu_A = 2(M_A^{-\frac{1}{2}}F_A, d_Aa)\nu_A$$

because F_A is self-dual and $F_a = d_Aa$. Moreover, we know that $tr(M_A^{\frac{1}{2}}) = \Lambda$ which is constant so its derivative $tr(M_A^{-\frac{1}{2}}M_a)$ vanishes. We conclude that

$$\nu_a = \frac{2}{\Lambda}(\Phi_A, d_Aa)\nu_A.$$

Finally, as Φ_A is self-dual, this vanishes for all a if and only if $0 = d_A^*\Phi_A = - \star d_A\Phi_A$ as announced.

For the second part, start again with

$$M_A\ \nu_A = F_i \wedge F_j = M_\nu\ \nu.$$

If we take the square root on both sides and then apply the trace we get

$$\nu_A = \frac{1}{\Lambda^2}\left(tr(\sqrt{M_\nu})\right)^2 \nu.$$

By Cauchy–Schwarz, the self-adjoint positive definite matrix M_ν satisfies

$$\left(tr(\sqrt{M_\nu})\right)^2 \leq tr(Id_3) \cdot tr(M_\nu) = 3\ tr(M_\nu)$$

with equality if and only if M_ν and the identity matrix are colinear. We conclude that we can find once more a topological bound for the action in terms of the Pontrjagin class of the bundle

$$S(A) = \int_M \nu_A \leq \frac{3}{\Lambda^2}\int_M tr(M_\nu)\nu = \frac{3}{\Lambda^2}\int_M tr(F_A \wedge F_A) = k \cdot p_1(E)$$

for some constant k which could be evaluated by Chern–Weil theory. The upper bound is attained precisely when Cauchy–Schwarz is saturated, i.e. when M_ν (and thus also M_A) is a multiple of the identity. q.e.d.

This leads to the nice conclusion:

The critical points of the volume functional give rise to Einstein metrics. The maxima of the volume functional induce anti-self-dual Einstein metrics.

References

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