Introduction to Supersymmetry
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Chapter 1

Introduction

Disclaimer (2009): These notes are being written in the beginning of 2009. LHC should start running hopefully within a year. Then, it is likely that some data concerning supersymmetry will soon start to be available. The most favourable outcome is a Nobel prize awarded already in 2011. Then there would be little need to motivate further the subject of these lectures. On the other hand the outcome could be the opposite, ruling out supersymmetry in the (accessible) real world. Then these lectures are concerned with a beautiful mathematical construction which allows us to better understand the quantum properties of field theories. Of course, it is most likely that the real outcome of LHC will be just half way between the two options above..

Disclaimer (2015): Well, it turned out to be, of course, in the middle. SUSY is not ruled out, but its most straightforward versions are having a hard time, in particular to justify their usefulness.. Stay tuned!

1.1 Motivating supersymmetry

Supersymmetry is an organizing principle for quantum field theories which addresses both theoretical and technical aspects of them.

In all generality, symmetries (both internal and space-time) implement constraints on the structure of the theory, and on its quantum corrections. In particular, they help in answering the following questions:

- Why is the spectrum as it is?
- Why are there some couplings, while others are vanishing?

- Why some quantities are much smaller than others (i.e. there are *hierarchies*)?

Supersymmetry is a very powerful symmetry, which extends the usual Poincaré space-time symmetry. In this sense it is more powerful than a global internal symmetry. Particles are commonly divided into bosons (of integer spin, such as scalars and vectors) and fermions (of half-integer spin, which herein will always be spin 1/2). Supersymmetry (also called SUSY) relates these two kinds of particles. It must then mix non-trivially with the Poincaré space-time symmetry since it relates particles which belong to different representations of the Lorentz group, since the latter are denoted by their spin.

As we will see, SUSY implements strong constraints on the spectrum and on the couplings of a field theory. Thus, it goes towards the goal of formulating a *unique* theory of all interactions, where everything is constrained and nothing is left to (arbitrary) choice. Actually, it goes very close to this aim when gravity is taken into account and the theory becomes the one of *supergravity*. Moreover, if one wants a setting in which gravity is consistently quantized, one has to resort to string theory. It is amusing to note that in turn, string theories are fully consistent only when supersymmetry is present–one then talks of *superstring theory*. Actually, it is really in this context that SUSY appeared first in physics, during the early 70s.

At a more technical level, SUSY helps also in addressing the question of quantum corrections and hierarchies, which is related to the notion of *naturalness*, as opposite to *fine tuning*: One wants the parameters in the theory describing Nature to be close to generic values. This is intimately related to the symmetry in the spectrum relating bosons and fermions.

Take e.g. the vacuum energy in the most simple quantum mechanical model. For a bosonic oscillator, we have

$$H_B = \frac{1}{2}(a^\dagger a + a a^\dagger).$$

At the quantum level, i.e. when $[a, a^\dagger] = \hbar$, we have that the vacuum state, defined by $a|0\rangle = 0$, is such that

$$H_B|0\rangle = \frac{1}{2}aa^\dagger|0\rangle = \frac{1}{2}(a^\dagger a + \hbar)|0\rangle = \frac{1}{2}\hbar|0\rangle$$

so that the vacuum energy is

$$E_{vac,B} = \frac{1}{2}\hbar.$$

In field theory, we basically have oscillators at every space-time point or, in other words (and after a Fourier transform), of every frequency. Summing over all of them will give a very large vacuum energy density, roughly proportional to the 4th power of the cut off scale beyond which the theory is no longer well defined, usually taken to be the Plack scale M_p.[1]

Now consider a fermionic oscillator, with Hamiltonian and anti-commutation relations as follow

$$H_F = \frac{1}{2}(\alpha^\dagger \alpha - \alpha \alpha^\dagger), \qquad \{\alpha, \alpha^\dagger\} = \hbar.$$

If the vacuum is defined by $\alpha|0\rangle = 0$, we get

$$H_F|0\rangle = -\frac{1}{2}\alpha\alpha^\dagger|0\rangle = -\frac{1}{2}(\hbar - \alpha^\dagger\alpha)|0\rangle = -\frac{1}{2}\hbar|0\rangle$$

so that the vacuum energy is now

$$E_{vac,F} = -\frac{1}{2}\hbar.$$

It is negative.[2]

So, if for every frequency we have

$$H_{tot} = H_B + H_F = a^\dagger a + \alpha^\dagger \alpha$$

then

$$E_{tot} = 0$$

and the total energy density vanishes.

This is one instance of the simplifications due to SUSY, and also an example of how a symmetry (implicit but present in the above) can constrain a quantity such as the vacuum energy. The lesson is that *quantum corrections are sometimes vanishing*, or at least more constrained. As we will see, this is also true for radiative corrections, for instance to the mass of particles.

One vexing problem of the Standard Model (SM) is the hierarchy between the scale of Electro-weak interactions $M_{EW} \sim 100$ GeV and the Planck scale

[1] Indeed, trading energy scales for length scales, it makes no sense to consider any quantum field theory at distances shorter than the Planck length ($L_p = M_p^{-1}$ in the relevant units), where quantum gravity effects affect the notion of space-time itself.

[2] Note that $H_F\alpha^\dagger|0\rangle = (\alpha^\dagger\alpha - \frac{1}{2}\hbar)\alpha^\dagger|0\rangle = (\alpha^\dagger\hbar - \frac{1}{2}\hbar\alpha^\dagger)|0\rangle = \frac{1}{2}\hbar\alpha^\dagger|0\rangle$ so that indeed $|0\rangle$ is the ground state.

$M_p \sim 10^{19}$ GeV. Roughly speaking, and without entering into the details, the Electro-weak scale depends in particular on a parameter of the SM, the Higgs scalar mass, which has to be fine tuned by many orders of magnitude (with respect to M_p, of course) in order to give the experimentally observed scale. Even more troubling, quantum corrections tend to restore this mass to its "natural" scale M_p, due to quadratic divergencies. Namely, there are for instance one-loop corrections to the Higgs scalar two-point function, with quarks and leptons running in the loop. If H is the Higgs scalar, $\psi, \bar{\psi}$ are such representative fermions, and there is an interaction term $\mathcal{L}_{SM} \supset \lambda H \psi \bar{\psi}$, then the leading UV divergence in the two point function of H is proportional to

$$\lambda^2 \int^{\Lambda_{UV}} d^4p \frac{1}{p} \frac{1}{p-k} \sim \lambda^2 \Lambda_{UV}^2,$$

where p is the momentum running in the loop, k the incoming momentum and Λ_{UV} the UV cut-off of the theory.

Supersymmetry comes to the rescue by adding, for every fermion, a new scalar particle, call it S_ψ, along with a new vertex involving H, $\mathcal{L}_{SM}^{SUSY} \supset \mu H^2 S_\psi^2$. There is then a new one-loop correction to the two point function for H, with S_ψ running in the loop. It also has a quadratic divergence, proportional to:

$$\mu \int^{\Lambda_{UV}} d^4p \frac{1}{p^2} \sim \mu \Lambda_{UV}^2.$$

It is easy to conceive that by tuning the couplings $\mu \propto \lambda^2$, the two leading corrections can cancel. This is exactly what supersymmetry does!

So, SUSY can help in maintaining a hierarchy, protecting it from quantum corrections.

Explaining the hierarchy is on the other hand more involved. Let us just mention a natural way to generate large hierarchies in quantum field theory. Consider for instance a scale generated dynamically by dimensional transmutation along a renormalization group flow. This is the case for non-abelian gauge theories, where the one-loop beta function for the gauge coupling g reads

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\frac{b_0}{16\pi^2} g^3 + \mathcal{O}(g^5).$$

We have that for asymptotically free theories, such as QCD, $b_0 > 0$ and the above differential equation can be integrated so that

$$\frac{8\pi^2}{g^2(\mu)} = \frac{8\pi^2}{g^2(\Lambda_{UV})} + b_0 \log \frac{\mu}{\Lambda_{UV}}.$$

A dynamical scale is obtained by running downwards the above expression until the coupling becomes infinite. Reversing then the equation, we obtain that the scale defined in such a way is given by

$$\Lambda_{dyn} = \Lambda_{UV} e^{-\frac{8\pi^2}{b_0 g^2(\Lambda_{UV})}}.$$

The coefficient b_0 is of the order of the rank of the gauge group, while $g(\Lambda_{UV})$ can be naturally taken to be < 1 and reasonably small. Then the exponential factor in the expression above can be very small, even 10^{-15} to mention one interesting ratio.

The upshot is that some hierarchies, like the one between Λ_{dyn} and Λ_{UV}, can be naturally explained. For instance, the dynamical scale of QCD $\Lambda_{QCD} \sim 250$ MeV gives the order of magnitude of the masses of hadrons and baryons, and the hierarchy between Λ_{QCD} and M_p is explained by the asymptotic freedom of QCD. Thus, for SUSY to *explain* the hierarchy problem in the Electro-weak sector, the mass of the Higgs scalar must be related to a dynamically generated scale. We will see shortly how SUSY can be associated to such a scale.

Another, independent, motivation for supersymmetry is found in the phenomenological arena, and has to do with the idea of Grand Unification (a.k.a. GUT, for *Grand Unified Theory*). The idea is that the gauge group of the Standard Model is embedded in a simple gauge group:

$$G_{SM} = SU(3) \times SU(2) \times U(1) \subset G_{GUT}$$

One easily realizes that the two smallest simple groups are:

$$G_{GUT} = SU(5), SO(10), \ldots$$

Of course any larger group containing the above is also suitable, but the above groups are the ones yielding a minimal extension of SM physics. (In particular, matter representations of G_{GUT} split up into just the matter representations of G_{SM}.)

A necessary condition for unification to take place, is that the breaking of the gauge symmetry from G_{GUT} to G_{SM} is consistent. From first principles, it has to happen in the following way. In the UV, there is only one gauge group, and hence a single coupling, which runs as prescribed by its beta function (one assumes that the GUT is asymptotically free). Going towards the IR, at some scale M_{GUT} the gauge group G_{GUT} is broken to G_{SM} by the Brout-Englert-Higgs mechanism (just as $SU(2) \times U(1)_Y$ is broken to $U(1)_{EM}$ at M_{EW}). At this stage, the couplings of the 3 groups in G_{SM} start running independently, according to each one's beta function. However, it is important to note that they share the *same boundary conditions* at M_{GUT}. This is referred to as *gauge coupling unification*. Hence, in order for the GUT idea to work, one has to take the observed values of the gauge couplings in G_{SM} at, say, the Electro-weak scale M_{EW}, and run them upwards using their beta functions. Doing this will draw three lines[3] in the plot of $1/g_i^2$ against $\log \mu$. GUT is then an acceptable idea if the three lines meet.

Now, if one uses the beta functions of the SM (which implies conjecturing that there is no new physics from the M_{EW} scale onwards to a putative M_{GUT} scale, since new heavy charged particles would contribute to the beta functions at scales roughly higher than their mass), the three lines come close but actually miss. Now three lines in a plane do not have to meet at a common point, and do not even have to come close to that. Hence this close miss does not rule out completely GUTs, at least at the level of wishful thinking. But SUSY rescues the idea altogether. Indeed, using the spectrum of the minimal SUSY extension of the SM, called the MSSM (to be defined later..), which means that a precise set of new particles starts contributing to the beta functions at scales slightly higher than M_{EW} (recall that SUSY has to kick in around M_{EW} in order to address the hierarchy problem), then the three lines do meet! Of course, there are error bars, but the meeting seems indeed reasonable and it predicts a GUT scale of $M_{GUT} \sim 10^{16}$ GeV.

Let us also briefly mention that one last interesting outcome of SUSY is the prediction of the existence of a stable, neutral particle (the lightest supersymmetric particle, or LSP) which is usually a viable candidate for dark matter.

There are also theoretical motivations for studying supersymmetry. In particular, the study of SUSY gauge theories in general (and not only the MSSM) has had many successes. SUSY is such a strong constraint on the structure (both at the classical and quantum levels) of the theory, that is has been possible to obtain results in the strong coupling regime (i.e. at low-

[3] At least at one-loop.

energies for asymptotically free theories, when $E \sim \Lambda_{dyn}$ and $g \to \infty$) that cannot be obtained, for instance, in non-supersymmetric QCD. This motivation, though more remote from observational constraints, is on the other hand somewhat more stable, in that the beautiful mathematical structure of SUSY and its implications for quantum gauge theories are independent on the likelihood of the LHC to work properly..

Until now, we refrained to address one important aspect, that should however be obvious: supersymmetry is not yet observed. Again, LHC aside, the important point here is that the SM, which reproduces so well all the observations done until now (say, below few TeVs), is *not* supersymmetric. In particular, SUSY must relate bosons and fermions, but as we will see it commutes with any gauge group (and, for that matters, also with generic global symmetry groups), so that superpartners must belong to the same representations. It is obvious that bosons and fermions in the SM do not belong to even slightly similar representations. Then, the most pressing question to be addressed is 'how is SUSY broken?'

There are essentially two options:

i) SUSY is broken explicitely, i.e. by non-SUSY terms in the Lagrangian, but in a way that its virtues are not (completely) spoiled. Such breaking is called *soft SUSY breaking*.

ii) SUSY is broken spontaneously (or, equivalently, dynamically, where the two terminologies subtly refer to the breaking being due, respectively, to a classical or a quantum mechanism): the theory is SUSY but the (classical or quantum) vacuum is not. This is somewhat nicer because in formulating the theory we can use all the constraints imposed by SUSY. Many of its consequences are also still valid. Such breaking is called *spontaneous* (or *dynamical*) *SUSY breaking*.

In both cases, it is important to note that there will be a scale associated to the SUSY breaking effects, which we will call M_{SUSY}. It can be for instance the scale of a mass (soft), or of a VEV (spontaneous). For $E > M_{SUSY}$ the theory is SUSY, while for $E < M_{SUSY}$ it is not. In a way, M_{SUSY} will work as an effective cut-off for the corrections which are protected by SUSY.

In particular, it will be phenomenologically important to obtain $M_{SUSY} \ll M_{GUT}, M_p$ so as to protect (and possibly implement) the hierarchy. Naturalness thus seems to point towards dynamical SUSY breaking, where M_{SUSY} is linked to Λ_{dyn} of some gauge group, possibly hidden.

To conclude, the motivation to study supersymmetry should be matched by an equal motivation to study how supersymmetry can be broken.

1.2 Plan of the lecture notes

These lecture notes are organized as follows.

In Chapter 2 we review the superalgebra as an extension of the Poincaré algebra, and its representations, i.e. the supermultiplets containing bosons and fermions.

In Chapter 3 we implement supersymmetry at the level of Lagrangian field theories, and provide simple examples of such theories with and without gauge fields.

In Chapter 4 we turn to formulating manifestly SUSY invariant actions for the same field theories, thus introducing superspace and superfields.

In Chapter 5 we discuss the most general SUSY gauge theory with matter, reviewing its classical properties such as the moduli space of SUSY vacua, and providing the examples of Super-QCD (SQCD) and the MSSM.

In Chapter 6 we review the perturbative quantization of SUSY field theories, discussing radiative corrections and non-renormalization theorems.

In Chapter 7 we discuss various mechanisms of supersymmetry breaking.

1.3 Further reading

It can be useful to list some references, essentially reviews or books, that can help in comprehending the subject of these notes. Most of the references below were indeed used to form the material presented here. Additional references on specific issues may be given at a later stage.

- A. Bilal, “Introduction to supersymmetry,” arXiv:hep-th/0101055.
- P. C. West, “Introduction to supersymmetry and supergravity,” *Singapore, Singapore: World Scientific (1990) 425 p.*
- J. Wess and J. Bagger, “Supersymmetry and supergravity,” *Princeton, USA: Univ. Pr. (1992) 259 p.*
- S. Weinberg, “The quantum theory of fields. Vol. 3: Supersymmetry,” *Cambridge, UK: Univ. Pr. (2000) 419 p.*

- P. Argyres, "Lectures on Supersymmetry," available at `http://www.physics.uc.edu/~argyres/661/index.html`.
- J. Terning, "Modern supersymmetry: Dynamics and duality," *Oxford, UK: Clarendon (2006) 324 p.*
- S. P. Martin, "A Supersymmetry Primer," arXiv:hep-ph/9709356.
- R. Argurio, G. Ferretti and R. Heise, "An introduction to supersymmetric gauge theories and matrix models," Int. J. Mod. Phys. A **19** (2004) 2015 [arXiv:hep-th/0311066].
- K. A. Intriligator and N. Seiberg, "Lectures on Supersymmetry Breaking," Class. Quant. Grav. **24** (2007) S741 [arXiv:hep-ph/0702069].

The references are essentially ordered by closeness in spirit to these notes.

Chapter 2

The superalgebra

In this chapter we introduce the algebraic structure on which supersymmetry is based, that is the superalgebra. Since it is an extension of the space-time Poincaré symmetry, we first review some basic notions on the representations of the Lorentz group. We then present the superalgebra, and derive the first simple physical consequences for the spectrum of a supersymmetric theory. We then construct representations of the superalgebra, also called supermultiplets, both massless and massive. In this chapter, we will also briefly mention *extended* superalgebras, which however will not be dealt with in the rest of the notes.

2.1 A graded extension of the Poincaré algebra

In general, under mild assumptions, the Poincaré algebra cannot mix non-trivially with other symmetries. In other words, it cannot be embedded in a larger symmetry algebra. This is the main conclusion of the Coleman-Mandula theorem of the '60s. In a nutshell, it says that if there are extra space-time symmetries (i.e. extra symmetries mixing non-trivially with the Poincaré generators, which means that they are in non-trivial representations of the Lorentz group), then they would constrain so much the S-matrix that it would necessarily be trivial. It has to be noted that this theorem holds in four dimensions.[1] Of course, a trivial S-matrix means that the theory is free, and no interactions make the theory a rather dull one. Hence, the only extensions which are allowed are the ones by global internal symmetries, i.e. whose generators are spin 0 scalars.

Now, the Coleman-Mandula theorem is true for bosonic symmetries. It has one exception if one considers *graded* extensions of the Poincaré algebra, that is algebras involving even and odd operators such that

$$[E, E] = E, \qquad [E, O] = O, \qquad \{O, O\} = E,$$

with E any even operator, O any odd operator and $\{,\}$ the anticommutator. It then turns out that allowing for odd operators, the Coleman-Mandula theorem is still at work and implies that the only such extra generators one can add have spin 1/2. This is a theorem by Haag, Lopuszanski and Sohnius.

Very schematically, one can introduce a spin-1/2 generator Q which satisfies the following (anti)commutation relations with the momentum P:

$$[P, Q] = 0, \qquad \{Q, Q\} = P.$$

The operator Q being of spin-1/2, it relates states belonging to representations of spin s to ones of spin $s \pm \frac{1}{2}$. By the relation between spin and statistics, it then relates bosons to fermions and vice-versa.

Since fermions, and spinors, are crucial to SUSY, we need to review some facts and notation about representations of the Poincaré group, in order to proceed and be more precise.

2.2 Representations of the Lorentz group

Let us anticipate that we will use consistently 2-component Weyl spinors throughout these notes. Indeed, the usual Dirac spinor is actually a reducible representation of the Lorentz group. This said, we can start reviewing the Poincaré algebra.

The commutation relations are ($\mu = 0, 1, 2, 3$):

$$[P_\mu, P_\nu] = 0 \tag{2.1}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \tag{2.2}$$

$$[M_{\mu\nu}, P_\rho] = -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu \tag{2.3}$$

[1] It has an important exception concerning theories where all degrees of freedom are massless (the space-time symmetry becomes then the conformal group).

Here the generators satisfy:

$$M_{\mu\nu} = -M_{\nu\mu}, \qquad M_{ij}^\dagger = M_{ij}, \qquad M_{0i}^\dagger = -M_{0i}, \qquad P_\mu^\dagger = P_\mu, \tag{2.4}$$

i.e. they are hermitian except for the boost generators (this is due to the non-compactness of the Lorentz group). Moreover, we take the metric to be

$$\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-). \tag{2.5}$$

Let us now consider more closely the algebra of the $M_{\mu\nu}$, that is the $SO(1,3)$ Lorentz algebra. Consider the split to the spatial indices $i = 1,2,3$ (and notice that $\eta_{ij} = -\delta_{ij}$). The M_{ij} are 3 generators satisfying

$$[M_{ij}, M_{kl}] = i\delta_{ik}M_{jl} + i\delta_{jl}M_{ik} - i\delta_{il}M_{jk} - i\delta_{jk}M_{il} \tag{2.6}$$

If we write

$$M_{ij} = \epsilon_{ijk}J_k \tag{2.7}$$

that is, $M_{12} = J_3, M_{23} = J_1, M_{31} = J_2$, we obtain

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{2.8}$$

the algebra of $SU(2)$.

Now we can also rename the other generators as $M_{0i} = K_i$ and find for their commutation relations

$$[M_{ij}, M_{0k}] = i\delta_{jk}M_{i0} - i\delta_{ik}M_{j0}$$

which gives

$$\epsilon_{ijl}[J_l, K_k] = -i\delta_{jk}K_i + i\delta_{ik}K_j$$

and finally

$$[J_i, K_j] = i\epsilon_{ijk}K_k.$$

We also have

$$[M_{0i}, M_{0j}] = -iM_{ij}$$

which gives

$$[K_i, K_j] = -i\epsilon_{ijk}J_k.$$

To summarize, we have rewritten the commutation relations of $SO(1,3)$ as:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.9}$$

We can now form the following complex generators

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i), \tag{2.10}$$

which *are* hermitian, and satisfy $(J_i^{\pm})^* = -J_i^{\mp}$ (we must take all the J_i and K_i to be imaginary, in order for all rotation and boost parameters to be real). Let us compute their own commutation relations:

$$\begin{aligned}
[J_i^{\pm}, J_j^{\pm}] &= \frac{1}{4}\{i\epsilon_{ijk}J_k - (-i\epsilon_{ijk}J_k) \pm i(i\epsilon_{ijk}K_k) \pm i(-i\epsilon_{jik}K_k)\} \\
&= i\epsilon_{ijk}\frac{1}{2}(J_i \pm iK_i) \\
&= i\epsilon_{ijk}J_k^{\pm}
\end{aligned}$$

while

$$\begin{aligned}
[J_i^{\pm}, J_j^{\mp}] &= \frac{1}{4}\{i\epsilon_{ijk}J_k + (-i\epsilon_{ijk}J_k) \mp i(i\epsilon_{ijk}K_k) \pm i(-i\epsilon_{jik}K_k)\} \\
&= 0
\end{aligned}$$

We have thus established that the Lorentz group can be rewritten as

$$SO(1,3) = SU(2) \times SU(2)^*.$$

This is of course a very well-known fact in the theory of Lie algebras, where at the level of complex algebras one writes that $SO(4) = SU(2) \times SU(2)$. Here we are concerned with a specific real form of $SO(4, \mathbb{C})$. It is also sometimes useful to note (or familiar) that

$$SU(2) \times SU(2)^* = SL(2, \mathbb{C})$$

the group of the 2×2 complex matrices of unit determinant.

Here, it will suffice to recall that representations of the Lorentz group can be classified by two $SU(2)$ spins (s, s') of $SU(2) \times SU(2)^*$. Then, the complex conjugation of the second $SU(2)$ is understood with the following meaning, when considering an arbitrary representation:

$$(s, s')^* = (s', s).$$

The smallest representations are the spinorial ones:

$$(\tfrac{1}{2}, 0) \qquad \text{and} \qquad (0, \tfrac{1}{2}).$$

We will denote them as 2-vectors with indices, respectively

$$\alpha = 1, 2 \qquad \text{and} \qquad \dot{\alpha} = 1, 2.$$

We will thus write our *Weyl spinors* as ψ_α and $\bar{\psi}_{\dot{\alpha}}$, with the understanding that

$$(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}.$$

Higher dimensional representations appear taking (tensor) products of this two basic (fundamental) representations. For instance:

$$(\tfrac{1}{2}, 0) \otimes (0, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}),$$

which is an irreducible 4-dimensional representation of $SO(1,3)$, so it cannot be anything else than a vector v_μ.

Indeed, we can write the product of two spinors as

$$\psi_\alpha \bar{\psi}_{\dot{\alpha}} \equiv v_{\alpha\dot{\alpha}}$$

a 2×2 matrix, thus with 4 components. The latter can in turn be decomposed into a basis of 2×2 matrices. As such a basis, we can choose the Pauli matrices supplemented by the identity:

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\mathbb{I}, -\tau_i), \tag{2.11}$$

which means explicitly

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.12}$$

We now see that the product of two spinors is indeed a vector:

$$\psi_\alpha \bar{\psi}_{\dot{\alpha}} \equiv v_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} v_\mu.$$

Another example is the following product:

$$(\tfrac{1}{2}, 0) \otimes (\tfrac{1}{2}, 0) = (0, 0) \oplus (1, 0).$$

The right hand side thus corresponds to a singlet together with a 3-dimensional representation of $SO(1,3)$. As a product of two spinors, we can write:

$$\psi_\alpha \chi_\beta = \epsilon_{\alpha\beta} s + t_{\alpha\beta}, \quad \text{with} \quad t_{\alpha\beta} = t_{\beta\alpha}.$$

$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ is the invariant tensor of $SU(2)$ (one can check that, using the fact that the $SU(2)$ matrices have determinant 1).

Then, what is $t_{\alpha\beta}$? It should be an integer spin representation of $SO(1,3)$, so it should be associated with a bosonic field with some Lorentz symmetry. Consider for instance an antisymmetric tensor $T_{\mu\nu} = -T_{\nu\mu}$. It has 6 independent components. However, in $SO(1,3)$ we also have an invariant completely antisymmetric tensor, written as $\epsilon_{\mu\nu\rho\sigma}$ and sometimes called the Levi-Civita tensor. We adopt the convention that

$$\epsilon_{0123} = \epsilon^{0123} = 1.$$

It can be used to define a *dual* tensor

$$\tilde{T}_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}.$$

Note that

$$\tilde{\tilde{T}}_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{T}^{\rho\sigma} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\left(-\frac{i}{2}\epsilon^{\rho\sigma\lambda\tau}T_{\lambda\tau}\right) = T_{\mu\nu}.$$

The dual tensor is also antisymmetric $\tilde{T}_{\mu\nu} = -\tilde{T}_{\nu\mu}$ and hence also has 6 components. We can define:

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2}\left(T_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}\right) + \frac{1}{2}\left(T_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}\right) \\ &= \frac{1}{2}T^{+}_{\mu\nu} + \frac{1}{2}T^{-}_{\mu\nu}. \end{aligned}$$

We have thus

$$\tilde{T}^{\pm}_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}T^{\pm\rho\sigma} = \pm T^{\pm}_{\mu\nu}$$

which means that these two tensors are *(anti)self-dual*. Each one has only 3 independent (generally complex) components.

The representation $(1,0)$ corresponds to a self-dual tensor like $T^{+}_{\mu\nu}$, while $(0,1) = (1,0)^*$ corresponds to an anti-self-dual tensor like $T^{-}_{\mu\nu}$ or $(T^{+}_{\mu\nu})^*$. Indeed, one has

$$(\tfrac{1}{2},\tfrac{1}{2}) \otimes_A (\tfrac{1}{2},\tfrac{1}{2}) = (1,0) \oplus (0,1).$$

Going back now to $\epsilon_{\alpha\beta}$, we note that it can be used to raise and lower indices. In the process we will fix a good deal of notation and conventions.

First of all, we define the $SU(2)$ antisymmetric tensor to have the following values:

$$\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = 1.$$

Then, we define:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \qquad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \qquad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}.$$

In other words, adjacent indices are always contracted with the epsilon tensor *on the left*. [Exercise: count the number of arbitrary signs that one is fixing with these conventions..] Note that the fact that, for instance, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, is really the statement that the fundamental and the antifundamental representations of $SU(2)$ are equivalent. This is *not* true for $SU(N)$, with $N > 2$, where there are two, distinct, conjugate N-dimensional representations, called the fundamental and the anti-fundamental (and often denoted by $\mathbf{N}$ and $\bar{\mathbf{N}}$).

As already seen, $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ can be used to form scalars from fermionic bilinears. The notation and sign conventions are the following:

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\chi_\alpha\psi_\beta = \epsilon^{\beta\alpha}\chi_\alpha\psi_\beta = \chi^\beta\psi_\beta = \chi\psi.$$

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}\bar{\psi}_{\dot{\alpha}} = \epsilon^{\dot{\beta}\dot{\alpha}}\bar{\chi}_{\dot{\beta}}\bar{\psi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}} = \bar{\chi}\bar{\psi}.$$

Note that the sign in the third equality in both expressions is due to the Grassmann nature of the half-integer spin fields ψ_α etc, as it befits fermions.

Other relevant identities are as follow. To form a vector from two spinors, one writes

$$\begin{aligned}
\psi\sigma^\mu\bar{\chi} &= \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \quad = \quad -\bar{\chi}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\psi^\alpha \\
&= -\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\epsilon^{\alpha\beta}\psi_\beta \quad = \quad -\bar{\chi}_{\dot{\beta}}(\epsilon^{\dot{\beta}\dot{\alpha}}\epsilon^{\beta\alpha}\sigma^\mu_{\alpha\dot{\alpha}})\psi_\beta \\
&= -\bar{\chi}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}\psi_\beta \quad = \quad -\bar{\chi}\bar{\sigma}^\mu\psi,
\end{aligned}$$

where we have defined

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\mathbb{I}, +\tau_i). \tag{2.13}$$

Under complex (i.e. hermitian) conjugation, we apply the rule

$$(\psi_\alpha\chi_\beta)^* = \chi^*_\beta\psi^*_\alpha = \bar{\chi}_{\dot{\beta}}\bar{\psi}_{\dot{\alpha}},$$

so that

$$(\psi\chi)^* = (\psi^\alpha\chi_\alpha)^* = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}.$$

Finally, let us make the connection with Dirac (i.e. 4-component) spinors, and Clifford γ-matrices. Taking

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix},$$

we can compute

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \begin{bmatrix} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu \end{bmatrix}.$$

Now, we have:

$$\begin{aligned} 2\sigma^0 \bar{\sigma}^0 &= 2\mathbb{I}_2 \\ \sigma^0 \bar{\sigma}^i + \sigma^i \bar{\sigma}^0 &= \tau_i - \tau_i = 0 \\ \sigma^i \bar{\sigma}^j + \sigma^j \bar{\sigma}^i &= -\{\tau_i, \tau_j\} = -2\delta_{ij}\mathbb{I}_2, \end{aligned}$$

and similarly for $\bar{\sigma}\sigma$. Hence we are reassured to have

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_4.$$

A Dirac spinor is then a spinor

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix},$$

in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ reducible representation.

The chirality matrix is

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}.$$

Hence we see that the Weyl spinors we have been dealing with are chiral, as expected:

$$\gamma^5 \Psi = \pm\Psi \qquad \text{iff} \qquad \Psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \quad \text{or} \quad \Psi = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

A Majorana ("real") spinor is one for which $\psi = \chi$, that is

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}.$$

The Lorentz generators are written as:

$$\begin{aligned}\Sigma^{\mu\nu} &= \frac{i}{2}\gamma^{\mu\nu} \equiv \frac{i}{4}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = \frac{i}{4}\begin{bmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{bmatrix} \\ &\equiv \begin{bmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & i\bar\sigma^{\mu\nu} \end{bmatrix}.\end{aligned}$$

This in particular means that $i\sigma^{\mu\nu}$ and $i\bar\sigma^{\mu\nu}$ act as Lorentz generators on ψ_α and $\bar\psi^{\dot\alpha}$, respectively.

For example:

$$(i\sigma^{12})_\alpha{}^\beta = \frac{i}{4}(\sigma^1\bar\sigma^2 - \sigma^2\bar\sigma^1) = -\frac{i}{4}(\tau_1\tau_2 - \tau_2\tau_1) = -\frac{i}{4}2i\tau_3 = \frac{1}{2}\tau_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and similarly for $(i\bar\sigma^{12})^{\dot\alpha}{}_{\dot\beta}$. This means that ψ_α and $\bar\psi^{\dot\alpha}$ correctly have eigenvalues $\pm\frac{1}{2}$ for J^3.

We can now close this long parenthesis on representations and notations/conventions.

2.3 A first look at the superalgebra

We are now armed with the required knowledge to write the superalgebra. First of all, we have to write the *supercharges*, i.e. the odd generators of the superalgebra. They must be in a spin-$\frac{1}{2}$ representation of the Lorentz group, we thus write them as

$$Q_\alpha.$$

If there are several supercharges, we can write Q^I_α, with $I = 1, \ldots \mathcal{N}$. Note that there are no independent charges like $\bar Q'_{\dot\alpha}$ because we can just take their hermitian conjugate as the "elementary" supercharges, $Q'_\alpha \equiv (\bar Q'_{\dot\alpha})^\dagger$. Minimal supersymmetry (in 4 dimensions) is achieved when $\mathcal{N} = 1$.

We should now write the basic anticommutation relations. Exploiting the fact that a vector is obtained through the product of two spinors of opposite chirality, $\psi_\alpha\chi_{\dot\alpha} = v_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}v_\mu$, we postulate:

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu. \tag{2.14}$$

The commutation relations of the Q_α with the Poincaré generators or, in other words, their transformation laws, are given by

$$\begin{aligned} [P_\mu, Q_\alpha] &= 0, & (2.15) \\ [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, & (2.16) \\ [M_{\mu\nu}, \bar{Q}_{\dot\alpha}] &= -i(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}. & (2.17) \end{aligned}$$

Note that the latter two equations are obtained by hermitian conjugation from each other, and they are the rightful transformations for a spin-$\frac{1}{2}$ operator. On the other hand, the relation (2.15) follows from the Jacobi identities.

First note that Jacobi identities must be adapted to graded algebras:

$$[E_1, [E_2, O_3]] + [E_2, [O_3, E_1]] + [O_3, [E_1, E_2]] = 0,$$

$$[E_1, \{O_2, O_3\}] + \{O_2, [O_3, E_1]\} - \{O_3, [E_1, O_2]\} = 0,$$

$$[O_1, \{O_2, O_3\}] + [O_2, \{O_3, O_1\}] + [O_3, \{O_1, O_2\}] = 0,$$

where E are even and O odd operators. Then one sees that considering the identity for Q_α, P_μ and P_ν, and recalling that there should not be operators with spin higher than 1 in the superalgebra (the Coleman-Mandula theorem extended by Haag-Lopuszanski-Sohnius), P_μ and Q_α could only commute to $\bar{Q}_{\dot\alpha}$, but the proportionality constant must vanish (essentially, the Jacobi identity would be proportional to $\sigma_{\mu\nu}Q$, which does not vanish).

Next, we can infer from the Jacobi identity involving P_μ, Q_α and Q_β that:

$$\{Q_\alpha, Q_\beta\} = 0.$$

Thus, to recapitulate, the minimal supersymmetry algebra is given by:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu, & (2.18) \\ \{Q_\alpha, Q_\beta\} &= 0, & (2.19) \\ [P_\mu, Q_\alpha] &= 0, & (2.20) \\ [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta. & (2.21) \end{aligned}$$

If we have an extended SUSY algebra, i.e. we have supercharges Q^I_α, there is one more possibility which is consistent with the Jacobi identities and the Coleman-Mandula theorem, that is we can add to the algebra some *central charges* Z^{IJ}, which are Lorentz scalars and hence commute with all the other generators:

$$\begin{aligned}
\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ}, && (2.22)\\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ}, && (2.23)\\
[Q_\alpha, Z^{IJ}] &= 0, && (2.24)\\
[P_\mu, Z^{IJ}] &= 0, && (2.25)\\
[M_{\mu\nu}, Z^{IJ}] &= 0. && (2.26)
\end{aligned}$$

Note also that the anticommutation relation where they appear implies that $Z^{IJ} = -Z^{JI}$. This is why there is no central charge when $\mathcal{N} = 1$.

2.3.1 Basic consequences of the superalgebra

We state here the most basic consequences that the superalgebra has on its representations.

First of all, since $[P_\mu, Q_\alpha] = 0$, we also have

$$[P^2, Q_\alpha] = 0,$$

and thus any two states related by the operator Q (namely, $Q|\omega\rangle = |\omega'\rangle$, with $|\omega\rangle$ and $|\omega'\rangle$ of opposite statistics) must be *degenerate in mass.*

Secondly, *the energy of any state is always positive, and it is zero only in a supersymmetric ground state.* This can be easily demonstrated. Take

$$\begin{aligned}
\delta^{\alpha\dot{\alpha}}\langle\omega|\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}|\omega\rangle &= \sum_{\dot{\alpha}} \langle\omega|(\bar{Q}_{\dot{\alpha}})^\dagger \bar{Q}_{\dot{\alpha}}|\omega\rangle + \sum_\alpha \langle\omega|(Q_\alpha)^\dagger Q_\alpha|\omega\rangle\\
&= \sum_{\dot{\alpha}} \|\bar{Q}_{\dot{\alpha}}|\omega\rangle\|^2 + \sum_\alpha \|Q_\alpha|\omega\rangle\|^2 \geq 0.
\end{aligned}$$

Now, noting that $\delta^{\alpha\dot{\alpha}} \equiv \bar{(}\sigma^0)^{\dot{\alpha}\alpha}$, we can also write

$$\begin{aligned}
\delta^{\alpha\dot{\alpha}}\langle\omega|\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}|\omega\rangle &= 2(\bar{\sigma}^0)^{\dot{\alpha}\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\langle\omega|P_\mu|\omega\rangle\\
&= 4\langle\omega|P_0|\omega\rangle\\
&= 4E\||\omega\rangle\|^2. && (2.27)
\end{aligned}$$

We thus conclude that $E \geq 0$ for any state $|\omega\rangle$ in a supersymmetric theory. (Note that we adopt the reasonable convention that $P_0 = P^0 \geq 0$ for future directed physical particles.)

Conversely, we also note that $E = 0$ holds if and only if

$$Q_\alpha|\omega\rangle = 0, \qquad \bar{Q}_{\dot{\alpha}}|\omega\rangle = 0,$$

which means that a zero energy state must be invariant under *all* supersymmetries. Since we also have that $P_\mu|\omega\rangle = 0$, we conclude that it must be a (supersymmetric) ground state.

It should be noted that this is a powerful statement. In particular, it holds also at the quantum level, thus the exact quantum vacuum of a theory, if it is supersymmetric, has exactly vanishing energy. This is to be contrasted with the usual non supersymmetric theories, where the vacuum fluctuations give a non-zero value to the vacuum energy (recall the discussion on the harmonic oscillators in the previous chapter).

The third consequence that we are going to derive is that *in a supermultiplet, there is an equal amount of bosonic and fermionic degrees of freedom.* Consider the operator $(-1)^F$, which is such that

$$(-1)^F|\omega_b\rangle = |\omega_b\rangle, \qquad\qquad (-1)^F|\omega_f\rangle = -|\omega_f\rangle,$$

for $|\omega_b\rangle$ a bosonic and $|\omega_f\rangle$ a fermionic state. Obviously, since Q shifts the spin by $\frac{1}{2}$, it changes the statistics, which implies that it satisfies

$$(-1)^F Q = -Q(-1)^F. \tag{2.28}$$

For states such that P_0 is fixed and $\neq 0$, we have (the trace is over all such states in the theory):

$$\begin{aligned}
\mathrm{tr}\,(-1)^F P_0 &= \frac{1}{2}\delta^{\alpha\dot{\alpha}}\mathrm{tr}\,(-1)^F\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \\
&= \frac{1}{4}\delta^{\alpha\dot{\alpha}}\mathrm{tr}\,(-1)^F(Q_\alpha\bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}}Q_\alpha) \\
&= \frac{1}{4}\delta^{\alpha\dot{\alpha}}\mathrm{tr}\,\left[(-1)^F Q_\alpha\bar{Q}_{\dot{\alpha}} - \bar{Q}_{\dot{\alpha}}(-1)^F Q_\alpha\right] \\
&= \frac{1}{4}\delta^{\alpha\dot{\alpha}}\mathrm{tr}\,\left[(-1)^F Q_\alpha\bar{Q}_{\dot{\alpha}} - (-1)^F Q_\alpha\bar{Q}_{\dot{\alpha}}\right] \\
&= 0,
\end{aligned}$$

where in the one before the last equality we have used the cyclicity of the trace.

Hence, in particular, summing on any finite dimensional representation with non zero energy

$$\mathrm{tr}\,(-1)^F = 0. \tag{2.29}$$

This implies that there is an equal number of bosonic and fermionic degrees of freedom in such a representation, or *supermultiplet*.

Note that the only states which can be unpaired are the SUSY ground states, for which $P_0 = 0$. Indeed, the most typical case is when there is only one ground state, which is bosonic.

2.4 Representations of the superalgebra

In this section we work out the representations of the superalgebra, which are also called supermultiplets. Indeed, they can be thought as multiplets where we assemble together several different representations of the Lorentz algebra, since the latter is a subalgebra of the superalgebra. We start with massless representations, and then proceed to massive ones. On the path, we will make a short digression concerning supermultiplets of extended superalgebras.

2.4.1 Massless supermultiplets

If $P^2 = 0$, we can take P_μ to a canonical form by applying boosts and rotations until it reads

$$P_\mu = (E, 0, 0, E).$$

Then

$$\sigma^\mu_{\alpha\dot\alpha} P_\mu = (\sigma^0 + \sigma^3)E = \begin{bmatrix} 0 & 0 \\ 0 & 2E \end{bmatrix},$$

and the SUSY algebra becomes

$$\begin{bmatrix} \{Q_1, \bar Q_{\dot 1}\} & \{Q_1, \bar Q_{\dot 2}\} \\ \{Q_2, \bar Q_{\dot 1}\} & \{Q_2, \bar Q_{\dot 2}\} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4E \end{bmatrix}, \tag{2.30}$$

intended as acting on the states of the multiplet we are looking for.

In particular,

$$\{Q_1, \bar Q_{\dot 1}\} = 0,$$

which implies that

$$\|Q_1|\omega\rangle\|^2 = 0 = \|\bar Q_{\dot 1}|\omega\rangle\|^2$$

and thus

$$Q_1|\omega\rangle = 0 = \bar{Q}_{\dot{1}}|\omega\rangle.$$

This means that as operators, $Q_1 = \bar{Q}_{\dot{1}} = 0$ on this multiplet.

The only nontrivial anticommutation relation that is left is:

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 4E. \tag{2.31}$$

If we call

$$\alpha = \frac{1}{2\sqrt{E}} Q_2, \qquad \alpha^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{2}},$$

then the anticommutation relations take the normalized form

$$\{\alpha, \alpha^\dagger\} = 1,$$

with of course $\{\alpha, \alpha\} = 0$.

We can build the representation starting from a state $|\lambda\rangle$ such that

$$\alpha|\lambda\rangle = 0.$$

Suppose it has helicity λ:

$$M_{12}|\lambda\rangle \equiv J_3|\lambda\rangle = \lambda|\lambda\rangle.$$

Then we can compute the helicity of $\alpha^\dagger|\lambda\rangle$:

$$M_{12}\bar{Q}_{\dot{2}}|\lambda\rangle = (\bar{Q}_{\dot{2}}M_{12} + \tfrac{1}{2}\bar{Q}_{\dot{2}})|\lambda\rangle = (\lambda + \tfrac{1}{2})\bar{Q}_{\dot{2}}|\lambda\rangle,$$

where we have used $[M_{12}, \bar{Q}_{\dot{2}}] = \frac{1}{2}\bar{Q}_{\dot{2}}$. Thus we learn that

$$\alpha^\dagger|\lambda\rangle = |\lambda + \tfrac{1}{2}\rangle.$$

It stops here since $(\alpha^\dagger)^2 = 0$ and hence

$$\alpha^\dagger|\lambda + \tfrac{1}{2}\rangle = 0.$$

Massless multiplets are thus composed of one boson and one fermion. Since physical particles must come in CPT conjugate representations (or, there are no spin-$\frac{1}{2}$ one-dimensional representations of the massless little group of the Lorentz group), one must add the CPT conjugate multiplet where helicities are flipped.

Let us give some examples:

- The *scalar* multiplet is obtained setting $\lambda = 0$. Then we have

$$\alpha^\dagger|0\rangle = |\tfrac{1}{2}\rangle.$$

The full multiplet is composed of two states with $\lambda = 0$ and a doublet with $\lambda = \pm\frac{1}{2}$. These are the degrees of freedom of a complex scalar and a Weyl (chiral) fermion.

- The *vector* multiplet is obtained starting from a $\lambda = \frac{1}{2}$ state. We get

$$\alpha^\dagger|\tfrac{1}{2}\rangle = |1\rangle.$$

To this we add the CPT conjugate multiplet, to obtain two pairs of states, one with $\lambda = \pm\frac{1}{2}$ and the other with $\lambda = \pm 1$. These are the degrees of freedom of a Weyl fermion and of a massless vector. The latter is usually interpreted as a gauge boson.

- Another multiplet is obtained starting from $\lambda = \frac{3}{2}$:

$$\alpha^\dagger|\tfrac{3}{2}\rangle = |2\rangle.$$

Adding the CPT conjugate, one has a pair of bosonic degrees of freedom with $\lambda = \pm 2$, which we interpret as the *graviton*, and a pair of fermionic degrees of freedom with $\lambda = \pm\frac{3}{2}$, which correspond to a massless spin-$\frac{3}{2}$ Rarita-Schwinger field, also called the *gravitino*, since it is the SUSY partner of the graviton, as we have just showed.

2.4.2 Supermultiplets of extended supersymmetry

Let us very briefly mention that having extended SUSY, the massless supermultiplets are longer. Take the superalgebra to be:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^{IJ},$$

where for simplicity we suppose that $Z^{IJ} = 0$ on these states. For massless states, $P_\mu = (E, 0, 0, E)$ and as before we have that

$$\{Q^I_1, \bar{Q}^J_{\dot 1}\} = 0,$$

which implies the operator equations $Q^I_1 = 0$ and $\bar{Q}^I_{\dot 1} = 0$, for $I = 1, \ldots \mathcal{N}$, on these states. The nontrivial relations are:

$$\{Q^I_2, \bar{Q}^J_{\dot 2}\} = 4E\delta^{IJ},$$

so that we can define

$$\alpha_I = \frac{1}{2\sqrt{E}} Q_2^I$$

and obtain the canonical anticommutation relations for $\mathcal{N}$ fermionic oscillators

$$\{\alpha_I, \alpha_J^\dagger\} = \delta_{IJ}.$$

If we start now from a state $|\lambda\rangle$ of helicity λ which satisfies $\alpha_I|\lambda\rangle = 0$, we build a multiplet as follows:

$$\begin{aligned} \alpha_I^\dagger|\lambda\rangle &= |\lambda + \tfrac{1}{2}\rangle_I, \\ \alpha_I^\dagger\alpha_J^\dagger|\lambda\rangle &= |\lambda + 1\rangle_{[IJ]}, \\ &\vdots \\ \alpha_1^\dagger \ldots \alpha_{\mathcal{N}}^\dagger|\lambda\rangle &= |\lambda + \tfrac{\mathcal{N}}{2}\rangle. \end{aligned} \tag{2.32}$$

Note that there are $\mathcal{N}$ states with helicity $\lambda + \frac{1}{2}$, $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ states with helicity $\lambda + 1$ and so on, until a single state with helicity $\lambda + \frac{\mathcal{N}}{2}$ (it is totally antisymmetric in $\mathcal{N}$ indices I).

In total, the supermultiplet is composed of

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}$$

states, half of them bosonic and half of them fermionic (as can be ascertained by computing $\sum_{k=0}^{\mathcal{N}}(-1)^k\binom{\mathcal{N}}{k} = (1-1)^{\mathcal{N}} = 0$).

Interestingly, in this case we can now have self-CPT conjugate multiplets. Take e.g. $\mathcal{N} = 4$ and start from $\lambda = -1$. Then $\lambda + \frac{\mathcal{N}}{2} = 1$ and the multiplet spans states of opposite helicities, thus filling complete representations of the Lorentz group. Indeed, it contains one pair of states with $\lambda = \pm 1$ (a vector, i.e. a gauge boson), 4 pairs of states with $\lambda = \pm\frac{1}{2}$ (4 Weyl fermions) and 6 states with $\lambda = 0$ (6 real scalars, or equivalently 3 complex scalars).

Another example is $\mathcal{N} = 8$ supersymmetry. Here if we start with $\lambda = -2$ we end up with $\lambda + \frac{\mathcal{N}}{2} = 2$. Thus in this case we have the graviton in the self-CPT conjugate multiplet, corresponding to the pair of states with $\lambda = \pm 2$. In addition, we have 8 massless gravitini with $\lambda = \pm\frac{3}{2}$, 28 massless vectors with $\lambda = \pm 1$, 56 massless Weyl fermions with $\lambda = \pm\frac{1}{2}$ and finally 70 real scalars with $\lambda = 0$. This is the content of $\mathcal{N} = 8$ *supergravity*, which is the only multiplet of $\mathcal{N} = 8$ supersymmetry with $|\lambda| \leq 2$. The latter condition is necessary in order to have consistent couplings (higher spin fields cannot be coupled in a consistent way with gravity and lower spin fields).

From the theoretical perspective, this is a very nice result, because we have a theory where *everything is determined* from symmetry alone: the complete spectrum and (as we will see) all the couplings. Unfortunately, this theory is also completely unphysical... To mention one problem, it has no room for fermions in complex representations of the gauge group, which are on the other hand present in the Standard Model.

2.4.3 Massive supermultiplets

When $P^2 = M^2 > 0$, by boosts and rotations P_μ can be put in the following form

$$P_\mu = (M, 0, 0, 0).$$

Then we have

$$\sigma^\mu_{\alpha\dot\alpha} P_\mu = M\sigma^0 = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix},$$

so that the superalgebra reads

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2M\delta_{\alpha\dot\alpha}. \tag{2.33}$$

Note that $[M_{12}, Q_1] = i(\sigma_{12})_1{}^1 Q_1 = \frac{1}{2}Q_1$, thus it is Q_1 that raises the helicity, in the same way as $\bar{Q}_{\dot 2}$. We make the redefinition

$$\begin{aligned} \alpha_1 &= \frac{1}{\sqrt{2M}}\bar{Q}_{\dot 1}, & \alpha_1^\dagger &= \frac{1}{\sqrt{2M}}Q_1, \\ \alpha_2 &= \frac{1}{\sqrt{2M}}Q_2, & \alpha_2^\dagger &= \frac{1}{\sqrt{2M}}\bar{Q}_{\dot 2}, \end{aligned} \tag{2.34}$$

so that we have the canonical anticommutation relations of two fermionic oscillators:

$$\{\alpha_a, \alpha_b^\dagger\} = \delta_{ab}, \qquad a, b = 1, 2.$$

If we start with $\alpha_a|\lambda\rangle = 0$, $M_{12}|\lambda\rangle = \lambda|\lambda\rangle$, then we build the multiplet as:

$$\begin{aligned} \alpha_1^\dagger|\lambda\rangle &= |\lambda + \tfrac{1}{2}\rangle_1, \\ \alpha_2^\dagger|\lambda\rangle &= |\lambda + \tfrac{1}{2}\rangle_2, \\ \alpha_1^\dagger\alpha_2^\dagger|\lambda\rangle &= |\lambda + 1\rangle. \end{aligned}$$

There are 4 states now (compared to the 2 in the massless case), two bosons and 2 fermions.

Let us consider again two examples.

- In the case of the *massive scalar multiplet*, we start from $\lambda = -\frac{1}{2}$ and obtain two states with $\lambda = 0$ and one state with $\lambda = \frac{1}{2}$. These are the degrees of freedom of one massive complex scalar and one massive Weyl fermion. Note that the latter might not be familiar. Indeed, one cannot write the usual Dirac mass term for a Weyl fermion. Instead, one can write what is called a Majorana mass term:

$$\mathcal{L} \supset m\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta + h.c.$$

  We will come back to this later on. Note also that the total number of degrees of freedom of a massive scalar multiplet is the same as that of a massless one.

- For a *massive vector multiplet*, start from $\lambda = 0$ to obtain 2 states with $\lambda = \frac{1}{2}$ and one state with $\lambda = 1$. To this we add the CPT conjugate multiplet so that in the end we have one pair with $\lambda = \pm 1$, two pairs with $\lambda = \pm\frac{1}{2}$ and 2 states with $\lambda = 0$. According to the massive little group, this corresponds to 1 massive vector (with $\lambda = \pm 1, 0$), 1 real scalar and 1 massive Dirac fermion. Note however that the content in degrees of freedom is the same as that of one massless vector multiplet together with one massless scalar multiplet. This is a hint that the consistent way to treat massive vectors in a supersymmetric field theory will be through a SUSY version of the Brout-Englert-Higgs mechanism.

We are now done with describing the degrees of freedom in the supermultiplets, and we move on to consider the field theories describing them.

The main message to take away from this section is that to every degree of freedom (i.e. particle) one considers, SUSY gives it (at least) a partner, i.e. its superpartner. So for instance, for every (chiral) fermion of the SM, there is a complex scalar with the same quantum numbers (for instance, gauge charges). The superpartners of the quarks are called *squarks*, while the superpartners of the leptons are called *sleptons*. If the SM particle is itself a complex scalar, like the Higgs, then its superpartner is a chiral fermion, called in this case the *higgsino*. On the other hand, we have seen that the gauge bosons (massless vectors) also have a chiral fermion superpartner, which is called the *gaugino*. In particular, the superpartners of the gluons, W, Z and the photon are called gluinos, Wino, Zino and photino. Last but not the least, in supergravity the superpartner of the graviton is the gravitino, as we already mentioned. All this might sound a little ridiculous but we all get used to it eventually...

Chapter 3

Supersymmetric field theories

In this chapter, we discuss how the supersymmetry algebra is represented on fields, with the aim of building field theory actions invariant under supersymmetry. We will use a constructive approach, starting from the variations of the fields, proceeding to the action for free fields and then introducing the interactions. We repeat the exercise both for the scalar and the vector supermultiplet. The hidden goal of the present chapter is, while making the reader familiar with supersymmetric field theories, to convince her/him that the present approach has to be replaced by a better formalism where supersymmetry is manifestly realized.

3.1 The theory for the scalar multiplet

Our aim is to construct field theories which are supersymmetric, i.e. such that their action is invariant under supersymmetry.

We have seen how supercharges act on single particle states. Our goal now is to implement how supercharges act on the fields that create those particles. Since we know that there must be bosonic and fermionic particles, there will also be *bosonic and fermionic fields*.

The action of any symmetry generator on a field is usually encoded in a variation of the field. The relation to the action on a state is as follows. Take e.g. a $U(1)$ charge:

$$\mathcal{Q}|q\rangle = q|q\rangle,$$

so that $|q\rangle$ is a state of a particle of charge q, and $\mathcal{Q}$ is a Lie algebra generator. Then, the field ϕ creating the state $|q\rangle$ must be in a representation of the group generated by $\mathcal{Q}$. It transforms as

$$\phi \to \phi' = e^{i\mathcal{Q}\alpha}\phi = e^{iq\alpha}\phi,$$

where α is the parameter of the transformation. For *small* α, we can write the transformation above as a *variation*:

$$\delta_\alpha \phi = \phi' - \phi = i\mathcal{Q}\alpha\phi = iq\alpha\phi,$$

where we have expanded the exponential at first order. These are the variations that are usually written in field theory, and which represent the algebra of symmetries on the fields.

3.1.1 Supersymmetry variations on scalar and fermionic fields

For supersymmetry, the generators act in the following way in the simplest case of a massless scalar multiplet:

$$\begin{aligned}
\frac{1}{2\sqrt{E}}\bar{Q}_{\dot{2}}|\lambda = 0\rangle &= |\lambda = \tfrac{1}{2}\rangle \\
\frac{1}{2\sqrt{E}}Q_2|\lambda = \tfrac{1}{2}\rangle &= |\lambda = 0\rangle
\end{aligned}$$

Considering now the CPT complete multiplet, the two $|\lambda = 0\rangle$ states will be created by a compex scalar field ϕ, while the $|\lambda = \pm\frac{1}{2}\rangle$ helicity states will be created by a fermionic spin-$\frac{1}{2}$ field ψ, which is a Weyl spinor.

Clearly, from the relations above, the variation of the scalar ϕ must be proportional to the spinor ψ, and vice-versa. We deduce that the parameter ϵ of the variation must be a Grassmann-odd, spinor variable in order for the relation to be consistent (i.e. for it to conserve spin and statistics). Also, the parameter ϵ must have some mass dimension.

Indeed, recall that $[P] = M$, i.e. the momentum generator has the dimension of mass. The fact that (schematically) $\{Q, \bar{Q}\} = P$ and $\bar{Q} = Q^\dagger$ implies that the supercharges have a dimesion given by $[Q] = M^{1/2}$. So, if we want to be able to exponentiate the action of the supercharge, by writing $e^{\epsilon Q}$, we see that we should have $[\epsilon Q] = M^0$ and hence $[\epsilon] = M^{-1/2}$.

Now, the fields have canonical dimensions given by $[\phi] = M$ and $[\psi] = M^{3/2}$. Thus, the only possibility we are left with (up to a numerical constant) for the variation of the scalar is:

$$\delta_\epsilon \phi = \sqrt{2}\epsilon^\alpha \psi_\alpha. \tag{3.1}$$

For the variation of the fermion, we guess the following:

$$\delta_\epsilon \psi_\alpha = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu \phi. \tag{3.2}$$

Let us see if the above variations indeed lead to the SUSY algebra being realized on the fields ϕ and ψ. In order to do this, we need to commute two variations. First note that in terms of the supercharges, the variations are written as:

$$\delta\phi = i(\epsilon^\alpha Q_\alpha + \bar{Q}_{\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}})\phi, \tag{3.3}$$

where we have written the variation of ϕ but the same relation should be true for any field. Then, the SUSY algebra dictates that the commutator of two variations should satisfy:

$$\begin{aligned}
[\delta_1, \delta_2]\phi &= -[\epsilon_1^\alpha Q_\alpha + \bar{Q}_{\dot{\alpha}}\bar{\epsilon}_1^{\dot{\alpha}}, \epsilon_2^\beta Q_\beta + \bar{Q}_{\dot{\beta}}\bar{\epsilon}_2^{\dot{\beta}}]\phi \\
&= -\left(\epsilon_1^\alpha \{Q_\alpha, \bar{Q}_{\dot{\beta}}\}\bar{\epsilon}_2^{\dot{\beta}} - \epsilon_2^\beta\{Q_\beta, \bar{Q}_{\dot{\alpha}}\}\bar{\epsilon}_1^{\dot{\alpha}}\right)\phi \\
&= -2\left(\epsilon_1\sigma^\mu\bar{\epsilon}_2 P_\mu - \epsilon_2\sigma^\mu\bar{\epsilon}_1 P_\mu\right)\phi \\
&= 2i\left(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\right)\partial_\mu\phi
\end{aligned} \tag{3.4}$$

where we recall that the action of the momentum operator on a field is

$$P_\mu \phi = -i\partial_\mu \phi.$$

Again, we stress that the above consequence of the SUSY algebra holds for any field.

We then check whether the variations (3.1) and (3.2) do indeed reproduce the above requirement. On the scalar field we have:[1]

$$\begin{aligned}
[\delta_1, \delta_2]\phi &= \delta_1\delta_2\phi - \delta_2\delta_1\phi \\
&= \sqrt{2}\epsilon_1^\alpha \delta_2\psi_\alpha - \sqrt{2}\epsilon_2^\alpha\delta_1\psi_\alpha \\
&= 2i\epsilon_1\sigma^\mu\bar{\epsilon}_2\partial_\mu\phi - 2i\epsilon_2\sigma^\mu\bar{\epsilon}_1\partial_\mu\phi \\
&= 2i\left(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\right)\partial_\mu\phi
\end{aligned} \tag{3.5}$$

[1] We adopt the following convention for the action of two successive variations. Given a set of fields φ^i transforming as $\delta\varphi^i = R^i{}_j\varphi^j$, we take $\delta_1\delta_2\varphi^i = R^i_{1j}\delta_2\varphi^j = R^i_{1j}R^j_{2k}\varphi^k = (R_1R_2)^i{}_j\varphi^j$. This is consistent with the superspace approach to compute such variations that we will introduce in the next chapter.

as it should.

Encouraged by this positive result, let us jump ahead and write already an action which is invariant under the above variations. In order to do this, we have however to first recall how to write an action for Weyl fermions.

Take a Dirac spinor to be

$$\Psi = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}.$$

Its Lagrangian density is

$$\mathcal{L} = -i\bar{\Psi}\gamma^\mu\partial_\mu\Psi.$$

Recall now that

$$\bar{\Psi} = \Psi^\dagger\gamma^0 = \begin{pmatrix} \bar{\psi} & \chi \end{pmatrix} \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} = \begin{pmatrix} \chi & \bar{\psi} \end{pmatrix}.$$

Then the Lagrangian density can be rewritten

$$\mathcal{L} = -i \begin{pmatrix} \chi & \bar{\psi} \end{pmatrix} \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \begin{pmatrix} \partial_\mu\psi \\ \partial_\mu\bar{\chi} \end{pmatrix} = -i\chi\sigma^\mu\partial_\mu\bar{\chi} - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi.$$

We will thus take

$$\mathcal{L}_{\text{Weyl}} = -i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi. \tag{3.6}$$

Note also that

$$S = -\int d^4x\, i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi = \int d^4x\, i\partial_\mu\psi\sigma^\mu\bar{\psi} = -\int d^4x\, i\psi\sigma^\mu\partial_\mu\bar{\psi},$$

where the middle equality follows from the identity $\bar{\psi}\bar{\sigma}^\mu\chi = -\chi\sigma^\mu\bar{\psi}$ and the last equality is obtained integrating by parts. As for the reality of the action, it follows from:

$$S^* = \int d^4x\, i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi = -\int d^4x\, i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi = S.$$

Again, in the middle equality one integrates by parts.

We can now write the action of a complex scalar together with a Weyl fermion, both massless:

$$S_{\text{tot}} = \int d^4x\, \left(\partial_\mu\phi\partial^\mu\phi^* - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi\right). \tag{3.7}$$

Taking its SUSY variation gives:

$$\delta S_{\text{tot}} = \int d^4x \left(\partial_\mu \delta\phi \partial^\mu \phi^* + \partial_\mu \phi \partial^\mu \delta\phi^* - i\delta\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \delta\psi\right).$$

Now, recall that the variations of the complex conjugate fields read

$$\delta\bar{\psi}_{\dot{\alpha}} = -i\sqrt{2}\epsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \phi^*, \qquad \delta\phi^* = \sqrt{2}\bar{\psi}_{\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}}.$$

The variation is again

$$\delta S_{\text{tot}} = \sqrt{2} \int d^4x \left(\epsilon \partial_\mu \psi \partial^\mu \phi^* + \partial_\mu \phi \partial^\mu \bar{\psi}\bar{\epsilon} - \epsilon\sigma^\nu \bar{\sigma}^\mu \partial_\mu \psi \partial_\nu \phi^* + \bar{\psi}\bar{\sigma}^\mu \sigma^\nu \bar{\epsilon} \partial_\mu \partial_\nu \phi\right).$$

Integrating by parts the first three terms gives

$$\delta S_{\text{tot}} = \sqrt{2} \int d^4x \left(-\epsilon\psi \partial_\mu \partial^\mu \phi^* - \bar{\psi}\bar{\epsilon} \partial_\mu \partial^\mu \phi + \epsilon\sigma^\nu \bar{\sigma}^\mu \psi \partial_\mu \partial_\nu \phi^* + \bar{\psi}\bar{\sigma}^\mu \sigma^\nu \bar{\epsilon} \partial_\mu \partial_\nu \phi\right).$$

Now we notice that

$$\bar{\sigma}^\mu \sigma^\nu \partial_\mu \partial_\nu \phi = \frac{1}{2}(\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu) \partial_\mu \partial_\nu \phi = \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = \partial_\mu \partial^\mu \phi.$$

Hence

$$\begin{aligned} \delta S_{\text{tot}} &= \sqrt{2} \int d^4x \left(-\epsilon\psi \partial_\mu \partial^\mu \phi^* - \bar{\psi}\bar{\epsilon} \partial_\mu \partial^\mu \phi + \epsilon\psi \partial_\mu \partial^\mu \phi^* + \bar{\psi}\bar{\epsilon} \partial_\mu \partial^\mu \phi\right) \\ &= 0. \end{aligned}$$

The action is thus invariant under the SUSY variations.

This is all reassuring. There is however a subtlety: Let us go back one step and verify that the SUSY algebra is satisfied also on the field ψ. We know we should have

$$[\delta_1, \delta_2]\psi_\alpha = 2i\left(\epsilon_1 \sigma^\mu \bar{\epsilon}_2 - \epsilon_2 \sigma^\mu \bar{\epsilon}_1\right) \partial_\mu \psi_\alpha.$$

Using now the variations, we have

$$\begin{aligned} [\delta_1, \delta_2]\psi_\alpha &= \delta_1 \delta_2 \psi_\alpha - \delta_2 \delta_1 \psi_\alpha \\ &= i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}} \bar{\epsilon}_1^{\dot{\alpha}} \partial_\mu \delta_2 \phi - i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}} \bar{\epsilon}_2^{\dot{\alpha}} \partial_\mu \delta_1 \phi \\ &= 2i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\epsilon}_1^{\dot{\alpha}} \epsilon_2^\beta \partial_\mu \psi_\beta - 2i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\epsilon}_2^{\dot{\alpha}} \epsilon_1^\beta \partial_\mu \psi_\beta \end{aligned}$$

This is not quite close to what we want, because ψ is not carrying the right index on the r.h.s. (we would like to have expressions like $\epsilon_1^\beta \sigma^\mu_{\beta\dot\alpha} \bar\epsilon_2^{\dot\alpha} \psi_\alpha$, i.e. with no sum on the index carried by ψ). A way to rearrange the sums over the spinorial indices is through *Fierz identities*.

The essence of the Fierz rearrangements is the following. Note first that $\bar\psi^{\dot\alpha}\chi^\alpha$ is a 2×2 matrix. It can be expanded as

$$\bar\psi^{\dot\alpha}\chi^\alpha = v_\mu \bar\sigma^{\mu\dot\alpha\alpha},$$

where v_μ will have an expression in terms of the two fermions:

$$v_\mu \propto \chi^\alpha \sigma_{\mu\alpha\dot\alpha} \bar\psi^{\dot\alpha}.$$

The only thing that is left to determine is the constant of proportionality. We do this e.g. by specializing to a particular component:

$$\begin{aligned} \bar\psi^{\dot 1}\chi^1 &= a\chi^\alpha \sigma_{\mu\alpha\dot\alpha}\bar\psi^{\dot\alpha}\bar\sigma^{\mu\dot 1 1} && (\bar\sigma^{0\dot 1 1} = \bar\sigma^{3\dot 1 1} = 1) \\ &= a\chi^\alpha\bar\psi^{\dot\alpha}(\sigma_{0\alpha\dot\alpha} + \sigma_{3\alpha\dot\alpha}) && (\sigma_i = \tau_i) \\ &= a\chi^\alpha\bar\psi^{\dot\alpha}(\delta_{\alpha\dot\alpha} + \tau_{3\alpha\dot\alpha}) \\ &= 2a\chi^1\bar\psi^{\dot 1} \end{aligned}$$

so that we fix $a = -\frac{1}{2}$. Hence, the relevant Fierz identity reads

$$\bar\psi^{\dot\alpha}\chi^\alpha = -\frac{1}{2}\chi^\beta\sigma_{\mu\beta\dot\beta}\bar\psi^{\dot\beta}\bar\sigma^{\mu\dot\alpha\alpha}. \tag{3.8}$$

The interesting outcome of this manipulation is of course that the free indices formerly carried by the spinors are now carried by the $\bar\sigma$ matrix.

We can thus continue with the evaluation of the variation:

$$\begin{aligned} [\delta_1, \delta_2]\psi_\alpha &= -i\sigma^\mu_{\alpha\dot\alpha}\epsilon_2\sigma_\nu\bar\epsilon_1\bar\sigma^{\nu\dot\alpha\beta}\partial_\mu\psi_\beta + i\sigma^\mu_{\alpha\dot\alpha}\epsilon_1\sigma_\nu\bar\epsilon_2\bar\sigma^{\nu\dot\alpha\beta}\partial_\mu\psi_\beta \\ &= i(\epsilon_1\sigma_\nu\bar\epsilon_2 - \epsilon_2\sigma_\nu\bar\epsilon_1)\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta}\partial_\mu\psi_\beta \\ &= i(\epsilon_1\sigma_\nu\bar\epsilon_2 - \epsilon_2\sigma_\nu\bar\epsilon_1)(2\eta^{\mu\nu}\delta^\beta_\alpha - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta})\partial_\mu\psi_\beta \\ &= 2i(\epsilon_1\sigma^\mu\bar\epsilon_2 - \epsilon_2\sigma^\mu\bar\epsilon_1)\partial_\mu\psi_\alpha - i(\epsilon_1\sigma_\nu\bar\epsilon_2 - \epsilon_2\sigma_\nu\bar\epsilon_1)\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta. \end{aligned}$$

We thus obtain what we wanted, up to an extra term which however vanishes whenever the equations of motion of ψ are satisfied (indeed, the latter read $\bar\sigma^\mu\partial_\mu\psi = 0$), i.e. *on-shell*. This is not completely satisfying, because we would like to be able to close the SUSY algebra off-shell, that is even if the equations of motion are not satisfied. One motivation for that is for instance to be able to do quantum field theory at a later stage (where virtual particles are usually not on-shell).

A way out in order to be able to do so, is to introduce *auxiliary fields* in addition to the physical fields ϕ and ψ.

3.1.2 Introducing auxiliary fields

If we are to realize the SUSY algebra off-shell, we should balance the degrees of freedom off-shell, i.e. the number of components of the fields. Since we have 2 components from the complex ϕ and 4 components from the Weyl spinor ψ, we see that we need an additional 2 bosonic components.

We thus add another complex scalar f, of mass dimension two $[f] = M^2$, and modify the variation of ψ to account for an extra term:

$$\delta\psi_\alpha = i\sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\partial_\mu\phi + \sqrt{2}\epsilon_\alpha f. \tag{3.9}$$

Of course we also have to write a variation for f:

$$\delta f = i\sqrt{2}\bar\epsilon_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha. \tag{3.10}$$

Note that it is proportional to the equations of motion of ψ, i.e. it vanishes on-shell.

Then we can again verify the SUSY algebra on all fields. First we re-check it on ϕ, since we have an additional piece in (3.9):

$$\begin{aligned}[\delta_1, \delta_2]\phi &= \sqrt{2}\epsilon_1^\alpha\delta_2\psi_\alpha - \sqrt{2}\epsilon_2^\alpha\delta_1\psi_\alpha \\ &= \cdots + 2\epsilon_1^\alpha\epsilon_{2\alpha}f - 2\epsilon_2^\alpha\epsilon_{1\alpha}f \\ &= \cdots + 0,\end{aligned}$$

that is the we get what we already had. For ψ, we now have:

$$\begin{aligned}[\delta_1, \delta_2]\psi_\alpha &= \delta_1\delta_2\psi_\alpha - \delta_2\delta_1\psi_\alpha \\ &= \cdots + \sqrt{2}\epsilon_{1\alpha}\delta_2 f - \sqrt{2}\epsilon_{2\alpha}\delta_1 f \\ &= \cdots + 2i\epsilon_{1\alpha}\bar\epsilon_{2\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta - 2i\epsilon_{2\alpha}\bar\epsilon_{1\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta.\end{aligned}$$

We now use again a Fierz identity, which reads:

$$\psi_\alpha\bar\chi_{\dot\alpha} = \frac{1}{2}\psi\sigma_\mu\bar\chi\sigma^\mu_{\alpha\dot\alpha}.$$

Going on, and putting together with the previous result, we get:

$$\begin{aligned}[\delta_1,\delta_2]\psi_\alpha &= 2i(\epsilon_1\sigma^\mu\bar\epsilon_2-\epsilon_2\sigma^\mu\bar\epsilon_1)\partial_\mu\psi_\alpha - i(\epsilon_1\sigma_\nu\bar\epsilon_2-\epsilon_2\sigma_\nu\bar\epsilon_1)\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta \\ &\quad -i\epsilon_2\sigma_\nu\bar\epsilon_1\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta + i\epsilon_1\sigma_\nu\bar\epsilon_2\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta \\ &= 2i(\epsilon_1\sigma^\mu\bar\epsilon_2-\epsilon_2\sigma^\mu\bar\epsilon_1)\partial_\mu\psi_\alpha.\end{aligned}$$

Now the SUSY algebra is realized on ψ *off-shell*.

As a last step, we close the SUSY algebra also on the new field f:

$$\begin{aligned}[\delta_1,\delta_2]f &= \delta_1\delta_2 f-\delta_2\delta_1 f \\ &= i\sqrt{2}\bar\epsilon_{1\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\delta_2\psi_\alpha - i\sqrt{2}\bar\epsilon_{2\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\delta_1\psi_\alpha \\ &= -2\bar\epsilon_{1\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta}\bar\epsilon_2^{\dot\beta}\partial_\mu\partial_\nu\phi + 2i\bar\epsilon_{1\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{2\alpha}\partial_\mu f \\ &\quad +2\bar\epsilon_{2\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta}\bar\epsilon_1^{\dot\beta}\partial_\mu\partial_\nu\phi - 2i\bar\epsilon_{2\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{1\alpha}\partial_\mu f \\ &= -2\bar\epsilon_1\bar\epsilon_2\partial_\mu\partial^\mu\phi + 2i\bar\epsilon_1\bar\sigma^\mu\epsilon_2\partial_\mu f \\ &\quad +2\bar\epsilon_2\bar\epsilon_1\partial_\mu\partial^\mu\phi - 2i\bar\epsilon_2\bar\sigma^\mu\epsilon_1\partial_\mu f \\ &= 2i(\epsilon_1\sigma^\mu\bar\epsilon_2-\epsilon_2\sigma^\mu\bar\epsilon_1)\partial_\mu f,\end{aligned}$$

as desired (recall that $\bar\chi\bar\sigma^\mu\psi = -\psi\sigma^\mu\bar\chi$).

Thus we have a set of fields ϕ,ψ,f with variations which represent the SUSY algebra off-shell. As we already mentioned, the two complex scalars have a total of 4 off-shell real degrees of freedom, while one Weyl fermion also has two complex components and thus 4 off-shell real degrees of freedom. However on-shell the degrees of freedom of the Weyl fermion reduce to 2 (because the equations of motion are of first order), while the ones of the scalars, if they are dynamical, usually stay the same number. We conclude that the scalar f must have no dynamics.

Indeed, consider the variation of the action (3.7) with the new term in $\delta\psi_\alpha$ (and recall that $\delta\bar\psi_{\dot\alpha} = \cdots + \sqrt{2}\bar\epsilon_{\dot\alpha}f^*$). Given that without the additional term the variation was zero, we get:

$$\begin{aligned}\delta S_{\text{tot}} &= -\int d^4x\,\sqrt{2}i\left(f^*\bar\epsilon\bar\sigma^\mu\partial_\mu\psi + \bar\psi\bar\sigma^\mu\epsilon\partial_\mu f\right) \\ &= -\int d^4x\,\sqrt{2}i\left(f^*\bar\epsilon\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\bar\sigma^\mu\epsilon f\right).\end{aligned}$$

Since $\delta f = i\sqrt{2}\bar\epsilon\bar\sigma^\mu\partial_\mu\psi$, and $\delta f^* = -i\sqrt{2}\partial_\mu\bar\psi\bar\sigma^\mu\epsilon$, if we set:

$$S_{\text{aux}} = \int d^4x\, f^* f,$$

we have

$$\begin{aligned}\delta S_{\text{aux}} &= \int d^4x \, (f^* \delta f + \delta f^* f) \\ &= \int d^4x \, \sqrt{2} i \left(f^* \bar{\epsilon} \bar{\sigma}^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \bar{\sigma}^\mu \epsilon f \right)\end{aligned}$$

so that

$$\delta S_{\text{tot}} + \delta S_{\text{aux}} = 0,$$

i.e.

$$S'_{\text{tot}} = S_{\text{tot}} + S_{\text{aux}} = \int d^4x \, \left(\partial_\mu \phi \partial^\mu \phi^* - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + f^* f \right). \tag{3.11}$$

is invariant under SUSY variations.

Clearly, the equations of motion for f are

$$f = 0 = f^*,$$

so there are no propagating degrees of freedom for f. This is a confirmation that it is indeed an auxiliary field.

The above is the simplest SUSY field theory, of a massless free scalar and a massless free Weyl fermion. As such, it is rather empty. We would now like to introduce possibly mass terms, and most of all interactions.

3.1.3 Supersymmetric mass terms

It is easy to notice that it is not completely straightforward to introduce mass terms in a supersymmetric action. Consider the usual mass term for a complex scalar:

$$\mathcal{L}_{m\phi} = -|m|^2 \phi^* \phi.$$

It variation reads:

$$\delta \mathcal{L}_{m\phi} = -\sqrt{2} |m|^2 (\phi^* \epsilon \psi + \bar{\epsilon} \bar{\psi} \phi).$$

On the other hand, for a Weyl fermion the (Majorana) mass term is:

$$\mathcal{L}_{m\psi} = \frac{1}{2} m \psi^\alpha \psi_\alpha + \frac{1}{2} m^* \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}},$$

so that the variation of the corresponding term in the action is:

$$\begin{aligned}
\delta \int d^4x\, \mathcal{L}_{m\psi} &= \int d^4x\, (m\psi^\alpha \delta\psi_\alpha + m^* \delta\bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) \\
&= \int d^4x\, \sqrt{2}(im\psi\sigma^\mu\bar{\epsilon}\partial_\mu\phi + m\psi\epsilon f - im^*\epsilon\sigma^\mu\bar{\psi}\partial_\mu\phi^* + m^* f^*\bar{\epsilon}\bar{\psi}) \\
&= \int d^4x\, \sqrt{2}(m\psi\epsilon f + m^* f^*\bar{\epsilon}\bar{\psi} - im\partial_\mu\psi\sigma^\mu\bar{\epsilon}\phi + im^*\phi^*\epsilon\sigma^\mu\partial_\mu\bar{\psi})
\end{aligned}$$

It is obvious that the above variation cannot be compensated off-shell by $\delta\mathcal{L}_{m\phi}$. However it is also rather easy to see that it is compensated by the variation of

$$\mathcal{L}_{m\mathrm{bos}} = -m\phi f - m^* f^* \phi^*.$$

We can thus write a tentative Lagrangian for the mass terms:

$$\mathcal{L}_m = \frac{1}{2} m\psi\psi - m\phi f + \frac{1}{2} m^* \bar{\psi}\bar{\psi} - m^* f^* \phi^*, \tag{3.12}$$

which satisfies

$$\delta \int d^4x\, \mathcal{L}_m = 0.$$

In order to see that it is indeed a mass term for the bosons, let us consider the full bosonic Lagrangian:

$$\mathcal{L}_{\mathrm{bos}} = \partial_\mu\phi^*\partial^\mu\phi + ff^* - m\phi f - m^* f^* \phi^*.$$

Imposing the equations of motion for the auxiliary fields f (i.e. "integrating them out"), we obtain:

$$\frac{\delta\mathcal{L}_{\mathrm{bos}}}{\delta f} = f^* - m\phi = 0.$$

This implies

$$f^* = m\phi, \qquad f = m^*\phi^*,$$

so that the bosonic Lagrangian becomes

$$\mathcal{L}_{\mathrm{bos}}|_f = \partial_\mu\phi^*\partial^\mu\phi - |m|^2\phi\phi^*,$$

indeed the Lagrangian of a massive scalar.

In other words, we have computed the potential for the scalar:

$$V(\phi, \phi^*) = |m|^2\phi\phi^* = ff^*.$$

Note that supersymmetry imposes the mass degeneracy between the fermions and the bosons also at the level of the action, of course.

3.1.4 Supersymmetric interactions

Let us now consider the addition of interactions. We have many options, but it is here that we expect to find new constraints from supersymmetry.

Consider interaction terms of dimension 3 or 4 (i.e. we consider renormalizable interactions). For interaction terms involving the fermions, the only option is to have a Yukawa-like interaction involving two fermions and a scalar, schematically $\phi\psi\psi$. For interactions among bosons only, we have at dimension 3 only a term trilinear in the scalars ϕ^3, while at dimension 4 we have two options, $f\phi^2$ and ϕ^4.

Now, the only interaction term at dimension 3 cannot be supersymmetric by itself (off shell), and hence cannot be present in a SUSY Lagrangian. We are left with the dimension 4 terms, and the only hope is to play the ones with the fermions against the bosonic ones. Again, the ϕ^4 is not promising since it variates to $\phi^3\epsilon\psi$, with no derivatives, which will not be cancelled by any term from $\delta\phi\psi\psi$. Hence it turns out that the only possibility is to play $\phi\psi\psi$ against $f\phi^2$ (where now we intend precisely the *holomorphic* fields, while the preceding discussion is valid also for terms mixing complex conjugates).

Consider the following variation:

$$\begin{aligned}\delta(\phi\psi^\alpha\psi_\alpha) &= \sqrt{2}\left\{\epsilon^\alpha\psi_\alpha\psi^\beta\psi_\beta + 2\phi\psi^\alpha(i\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\partial_\mu\phi + \epsilon_\alpha f)\right\}\\ &= 2\sqrt{2}(i\phi\psi\sigma^\mu\bar\epsilon\partial_\mu\phi + \phi\psi\epsilon f).\end{aligned}$$

Indeed, note that $\psi_\alpha\psi^\beta\psi_\beta = 0$ because in every term of the sum we will have ψ_1 or ψ_2 appearing twice, and $(\psi_1)^2 = 0 = (\psi_2)^2$ by Grassmann parity. Notice also that a similar term trilinear in the fermions would *not* vanish if we computed the variation $\delta(\phi^*\psi^\alpha\psi_\alpha)$. This is what constrains us to consider only the purely holomorphic interaction term.

Take then:

$$\begin{aligned}\delta\int d^4x\, f\phi^2 &= \sqrt{2}\int d^4x\,(i\bar\epsilon\bar\sigma^\mu\partial_\mu\psi\phi^2 + 2f\phi\epsilon\psi)\\ &= \sqrt{2}\int d^4x\,(-2i\bar\epsilon\bar\sigma^\mu\psi\phi\partial_\mu\phi + 2f\phi\epsilon\psi)\\ &= 2\sqrt{2}\int d^4x\,(i\psi\sigma^\mu\bar\epsilon\phi\partial_\mu\phi + f\phi\epsilon\psi).\end{aligned}$$

We have thus constructed the Lagrangian interaction terms:

$$\mathcal{L}_{\text{int}} = \lambda\phi\psi\psi - \lambda f\phi^2 + h.c., \tag{3.13}$$

where λ is the coupling. They satisfy

$$\delta \int d^4x\, \mathcal{L}_{\rm int} = 0.$$

An important fact is the following. The above variation vanishes without mixing between the holomorphic fields ϕ, ψ and f and the anti-holomorphic ones ϕ^*, $\bar{\psi}$ and f^*. This was actually also true for the mass terms in $\mathcal{L}_m$. As we will see in the next chapter, this fact has a deep reason.

We are now ready to sum up all the pieces together, and we write the complete supersymmetric Lagrangian for a massive complex scalar and massive Weyl fermion in interaction:

$$\mathcal{L}''_{\rm tot} = \partial_\mu\phi^*\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + ff^* + \left(\frac{1}{2}m\psi\psi - m\phi f + \lambda\phi\psi\psi - \lambda f\phi^2 + h.c.\right). \tag{3.14}$$

As before, we integrate out the auxiliary fields by imposing their algebraic (non-dynamical) equations of motion:

$$\frac{\delta\mathcal{L}''_{\rm tot}}{\delta f} = f^* - m\phi - \lambda\phi^2.$$

After substitution, we obtain:

$$\begin{aligned}\mathcal{L}''_{\rm tot}|_{f\,{\rm on-shell}} &= \partial_\mu\phi^*\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \left(\frac{1}{2}m\psi\psi + \lambda\phi\psi\psi + h.c.\right)\\ &\quad -|m|^2|\phi|^2 - |\lambda|^2|\phi|^4 - (m^*\lambda\phi^*\phi^2 + h.c.).\end{aligned} \tag{3.15}$$

We recognize a Yukawa term $\phi\psi\psi$ with coupling λ, and a quartic interaction for the scalar $|\phi|^4$ with coupling $|\lambda|^2$. In other words, as anticipated in the first chapter, there are relations among the couplings such as $\lambda_{\rm quartic} = |\lambda_{\rm Yukawa}|^2$. There are also couplings which are strictly set to zero, such as $\phi^*\psi\psi$ and ϕ^3.

We will see in the next chapter that there is a much simpler and natural way to encode all these facts.

3.2 The theory for the vector multiplet

Let us now briefly consider the supersymmetric field theory of the massless vector multiplet, which is obviously relevant to supersymmetric gauge theories. In this simple approach, we should already warn the reader that gauge symmetry might mix non-trivially, in the sense that sometimes results will be obtained up to gauge transformations. Indeed, gauge invariance is an important element in the counting of degrees of freedom, and SUSY is all about matching exactly degrees of freedom.

Since

$$\frac{1}{2\sqrt{E}}\bar{Q}_{\dot{2}}|\lambda = \tfrac{1}{2}\rangle = |\lambda = 1\rangle$$

we will have in the field content a spin 1 vector gauge field A_μ and a Weyl fermion conventionally denoted as λ_α. It turns out that we will need, as for the scalar multiplet, an auxiliary field D, which is here a real scalar. The SUSY transformations are as follows:

$$\delta A^\mu = -i\bar{\lambda}\bar{\sigma}^\mu\epsilon + i\bar{\epsilon}\bar{\sigma}^\mu\lambda, \tag{3.16}$$

$$\delta\lambda_\alpha = -\sigma^{\mu\nu}{}_\alpha{}^\beta\epsilon_\beta F_{\mu\nu} + i\epsilon_\alpha D, \tag{3.17}$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\beta}}\bar{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}}F_{\mu\nu} - i\bar{\epsilon}_{\dot{\alpha}}D, \tag{3.18}$$

$$\delta D = -\epsilon\sigma^\mu\partial_\mu\bar{\lambda} - \partial_\mu\lambda\sigma^\mu\bar{\epsilon}. \tag{3.19}$$

One can then start checking that the above SUSY transformations indeed represent the SUSY algebra on all fields. For instance:

$$\begin{aligned}
[\delta_1, \delta_2]A^\mu &= \delta_1\delta_2 A^\mu - \delta_2\delta_1 A^\mu \\
&= -i\delta_2\bar{\lambda}\bar{\sigma}^\mu\epsilon_1 + i\bar{\epsilon}_1\bar{\sigma}^\mu\delta_2\lambda + i\delta_1\bar{\lambda}\bar{\sigma}^\mu\epsilon_2 - i\bar{\epsilon}_2\bar{\sigma}^\mu\delta_1\lambda \\
&= -i\bar{\epsilon}_2\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu\epsilon_1 F_{\rho\sigma} - \bar{\epsilon}_2\bar{\sigma}^\mu\epsilon_1 D - i\bar{\epsilon}_1\bar{\sigma}^\mu\sigma^{\rho\sigma}\epsilon_2 F_{\rho\sigma} - \bar{\epsilon}_1\bar{\sigma}^\mu\epsilon_2 D \\
&\quad + i\bar{\epsilon}_1\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu\epsilon_2 F_{\rho\sigma} + \bar{\epsilon}_1\bar{\sigma}^\mu\epsilon_2 D + i\bar{\epsilon}_2\bar{\sigma}^\mu\sigma^{\rho\sigma}\epsilon_1 F_{\rho\sigma} + \bar{\epsilon}_2\bar{\sigma}^\mu\epsilon_1 D \\
&= i\bar{\epsilon}_1(\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu - \bar{\sigma}^\mu\sigma^{\rho\sigma})\epsilon_2 F_{\rho\sigma} - i\bar{\epsilon}_2(\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu - \bar{\sigma}^\mu\sigma^{\rho\sigma})\epsilon_1 F_{\rho\sigma}.
\end{aligned}$$

Now we use the identity

$$\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu - \bar{\sigma}^\mu\sigma^{\rho\sigma} = -\eta^{\mu\rho}\bar{\sigma}^\sigma + \eta^{\mu\sigma}\bar{\sigma}^\rho$$

so that we get:

$$\begin{aligned}
[\delta_1, \delta_2]A^\mu &= -2i\bar{\epsilon}_1\bar{\sigma}_\nu\epsilon_2 F^{\mu\nu} + 2i\bar{\epsilon}_2\bar{\sigma}_\nu\epsilon_1 F^{\mu\nu} \\
&= 2i(\epsilon_2\sigma_\nu\bar{\epsilon}_1 - \epsilon_1\sigma_\nu\bar{\epsilon}_2)F^{\mu\nu} \\
&= 2i(\epsilon_1\sigma^\nu\bar{\epsilon}_2 - \epsilon_2\sigma^\nu\bar{\epsilon}_1)\partial_\nu A^\mu + \partial^\mu\left[2i(\epsilon_2\sigma_\nu\bar{\epsilon}_1 - \epsilon_1\sigma_\nu\bar{\epsilon}_2)A^\nu\right].
\end{aligned}$$

The first term in the result is what we indeed expect for a realization of the SUSY algebra on A_μ. The second term is an extra, but not worrysome, since it takes the form of a gauge transformation on the vector field A_μ. Since A_μ is defined up to gauge transformations, we see that upon modding out by this gauge freedom the superalgebra indeed closes on the vector field.

If we count on-shell degrees of freedom, we know that we have 2 for A_μ, 2 for λ and 0 for D since the latter has $[D] = M^2$ and hence has no dynamics. However, we are aiming here at balancing the degrees of freedom also off-shell, and the counting goes as follows. The are 4 d.o.f. for λ, 1 for D (recall that it is real), and hence we are left with 3 for A_μ. We can understand this number by the fact that as shown above, it is really on equivalence classes under the gauge transformations that the superalgebra closes. Off-shell, one such equivalence class is specified by the choice of the 4 components of A_μ, up to a transformation involving one scalar function $A_\mu \to A_\mu + \partial_\mu \alpha$, which makes the 3 independent components that we were seeking.

We will later see that there is a formalism where we do not need to go through these subtleties, the price to pay being the introduction of additional non-physical degrees of freedom.

Let us now skip the other (rather tedious) commutators of SUSY variations, and proceed to check the invariance of the action:

$$S = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \right). \tag{3.20}$$

Under a SUSY transformation we have:

$$\begin{aligned}
\delta S &= \int d^4x \left(-F^{\mu\nu}\partial_\mu \delta A_\nu - i\delta\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda - i\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \delta\lambda + D\delta D \right) \\
&= \int d^4x \left(iF^{\mu\nu}\partial_\mu \bar{\lambda}\bar{\sigma}_\nu \epsilon - iF^{\mu\nu}\bar{\epsilon}\bar{\sigma}_\nu \partial_\mu \lambda - i\bar{\epsilon}\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu \partial_\mu \lambda F_{\rho\sigma} - \bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \lambda D \right. \\
&\qquad \left. +i\,\bar{\lambda}\bar{\sigma}^\mu \sigma^{\rho\sigma}\epsilon \partial_\mu F_{\rho\sigma} + \bar{\lambda}\bar{\sigma}^\mu \epsilon \partial_\mu D - D\epsilon\sigma^\mu \partial_\mu \bar{\lambda} - D\partial_\mu \lambda \sigma^\mu \bar{\epsilon} \right) \\
&= \int d^4x \left(iF^{\mu\nu}\partial_\mu \bar{\lambda}\bar{\sigma}_\nu \epsilon - iF^{\mu\nu}\bar{\epsilon}\bar{\sigma}_\nu \partial_\mu \lambda + i\bar{\epsilon}\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu \lambda \partial_\mu F_{\rho\sigma} + \partial_\mu \lambda \sigma^\mu \bar{\epsilon} D \right. \\
&\qquad \left. +i\,\bar{\lambda}\bar{\sigma}^\mu \sigma^{\rho\sigma}\epsilon \partial_\mu F_{\rho\sigma} + \epsilon\sigma^\mu \partial_\mu \bar{\lambda} D - D\epsilon\sigma^\mu \partial_\mu \bar{\lambda} - D\partial_\mu \lambda \sigma^\mu \bar{\epsilon} \right).
\end{aligned}$$

The terms containing D simplify, while for the others one has to use the following identity:

$$\bar{\lambda}\bar{\sigma}^\mu \sigma^{\rho\sigma}\epsilon \partial_\mu F_{\rho\sigma} = \bar{\lambda}\bar{\sigma}^\nu \epsilon \partial^\mu F_{\mu\nu}$$

and its complex conjugate

$$\bar{\epsilon}\bar{\sigma}^{\rho\sigma}\bar{\sigma}^\mu \lambda \partial_\mu F_{\rho\sigma} = -\bar{\epsilon}\bar{\sigma}^\nu \lambda \partial^\mu F_{\mu\nu}$$

where in deriving them we have used for instance that $\partial_{[\mu}F_{\rho\sigma]} = 0$.

Eventually we get:

$$\delta S = \int d^4x \, \left(iF^{\mu\nu}\partial_\mu \bar{\lambda}\bar{\sigma}_\nu \epsilon - iF^{\mu\nu}\bar{\epsilon}\bar{\sigma}_\nu \partial_\mu \lambda - i\bar{\epsilon}\bar{\sigma}^\nu \lambda \partial^\mu F_{\mu\nu} + i\bar{\lambda}\bar{\sigma}^\nu \epsilon \partial^\mu F_{\mu\nu} \right) = 0,$$

so that the action is invariant up to a boundary term as usual. We have thus shown that the action (3.20) is the correct one for the supersymmetrization of the action of an abelian (free) vector field.

It is important to note that promoting the above discussion to a vector field taking values in a non-abelian gauge group is non-trivial. This is to be expected since by doing so one is actually introducing interactions and we have seen that interactions must be introduced delicately in a supersymmetric field theory. In particular, all the members of the vector multiplet A_μ, λ and D must belong to the adjoint representation of the gauge group, which means that they transform non-trivially under gauge transformations, e.g.

$$\delta_{\text{gauge}}\lambda = ig[\lambda, \alpha]$$

and hence in turn they couple non-trivially with the gauge boson.

However, we will not pursue the construction of a non-abelian gauge theory along the lines discussed in the present chapter. Indeed, it should be by now obvious that building SUSY field theories in this way is rather painful: one has to check first that the SUSY algebra is realized on every field, and then check that the action that one has guessed is also invariant under SUSY transformations. This is *not* the way that one builds Lorentz invariant actions for instance! Rather, we use a formulation that is manifestly Lorentz covariant. It is then clear that we need an organizing principle also for supersymmetry. This will be the subject of the next chapter.

Chapter 4

Superspace and superfields

The aim of this chapter is to present a formulation of supersymmetric field theories which is manifestly supersymmetric. The key in achieving this is to see supersymmetric variations as the result of generalized translations in *superspace*, which is an extension of ordinary spacetime by Grassmannian (i.e. "fermionic") coordinates. Then, the various fields which are mapped to each other through supersymmetric variations are assembled together into *superfields*, which depend on all the coordinates of superspace. It is in terms of the superfields that we are able to write manifestly SUSY invariant actions. It will also turn out that we need to constrain the most general superfield, in two different ways, in order to reproduce the field content of the theories for both the scalar and the vector multiplet, respectively.

4.1 Introducing superspace and superfields

In the previous chapter, we have constructed supersymmetric field theories by enforcing that they be invariant under supersymmetry transformations, that we had defined as acting on the fields. This approach is quite laborious, and we would thus like to formulate SUSY field theories in a way that SUSY is a manifest symmetry, so that we could avoid to have to check invariance under it at every step in the construction.

Let us have a brief interlude to illustrate a similar need for manifest invariance in the presumably familiar example of Lorentz symmetry.

The vacuum Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \times \vec{B} = -\partial_t \vec{E}, \quad \vec{\nabla} \times \vec{E} = \partial_t \vec{B}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

are invariant under (infinitesimal) boosts

$$\delta x^0 = \omega x^3, \qquad \delta x^3 = \omega x^0$$

provided the electric and magnetic fields also transform as

$$\delta E_1 = \omega B_2, \quad \delta E_2 = -\omega B_1, \quad \delta B_1 = -\omega E_2, \quad \delta B_2 = \omega E_1.$$

The Lorentz transformation mixes E_i and B_i, so that an invariant term, suitable to appear in the action, is

$$E_i^2 - B_i^2,$$

where the sign is of course crucial for invariance.

All of this is made trivial using Lorentz covariant objects. The electric and magnetic fields are assembled in one field strength $F_{\mu\nu}$ such that

$$F_{0i} = E_i, \qquad F_{ij} = \epsilon_{ijk} B_k$$

and the Lorentz transformation of $F_{\mu\nu}$ reads

$$\delta F_{\mu\nu} = F_{\mu\rho}\omega^\rho{}_\nu - F_{\nu\rho}\omega^\rho{}_\mu.$$

The Lagrangian is then trivially proportional to the only quadratic Lorentz scalar one can build from $F_{\mu\nu}$:

$$\mathcal{L} \propto F^{\mu\nu} F_{\mu\nu}.$$

Our aim is to pursue a similar route as far as supersymmetry is concerned.

First of all, we should ask ourselves what kind of spacetime symmetry is closest to supersymmetry. From the SUSY algebra

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu, \tag{4.1}$$

it is obvious: the translations, which are generated by P_μ.

The action of a translation $x^\mu \to x^\mu + a^\mu$ on a field $\phi(x)$ yields the following variation:

$$\delta\phi(x) = i a^\mu P_\mu \phi(x) = a^\mu \partial_\mu \phi(x). \tag{4.2}$$

This can be seen as the limit of small a^μ of a finite translation which, at the operator level, reads $\phi(x+a) = e^{ia^\mu P_\mu}\phi(x) e^{-ia^\mu P_\mu}$.

This is well-known, but it is important here to adopt a point of view which is to look at a specific fixed spacetime point x. At that point, we see that a translation maps one field ϕ, to another field $\partial_\mu\phi$, in a different Lorentz representation, but of the same Grassmann G-parity. This is because the parameter of the translation is a^μ, a Grassmann even (bosonic) quantity, as the coordinates of spacetime x^μ are.

Now, supersymmetry maps instead G-even fields ϕ to G-odd fields ψ (at the same spacetime point of course), and the SUSY parameter ϵ (or $\bar{\epsilon}$) is G-odd. It is tempting to interpret ϵ_α as parametrizing a translation in a *G-odd coordinate* θ_α. Spacetime is thus extended to *superspace* parametrized by

$$x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}.$$

The Grassmann coordinates being complex, the full superspace is indeed parametrized also by their complex conjugates (just as $\mathbb{R}^2 = \mathbb{C}$ is parametrized by z and $\bar{z}$).

In the same way as we expand a spacetime field in its derivatives

$$\phi(x) = \phi(0) + x^\mu \partial_\mu \phi(0) + \dots$$

we expand a *superfield* both in x^μ and in $\theta, \bar{\theta}$. However the coefficients in this expansion really are different spacetime fields, with alternate G-parity:

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \dots$$

A crucial and important difference is that since $\theta, \bar{\theta}$ are anticommuting, the expansion is *finite*: $(\theta_1)^2 = 0$ and similarly for the other Grassmann coordinates. The expansion stops when we have all four of them: $\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$.

The most general superfield is thus:

$$\begin{aligned} Y(x, \theta, \bar{\theta}) &= \phi(x) + \theta\eta(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) \\ &\quad + \theta\sigma^\mu\bar{\theta} A_\mu(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\psi(x) + \theta^2\bar{\theta}^2 d(x). \end{aligned} \tag{4.3}$$

We have defined:

$$\begin{aligned} \theta^2 &= \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\beta\theta_\alpha = 2\theta_2\theta_1 = -2\theta_1\theta_2, \\ \bar{\theta}^2 &= \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = 2\bar{\theta}_1\bar{\theta}_2. \end{aligned} \tag{4.4}$$

Note also that

$$\begin{aligned} \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta^2, & \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2, \\ \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^2, & \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2, \end{aligned} \tag{4.5}$$

and recall that

$$\theta_\alpha \bar{\theta}_{\dot{\alpha}} = \frac{1}{2} \theta \sigma_\mu \bar{\theta} \sigma^\mu_{\alpha\dot{\alpha}}. \tag{4.6}$$

The superfield Y is composed of 4 complex scalars ϕ, m, n and d, 1 complex vector A_μ and 4 Weyl fermions $\eta, \psi, \bar{\chi}$ and $\bar{\lambda}$. In total we have 16 bosonic degrees of freedom and 16 fermionic degrees of freedom. These are all off-shell degrees of freedom.

Clearly, a generic superfield such as the one above forms a representation of the SUSY algebra (indeed, we have a balance between bosonic and fermionic degrees of freedom), but it must be reducible, since we have already encountered much smaller representations. Recall that the scalar multiplet had $4+4$ off-shell degrees of freedom, and the vector multiplet also had $4+4$ off-shell (but gauge fixed) degrees of freedom. We will see shortly how to reduce consistently the number of components in a superfield. However we have first to see how it represents the SUSY algebra.

4.1.1 Supersymmetry transformations as translations in superspace

We have established supersymmetry transformations as translations in the $\theta, \bar{\theta}$ coordinates of superspace. However, if they did just that two SUSY transformations would anticommute to nothing. Instead they must anticommute to a spacetime translation.

Recall that

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$$

implies that (see e.g. (3.4))

$$\begin{aligned}
[i(\epsilon_1 Q + \bar{Q}\bar{\epsilon}_1), i(\epsilon_2 Q + \bar{Q}\bar{\epsilon}_2)] &= -2(\epsilon_1 \sigma^\mu \bar{\epsilon}_2 - \epsilon_2 \sigma^\mu \bar{\epsilon}_1) P_\mu \\
&= i a^\mu P_\mu .
\end{aligned}$$

Thus the commutator of two translations

$$\theta \to \theta + \epsilon_1,\ \bar{\theta} \to \bar{\theta} + \bar{\epsilon}_1 \qquad \text{and} \qquad \theta \to \theta + \epsilon_2,\ \bar{\theta} \to \bar{\theta} + \bar{\epsilon}_2,$$

is a spacetime translation

$$x^\mu \to x^\mu + 2i(\epsilon_1 \sigma^\mu \bar{\epsilon}_2 - \epsilon_2 \sigma^\mu \bar{\epsilon}_1).$$

As a consequence, a translation in θ must also be accompanied by some transformation on x^μ. It turns out that a full supertranslation is the following:

$$\theta \to \theta + \epsilon, \qquad \bar\theta \to \bar\theta + \bar\epsilon$$
$$x^\mu \to x^\mu + i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta. \tag{4.7}$$

This is actually the most symmetrical definition, other definitions are possible and may be useful in some special cases. In these notes we will stick to the definition above.

It is easy to check indeed using the above that

$$[\delta_1, \delta_2]x^\mu = 2i(\epsilon_1\sigma^\mu\bar\epsilon_2 - \epsilon_2\sigma^\mu\bar\epsilon_1)$$

as required.

We now wish to implement SUSY translations on superfields in the same way as we are used to implement spacetime translations on ordinary fields, that is through the operator $P_\mu = -i\partial_\mu$.

Clearly, restricting to the operator that is multiplied by the parameter ϵ, in order to implement at the same time $\theta \to \theta + \epsilon$ and $x^\mu \to x^\mu - i\epsilon\sigma^\mu\bar\theta$, the operator will involve a "derivative" with respect to θ and a term like $-i\sigma^\mu\bar\theta\partial_\mu$ (recall that $x^\mu \to x^\mu + a^\mu$ is represented by $a^\mu\partial_\mu$).

We must then first define derivatives with respect to G-odd variables. We require:

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta \equiv \partial_\alpha\theta^\beta = \delta_\alpha^\beta \qquad \text{and} \qquad \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta} \equiv \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}, \tag{4.8}$$

where we have defined our shorthands. Notice this implies

$$\partial^\alpha\theta_\beta = -\delta^\alpha_\beta \qquad \text{and} \qquad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = -\delta^{\dot\alpha}_{\dot\beta}.$$

As usual with Grassmann variables, we must anticommute the variable to the left of the monomial before acting on it with the derivative. (Alternatively, passing through the monomial we must anticommute the derivative.) Under hermitian conjugation we have

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^\dagger = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}.$$

Indeed, under *complex* conjugation we have that

$$\bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} = (\delta_\alpha^\beta)^* = (\partial_\alpha\theta^\beta)^* = \bar\theta^{\dot\beta}(\overleftarrow{\partial_\alpha})^* = -(\partial_\alpha)^*\bar\theta^{\dot\beta}$$

so that $(\partial_\alpha)^* = -\bar\partial_{\dot\alpha}$. Hermitian conjugation implies an extra $-$ sign as for ∂_μ.

We can then define the differential operators generating the supersymmetry transformations as

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu,$$
$$\bar Q_{\dot\alpha} = i\bar\partial_{\dot\alpha} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \qquad (4.9)$$

It is easily checked that

$$(Q_\alpha)^\dagger = i\partial^\dagger_\alpha - (\bar\theta^{\dot\alpha})^*(\sigma^\mu_{\alpha\dot\alpha})^\dagger\partial^\dagger_\mu = i\bar\partial_{\dot\alpha} - \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}(-\partial_\mu) = \bar Q_{\dot\alpha}.$$

We can also check that we correctly obtain

$$\begin{aligned}\delta_\epsilon x^\mu &= i\epsilon^\alpha Q_\alpha x^\mu = -i\epsilon\sigma^\mu\bar\theta,\\ \delta_{\bar\epsilon} x^\mu &= i\bar Q_{\dot\alpha}\bar\epsilon^{\dot\alpha}x^\mu = i\theta\sigma^\mu\bar\epsilon.\end{aligned}$$

Most importantly, the supersymmetry algebra is correctly represented:

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu - i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu.$$

Thus, a general SUSY transformation on a superfield Y will be obtained by acting with the operators Q and $\bar Q$ on Y:

$$\begin{aligned}\delta_{\epsilon,\bar\epsilon}Y &= i(\epsilon Q + \bar\epsilon\bar Q)Y\\ &= \left[\epsilon^\alpha\partial_\alpha - \bar\epsilon_{\dot\alpha}\bar\partial^{\dot\alpha} + i(\theta\sigma^\mu\bar\epsilon - \epsilon\sigma^\mu\bar\theta)\partial_\mu\right]Y(x,\theta,\bar\theta). \qquad (4.10)\end{aligned}$$

It is obvious that δY is also a superfield (just because it is a function of x, θ and $\bar\theta$), thus it can be decomposed in components, and one can then read the variations component by component. Since the action of Q and $\bar Q$ implies derivatives and multiplication with respect to $\theta, \bar\theta$, the bosonic and fermionic components are mixed under the variation.

If we start from (4.3)

$$Y = \phi + \theta\eta + \bar\theta\bar\chi + \theta^2 m + \bar\theta^2 n + \theta\sigma^\mu\bar\theta A_\mu + \theta^2\bar\theta\bar\lambda + \bar\theta^2\theta\psi + \theta^2\bar\theta^2 d,$$

we obtain for the variation

$$\begin{aligned}\delta Y &= \epsilon\eta + 2\theta\epsilon m + \epsilon\sigma^\mu\bar\theta A_\mu + 2\theta\epsilon\bar\theta\bar\lambda + \bar\theta^2\epsilon\psi + 2\bar\theta^2\theta\epsilon d\\ &\quad + \bar\epsilon\bar\chi + 2\bar\theta\bar\epsilon n + \theta\sigma^\mu\bar\epsilon A_\mu + \theta^2\bar\epsilon\bar\lambda + 2\bar\theta\bar\epsilon\theta\psi + 2\theta^2\bar\theta\bar\epsilon d\\ &\quad + i\theta\sigma^\mu\bar\epsilon\partial_\mu\phi - i\epsilon\sigma^\mu\bar\theta\partial_\mu\phi + \ldots\\ &\quad + i\theta\sigma^\mu\bar\epsilon\bar\theta^2\theta\partial_\mu\psi - i\epsilon\sigma^\mu\bar\theta\theta^2\bar\theta\partial_\mu\bar\lambda.\end{aligned}$$

Thus we learn that the variations of the components are for instance

$$\begin{aligned}
\delta\phi &= \epsilon\eta + \bar\epsilon\bar\chi \\
\delta\eta &= i\sigma^\mu\bar\epsilon\partial_\mu\phi + 2\epsilon m + \sigma^\mu\bar\epsilon A_\mu \\
\delta\bar\chi &= -i\epsilon\sigma^\mu\partial_\mu\phi + 2\bar\epsilon n + \epsilon\sigma^\mu A_\mu \\
&\vdots \\
\delta d &= \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar\lambda - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar\epsilon, \qquad (4.11)
\end{aligned}$$

where we have used, e.g.

$$\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\theta^\beta\partial_\mu\psi_\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta^2\partial_\mu\psi_\beta\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} = -\tfrac{1}{2}\theta^2\partial_\mu\psi\sigma^\mu\bar\epsilon.$$

4.1.2 Manifestly supersymmetry invariant action

We now revert to the original goal, that is to write a manifestly SUSY invariant action.

Consider again for the sake of the example the Poincaré symmetry. An action which is invariant under Lorentz transformations and translations is

$$S = \int d^4x \mathcal{L}(\phi),$$

with $\mathcal{L}$ a Lorentz scalar function of the fields ϕ, and *no explicit dependence* on the spacetime coordinates x^μ.

Translations then generate a total derivative. Under $x^\mu \to x^\mu + a^\mu$, the action varies as

$$\delta S = \int d^4x\delta\mathcal{L} = \int d^4x a^\mu\partial_\mu\mathcal{L} = \int d^4x\partial_\mu(a^\mu\mathcal{L}).$$

Total derivatives do not affect the action principle, and thus the equations of motion, since they are boundary terms fixed by the boundary conditions one imposes on the fields. Hence the theory defined by S is invariant under translations.

Alternatively, we can obtain the same result in a slightly different (passive) way. Since there is no explicit x^μ dependence in $\mathcal{L}$, translations can be reabsorbed by the integration:

$$\begin{aligned}
\delta S &= \int d^4x\left[\mathcal{L}(\phi(x+a)) - \mathcal{L}(\phi(x))\right] = \int d^4x\mathcal{L}(\phi(x+a)) - \int d^4x\mathcal{L}(\phi(x)) \\
&= \int d^4x'\mathcal{L}(\phi(x')) - \int d^4x\mathcal{L}(\phi(x)) = 0.
\end{aligned}$$

In order to write $\int d^4(x'-a) \equiv \int d^4x'$ of course we also need to fix boundary conditions on ϕ.

We can now make the most natural guess for an action invariant under SUSY translations: we have to integrate over all of superspace

$$S \propto \int d^4x \int d^2\theta \int d^2\bar{\theta}(\dots).$$

Before continuing, we need therefore to give our definition of integrals over G-odd variables.

Let us consider first a single Grassmann variable θ (like, e.g., a single component θ_1 of θ_α). The most general function of θ is

$$f(\theta) = a + b\theta$$

since there cannot be terms like θ^2 or higher. Then we define in the most general way

$$\int d\theta(a + b\theta) = b.$$

Indeed, such a definition gives a translational invariant integral:

$$\int d\theta f(\theta + \epsilon) = \int d\theta(a + b\theta + b\epsilon) = b \qquad \text{since} \qquad \int d\theta\epsilon = 0.$$

We now define the integrals over $\int d^2\theta$ and $\int d^2\bar{\theta}$ in such a way that:

$$\int d^2\theta\theta^2 = 1, \qquad \int d^2\bar{\theta}\bar{\theta}^2 = 1. \tag{4.12}$$

Since $\theta^2 = 2\theta_2\theta_1$ and $\bar{\theta}^2 = 2\bar{\theta}_1\bar{\theta}_2$, we have

$$\int d^2\theta = \tfrac{1}{2}\int d\theta_1 d\theta_2, \qquad \int d^2\bar{\theta} = \tfrac{1}{2}\int d\bar{\theta}_2 d\bar{\theta}_1 \tag{4.13}$$

and also, eventually:

$$\int d^2\theta d^2\bar{\theta}\theta^2\bar{\theta}^2 = 1. \tag{4.14}$$

As a consequence, integrating a superfield over all of superspace will single out its highest component, which we called d before:

$$\int d^2\theta d^2\bar{\theta}\mathcal{Y}(x, \theta, \bar{\theta}) = d(x). \tag{4.15}$$

Here $\mathcal{Y}$ has to be considered more generally as a composite superfield, i.e. the result of a product of two (or possibly more) "elementary superfields". It is obvious that a product of superfields is still a superfield, and all its components, though being themselves composite, will still be related by the same supersymmetry transformations as given above.

Indeed, we have seen that the SUSY variation of d is a total derivative

$$\delta d = \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} = \partial_\mu\left(\frac{i}{2}\epsilon\sigma^\mu\bar{\lambda} - \frac{i}{2}\psi\sigma^\mu\bar{\epsilon}\right),$$

so that if the Lagrangian is just given by d,

$$\mathcal{L} = d \qquad \Rightarrow \qquad \delta\mathcal{L} = \partial_\mu v^\mu$$

and the action is invariant. In other words, we have shown that an action written as

$$S = \int d^4x d^2\theta d^2\bar{\theta}\mathcal{Y}, \tag{4.16}$$

with $\mathcal{Y}$ a scalar superfield (generically composite), is *automatically invariant under supersymmetry transformations*. This is what we refer to as manifest supersymmetry invariance.

As a last remark, note that here too there cannot be explicit θ dependence. Indeed, θ_α by itself is rather obviously *not* a superfield.

In order to write now sensible supersymmetric actions, we need to impose conditions on the generic superfield Y in such a way that it will eventually describe *irreducible* representations of the SUSY algebra.

4.2 Chiral superfields

The condition that we want to impose on a generic superfield must (anti)commute with the supercharges so that the constrained superfield is indeed mapped to itself by a SUSY transformation. In other words, the components which are projected out should *not* be re-generated by SUSY transformations.

One approach is to look for subsets of components of a general superfield Y that do not mix with each other when taking the variation δY. Concretely, one imposes a constraint on the superfield and then verifies that the variation of the superfield satisfies the same constraint. The outcome is as follows.

It turns out that it is convenient to introduce the following two operators:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$$

$$\bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{4.17}$$

which satisfy $(D_\alpha)^\dagger = \bar{D}_{\dot{\alpha}}$.

The two operators above look very similar to Q_α and $\bar{Q}_{\dot{\alpha}}$, except for a relative sign. It is because of this relative sign that they actually anticommute with Q_α and $\bar{Q}_{\dot{\alpha}}$. Indeed

$$\{D_\alpha, Q_\beta\} = 0 = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\},$$

$$\{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = \partial_\alpha(\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu) + i\bar{\partial}_{\dot{\alpha}}(i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu) = \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu - \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu = 0,$$

and similarly

$$\{\bar{D}_{\dot{\alpha}}, Q_\alpha\} = 0.$$

Moreover, among themselves these operators anticommute to

$$\{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\},$$

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.$$

Since D_α anticommutes with the supercharges, it can be used to write a condition that reduces the components of a general superfield. Indeed, if Y is such that $D_\alpha Y = 0$, then also δY satisfies

$$D_\alpha\delta Y = D_\alpha[i(\epsilon Q + \bar{Q}\bar{\epsilon})Y] = i(\epsilon Q + \bar{Q}\bar{\epsilon})D_\alpha Y = 0.$$

This means that the components that are eliminated with the constraint $D_\alpha Y = 0$ are not regenerated by a SUSY variation. The same can obviously be said of $\bar{D}_{\dot{\alpha}}Y = 0$.

We therefore define

- Chiral superfields
$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{4.18}$$

- Anti-chiral superfields
$$D_\alpha\bar{\Phi} = 0. \tag{4.19}$$

The names reflect the following facts. First note that

$$\bar{D}_{\dot{\alpha}}\theta^{\beta} = 0.$$

Another quantity that is annihilated by $\bar{D}_{\dot{\alpha}}$ is

$$\bar{D}_{\dot{\alpha}}(x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}) = -i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}} + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}} = 0.$$

Thus, if we call

$$y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}, \tag{4.20}$$

a chiral superfield only depends on θ and y:

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \qquad \Leftrightarrow \qquad \Phi = \Phi(y, \theta). \tag{4.21}$$

It is called chiral, or (equivalently) holomorphic, in the same sense as for a holomorphic function of z we have that

$$\frac{\partial}{\partial \bar{z}} f = 0 \qquad \Leftrightarrow \qquad f = f(z).$$

In components, we write:

$$\begin{aligned} \Phi(y, \theta) &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 f(y) \\ &= \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) - \tfrac{1}{2}\theta\sigma^{\mu}\bar{\theta}\theta\sigma^{\nu}\bar{\theta}\partial_{\mu}\partial_{\nu}\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) + i\sqrt{2}\theta\sigma^{\mu}\bar{\theta}\theta\partial_{\mu}\psi(x) + \theta^2 f(x). \end{aligned} \tag{4.22}$$

Recalling some identities like $\theta^{\alpha}\theta^{\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}$, $\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} = \bar{\sigma}^{\mu\alpha\dot{\alpha}}$ and $(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}{}^{\beta} = 2\eta^{\mu\nu}\delta^{\alpha}_{\beta}$, we have that

$$\theta\sigma^{\mu}\bar{\theta}\theta\sigma^{\nu}\bar{\theta}\partial_{\mu}\partial_{\nu}\phi = \tfrac{1}{2}\theta^2\bar{\theta}^2\partial_{\mu}\partial^{\mu}\phi$$

and

$$\theta\sigma^{\mu}\bar{\theta}\theta\partial_{\mu}\psi = \tfrac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^{\mu}\partial_{\mu}\psi.$$

Thus eventually, the chiral superfield in components reads

$$\begin{aligned} \Phi(y, \theta) &= \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial_{\mu}\partial^{\mu}\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^{\mu}\partial_{\mu}\psi(x) + \theta^2 f(x). \end{aligned} \tag{4.23}$$

We will not really often need the higher components, usually the component fields in terms of the shifted variable y are enough.

Let us now compute a SUSY variation of Φ and translate it to its components, as we did before for the general superfield $Y(x,\theta,\bar\theta)$. It is simplest to first recall that SUSY transformations are supertranslations:

$$\begin{aligned}\delta\theta^\alpha &= i(\epsilon Q+\bar\epsilon\bar Q)\theta^\alpha = \epsilon^\alpha,\\ \delta y^\mu &= i(\epsilon Q+\bar\epsilon\bar Q)y^\mu = \delta x^\mu + i\delta\theta\sigma^\mu\bar\theta + \theta\sigma^\mu\delta\bar\theta\\ &= i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\epsilon + i\epsilon\sigma^\mu\bar\theta = 2i\theta\sigma^\mu\bar\epsilon.\end{aligned}$$

Hence, we have

$$\delta\Phi = \partial_\mu\phi(y)\delta y^\mu + \sqrt2\delta\theta\psi(y) + \sqrt2\theta\partial_\mu\psi(y)\delta y^\mu + 2\theta\delta\theta f(y) + \theta^2\partial_\mu f(y)\delta y^\mu,$$

where we have used that $\delta F(y) = \partial_\mu F(y)\delta y^\mu$. The last term vanishes because it is cubic in θ. We thus obtain:

$$\begin{aligned}\delta\Phi &= 2i\theta\sigma^\mu\bar\epsilon\partial_\mu\phi(y) + \sqrt2\epsilon\psi(y) + 2\sqrt2 i\theta\sigma^\mu\bar\epsilon\theta\partial_\mu\psi(y) + 2\theta\epsilon f(y)\\ &= \sqrt2\epsilon\psi(y) + \sqrt2\theta\left(i\sqrt2\sigma^\mu\bar\epsilon\partial_\mu\phi(y) + \sqrt2\epsilon f(y)\right) + \theta^2\left(i\sqrt2\bar\epsilon\bar\sigma^\mu\partial_\mu\psi(y)\right).\end{aligned}$$

Correctly, we find an expression that depends only on θ and y, so that $\delta\Phi$ is a chiral superfield, $\bar D_{\dot\alpha}\delta\Phi = 0$. We can now read the variations of each component:

$$\begin{aligned}\delta\phi &= \sqrt2\epsilon\psi,\\ \delta\psi &= i\sqrt2\sigma^\mu\bar\epsilon\partial_\mu\phi(y) + \sqrt2\epsilon f,\\ \delta f &= i\sqrt2\bar\epsilon\bar\sigma^\mu\partial_\mu\psi.\end{aligned}\tag{4.24}$$

These are exactly the variations that we had postulated for the scalar (Wess-Zumino) model, see (3.1), (3.9) and (3.10).

4.2.1 Action for a chiral superfield

We have bundled together in a superfield the (off-shell) degrees of freedom of the scalar multiplet that we had discussed in the previous chapter. We have also checked that the suspersymmetry transformations are correctly reproduced, as expected for a manifestly SUSY formulation of the theory. What we are still missing is a way to write the action in terms of the superfields, so as to reproduce the action for the scalar supermultiplet theory.

Recall that an action is the spacetime integral of a Lagrangian

$$S = \int d^4x \mathcal{L}$$

where $\mathcal{L}$ is a real scalar, and in a manifestly SUSY formulation it must be the highest component of a general superfield, i.e. the coefficient of $\theta^2\bar{\theta}^2$.

Now, we expect the lowest terms in the action to be quadratic in the fields. Since the superfield Φ is linear in its component fields, we should thus look for an expression quadratic in Φ. Moreover, if we want the expression to be real, it should also involve the complex conjugate of Φ, that is $\bar{\Phi} = \Phi^\dagger$. It is an antichiral superfield, since $\bar{D}_{\dot{\alpha}}\Phi = 0$ implies

$$D_\alpha \bar{\Phi} = 0.$$

Hence it has an expansion given by

$$\bar{\Phi} = \phi^*(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) + \bar{\theta}^2 f^*(\bar{y}), \qquad \text{where} \qquad \bar{y}^\mu = (y^\mu)^* = x^\mu - i\theta\sigma^\mu\bar{\theta}.$$

Let us think one moment of dimensional analysis in order to write the action. Of course, we require $[\mathcal{L}] = M^4$ so that the action is dimensionless (we are using units where $\hbar = 1$). From the (free) action (3.11), we gather that the component fields have the usual canonical dimensions $[\phi] = M$, $[\psi] = M^{3/2}$ and $[f] = M^2$. At the beginning of Chapter 3 we also derived that the SUSY parameter should have dimension $[\epsilon] = M^{-1/2}$. Then, this must be the dimension of the Grassmann coordinates of superspace, $[\theta] = M^{-1/2}$. Thus, every term in the expansion of a chiral superfield has the same dimension, and the dimension of the superfield itself is given by

$$[\Phi] = M.$$

However, recall that in order to write a Lagrangian density in a manifestly SUSY invariant way, i.e. in terms of superfields, we need to integrate over superspace

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} Y. \tag{4.25}$$

Notice that $\int d\theta\theta = 1$ implies that the Grassmann differential has the *opposite* dimension with respect to the coordinate

$$[d\theta] = [\theta]^{-1} = M^{1/2},$$

so that the measure over all of superspace has dimension

$$[d^2\theta d^2\bar{\theta}] = M^2.$$

Given (4.25), this implies that

$$[\mathcal{L}] = M^4 \qquad \Leftrightarrow \qquad [Y] = M^2.$$

Collecting all the constraints derived from the above discussion, we learn that the action must be expressed in terms of a superfield Y which has dimension two, is quadratic in Φ and has to be real. There are only two options, $\Phi\bar{\Phi}$ and $(\Phi^2 + \bar{\Phi}^2)$. The latter however, after being integrated over all of superspace, becomes a total derivative (as we will see shortly, Φ^2 is also a chiral superfield, and hence its d-component is a total derivative, see eq. (4.23)). Hence the only possibility is really $Y = \Phi\bar{\Phi}$.

Let us then compute

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\Phi\bar{\Phi}.$$

In other words, we need to extract the $\theta^2\bar{\theta}^2$ component of $\Phi\bar{\Phi}$. Recall that we have

$$\begin{aligned}
\Phi &= \phi + \sqrt{2}\theta\psi + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi + \theta^2 f + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^\mu\partial_\mu\phi, \\
\bar{\Phi} &= \phi^* + \sqrt{2}\bar{\theta}\bar{\psi} - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^* + \bar{\theta}^2 f^* + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\psi} - \frac{1}{4}\theta^2\bar{\theta}^2\partial^\mu\partial_\mu\phi^*.
\end{aligned}$$

Then,

$$\begin{aligned}
\Phi\bar{\Phi} &= \phi\left(-\frac{1}{4}\theta^2\bar{\theta}^2\partial^\mu\partial_\mu\phi^*\right) + \sqrt{2}\theta\psi\left(\frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\psi}\right) \\
&\quad + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi\left(-i\theta\sigma^\nu\bar{\theta}\partial_\nu\phi^*\right) + \theta^2\bar{\theta}^2 f f^* + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(\sqrt{2}\bar{\theta}\bar{\psi}) \\
&\quad - \frac{1}{4}\theta^2\bar{\theta}^2\partial^\mu\partial_\mu\phi\cdot\phi^* + \text{terms with 3 } \theta\text{s or less}
\end{aligned}$$

We now use the following identities

$$\begin{aligned}
\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= \tfrac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu}, \\
\theta\psi\theta\sigma^\mu\partial_\mu\bar{\psi} &= -\tfrac{1}{2}\theta^2\psi\sigma^\mu\partial_\mu\bar{\psi}, \\
\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi\bar{\theta}\bar{\psi} &= -\tfrac{1}{2}\bar{\theta}^2\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi,
\end{aligned}$$

to obtain

$$\Phi\bar{\Phi} = \theta^2\bar{\theta}^2 \left\{ -\frac{1}{4}\phi\partial^\mu\partial_\mu\phi^* + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi^* - \frac{1}{4}\phi^*\partial^\mu\partial_\mu\phi + ff^* \right.$$
$$\left. -\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \right\} + \ldots$$

so that, eventually

$$\begin{aligned} S &= \int d^4x \int d^2\theta d^2\bar{\theta}\Phi\bar{\Phi} \\ &= \int d^4x \left\{ -\frac{1}{4}\phi\partial^\mu\partial_\mu\phi^* + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi^* - \frac{1}{4}\phi^*\partial^\mu\partial_\mu\phi + ff^* \right. \\ &\qquad \left. +\frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \right\} \\ &= \int d^4x \left(\partial^\mu\phi\partial_\mu\phi^* - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + ff^* \right). \end{aligned} \tag{4.26}$$

This is exactly the action (3.11) for the free massless theory of a scalar supermultiplet.

Since we are interested in theories which go beyond the massless, free limit, we see that we have obtained the kinetic term of the full action. We seem however to have a little problem: $\Phi\bar{\Phi}$ is the lowest dimension expression that we can write, and it eventually leads to a two-derivative Lagrangian density. How are we going to write non-derivative interactions? d-terms such as the one considered above do not leave much room for that. We have to look for another possibility.

4.2.2 Superpotential

Recall that we had postulated that $\mathcal{L} \propto d$ because a d-term has a SUSY variation that is a total derivative, $\delta_\epsilon d = \partial_\mu v^\mu_\epsilon$. Let us now go back to the expression for $\delta\Phi$ in components, or equivalently to the variations (4.24). We immediately realize that

$$\delta f = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi = \partial_\mu(i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\psi) = \partial_\mu\tilde{v}^\mu,$$

the variation of an f-term, the highest (independent) component of a chiral superfield, is also always a total derivative. This is true for the f-term of *any* chiral superfield.

Now, note that the product of 2 chiral superfields is itself a chiral superfield. Indeed if $\bar{D}_{\dot\alpha}\Phi = 0$ and $\bar{D}_{\dot\alpha}\Sigma = 0$, then

$$\bar{D}_{\dot\alpha}(\Phi\Sigma) = (\bar{D}_{\dot\alpha}\Phi)\Sigma + \Phi(\bar{D}_{\dot\alpha}\Sigma) = 0.$$

In components, take

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 f, \qquad \Sigma = \sigma + \sqrt{2}\theta\chi + \theta^2 g$$

where all fields depend on y; for the product we obtain

$$\Phi\Sigma = \phi\sigma + \sqrt{2}\theta(\psi\sigma + \chi\phi) + \theta^2(f\sigma + \phi g - \psi\chi)$$

(we have used that $\theta\psi\theta\chi = -\frac{1}{2}\theta^2\psi\chi$). It is thus obvious that $\Phi\Sigma$ is also a chiral superfield in its own right, and then the variation of its f-term has to be of the form given in (4.24) with all component fields replaced by the bilinears read from the expression above.

This is then true for any function of chiral superfields only. If $W(z)$ is a holomorphic function, then $W(\Phi)$, as defined by the Taylor expansion of $W(z)$, is a chiral superfield:

$$\bar{D}_{\dot\alpha}W(\Phi) = W'(\Phi)\bar{D}_{\dot\alpha}\Phi = 0.$$

Then, the f-term of $W(\Phi)$, i.e. the coefficient of θ^2, has a SUSY variation which is a total derivative. Since $\int d^2\theta\theta^2 = 1$, we can single out this component by integrating over only half of superspace

$$\int d^2\theta W(\Phi),$$

with the understanding that the resulting expression is evaluated at $y^\mu = x^\mu$.

For the action to be real, we must add the complex conjugate. Hence, another piece of a manifestly SUSY invariant action is:

$$\int d^4x \left(\int d^2\theta W(\Phi) + \int d^2\bar\theta \bar{W}(\bar\Phi)\right), \tag{4.27}$$

where by $\bar{W}$ we denote the function $\bar{W}(\bar{z}) = W(z)^*$, that is we take the complex conjugates of the expansion coefficients (i.e. the coupling constants).

Since the integral over half of superspace has a measure of dimension $[d^2\theta] = M$, we will have a Lagrangian density of canonical dimension $[\mathcal{L}] =$

M^4 only provided $[W] = M^3$. Therefore, we will have a renormalizable theory (all couplings are dimensionless or have positive mass dimensions) if the function $W(\Phi)$, called the *superpotential*, is at most cubic.

We take

$$W(\Phi) = -\frac{1}{2}m\Phi^2 - \frac{1}{3}\lambda\Phi^3. \tag{4.28}$$

According to the rule discussed above, we have

$$\begin{aligned}\Phi^2 &= \phi^2 + 2\sqrt{2}\theta\psi\phi + \theta^2(2\phi f - \psi\psi),\\ \Phi^3 &= \phi^3 + 3\sqrt{2}\theta\psi\phi^2 + \theta^2(3\phi^2 f - 3\phi\psi\psi).\end{aligned}$$

Hence,

$$\int d^2\theta W(\Phi) = -m\phi f + \tfrac{1}{2}m\psi\psi - \lambda\phi^2 f + \lambda\phi\psi\psi. \tag{4.29}$$

This is exactly the holomorphic piece of total Lagrangian (3.14) of the Wess-Zumino model discussed in the previous chapter. The holomorphicity of such terms, which was unexplained in the construction in components that we performed previously, has now a natural explanation in terms of chiral superfields. The importance of the present result cannot be stressed enough: The fact that supersymmetry imposes the mass and interaction terms for chiral superfields, i.e. the superpotential, to be holomorphic has enormous consequences. Indeed, such results based on symmetries apply to a theory in general, even after quantum corrections are taken into account (at the perturbative and even at the non-perturbative level). Hence, holomorphy of the superpotential will highly constrain the quantum radiative corrections and the renormalization of the theory, and also the non-perturbative corrections affecting the effective action.

It is possible to consider more general expressions for $W(\Phi)$, for instance in the context of effective theories. It is then obvious that if the superpotential is purely a polynomial in the chiral superfields, then the interactions are polynomial too.

4.2.3 Equations of motion for chiral superfields

We start by noticing a subtlety, which might have actually already puzzled the reader: The integrals over G-odd variables act as derivations. Indeed, we have

$$\int d\theta\theta = 1, \qquad \int d\theta 1 = 0$$

in the same way as

$$\frac{\partial}{\partial\theta}\theta = 1, \qquad \frac{\partial}{\partial\theta}1 = 0.$$

In an action (i.e. under a spacetime integral), we can even go further. We have that

$$\int d^4x \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}Y = \int d^4x(\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}Y + \partial_\mu v^\mu) \equiv \int d^4x \bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}Y = -4\int d^4x d^2\bar{\theta}Y,$$

where we have used the fact that $\bar{\partial}^2\bar{\theta}^2 = -4$ while $\int d^2\bar{\theta}\bar{\theta}^2 = 1$.

More interestingly, the converse is also true:

$$\int d^4x d^2\theta d^2\bar{\theta}Y = -\frac{1}{4}\int d^4x d^2\theta \bar{D}^2 Y. \tag{4.30}$$

The right hand side now looks like an f-term, being integrated over only chiral superspace. Indeed, $\bar{D}^2Y$ is a chiral superfield, since $\bar{D}_{\dot{\alpha}}\bar{D}^2Y = 0$ just because $\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}} \equiv 0$.

However, in the usual terminology for the action of a chiral superfield, we will not call the above a true f-term, since it can be reexpressed as a d-term. This is not always possible, of course. An f-term like Φ^2 *cannot* be rewritten (locally) as a d-term.

We can use the trick above to write the equations of motion in a SUSY covariant way. We rewrite the action as

$$\begin{aligned} S &= \int d^4x \left(\int d^2\theta d^2\bar{\theta}\Phi\bar{\Phi} + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}) \right) \\ &= \int d^4x \left[\int d^2\theta \left(-\frac{1}{4}\Phi\bar{D}^2\bar{\Phi} + W(\Phi) \right) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}) \right], \end{aligned}$$

so that

$$\frac{\delta S}{\delta\Phi} = 0 \qquad \Leftrightarrow \qquad -\frac{1}{4}\bar{D}^2\bar{\Phi} + W'(\Phi) = 0. \tag{4.31}$$

As we show below, it is a simple matter to see that $\bar{D}^2$ contains, for instance, a $\partial^\mu\partial_\mu$ operator. These are the equations of motion of the Wess-Zumino model, written in a manifestly SUSY covariant way.

Let us concentrate on the free case (that is, for $W = 0$). The equations of motion are:

$$D^2\Phi = 0.$$

Notice first that $D^2\Phi$ is an antichiral superfield, $D_\alpha D^2\Phi \equiv 0$. Hence it can be expanded as

$$D^2\Phi = a(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\chi}(\bar{y}) + \bar{\theta}^2 b(\bar{y}),$$

where

$$\bar{y}^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} = y^\mu - 2i\theta\sigma^\mu\bar{\theta}.$$

We can start by writing Φ in terms of $\bar{y}$:

$$\begin{aligned}
\Phi &= \phi(\bar{y} + 2i\theta\sigma^\mu\bar{\theta}) + \sqrt{2}\theta\psi(\bar{y} + 2i\theta\sigma^\mu\bar{\theta}) + \theta^2 f(\bar{y} + 2i\theta\sigma^\mu\bar{\theta}) \\
&= \phi(\bar{y}) + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(\bar{y}) - 4\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu\partial_\nu\phi(\bar{y}) \\
&\quad + \sqrt{2}\theta\psi(\bar{y}) + 2\sqrt{2}i\theta\sigma^\mu\bar{\theta}\theta\partial_\mu\psi(\bar{y}) + \theta^2 f(\bar{y}) \\
&= \phi(\bar{y}) + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(\bar{y}) - \theta^2\bar{\theta}^2\partial_\mu\partial^\mu\phi(\bar{y}) \\
&\quad + \sqrt{2}\theta\psi(\bar{y}) + i\sqrt{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(\bar{y}) + \theta^2 f(\bar{y})
\end{aligned}$$

Now, we can use the relations $D_\alpha\bar{\theta}^{\dot{\alpha}} = 0$, $D_\alpha\bar{y}^\mu = 0$ and $D_\alpha\theta^\beta = \delta_\alpha^\beta$.

We obtain

$$\begin{aligned}
D_\alpha\Phi &= 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\phi(\bar{y}) - 2\theta_\alpha\bar{\theta}^2\partial_\mu\partial^\mu\phi(\bar{y}) + \sqrt{2}\psi_\alpha(\bar{y}) \\
&\quad + 2\sqrt{2}i\theta_\alpha\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(\bar{y}) + 2\theta_\alpha f(\bar{y}), \\
D^\alpha D_\alpha\Phi &= 4\bar{\theta}^2\partial_\mu\partial^\mu\phi(\bar{y}) - 4\sqrt{2}i\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(\bar{y}) - 4f(\bar{y}).
\end{aligned}$$

This is indeed an antichiral superfield. Then the superfield equations of motion imply

$$D^2\Phi = 0 \qquad \Leftrightarrow \qquad \partial_\mu\partial^\mu\phi = 0, \quad \bar{\sigma}^\mu\partial_\mu\psi = 0, \quad f = 0. \tag{4.32}$$

As it should be obvious by now, there are no surprises and we recover what we expected.

4.3 Real superfields

We consider now a different restriction on a general superfield Y, that will eventually lead to the field content of the vector multiplet.

Recall that a gauge vector A_μ is real, while all components of a chiral superfield must be complex. In other words, it is obvious that we cannot build a chiral superfield such as $\Phi_\mu = A_\mu + \theta\lambda_\mu + \theta^2 f_\mu$.

Consider instead the real projection on Y: let us call V a (generically matrix-valued) superfield such that

$$V = V^\dagger. \tag{4.33}$$

On a general superfield

$$Y = \phi + \theta\eta + \bar\theta\bar\chi + \theta^2 m + \bar\theta^2 n + \theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\psi + \frac{1}{2}\theta^2\bar\theta^2 d,$$

the condition (4.33) implies

$$\phi = \phi^\dagger, \quad \eta = \chi, \quad m = n^\dagger, \quad A_\mu = A_\mu^\dagger, \quad \lambda = \psi, \quad d = d^\dagger.$$

Thus, the real superfield is given by

$$V = \phi + \theta\chi + \bar\theta\bar\chi + \theta^2 m + \bar\theta^2 m^\dagger + \theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 d, \tag{4.34}$$

with ϕ, A_μ and d real and m complex. (We have slightly adjusted the conventions for later convenience, i.e. for the sake of comparison with the notation in the previous chapter.) This gives off-shell a total of 8 bosonic and 8 fermionic degrees of freedom (χ and λ are Weyl spinors).

Though we have reduced by half the number of degrees of freedom with respect to a general superfield, it is still more than what we had in the (abelian) vector multiplet. However recall that, as we had discussed there, the gauge symmetry was crucial in giving the right balance of degrees of freedom. Since A_μ is in the game, gauge symmetry is bound to appear.

4.3.1 Gauge symmetry and the vector multiplet

Let us first recall some basics about gauge symmetries, and how they translate to a supersymmetric theory. A gauge symmetry is the invariance of the theory under local, i.e. spacetime dependent, transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x). \tag{4.35}$$

This should be true for all the components of a given multiplet, since a gauge symmetry is supposed to commute with supersymmetry (otherwise the latter has to become local too). Then, taking for instance a scalar muliplet, one would like to generalize the above transformation to a chiral superfield. However

$$\Phi(y,\theta) \to e^{i\alpha(x)}\Phi(y,\theta)$$

cannot work since it implies (4.35) but also, as its $\theta\bar{\theta}$ component, $\partial_\mu\phi(x) \to e^{i\alpha(x)}\partial_\mu\phi(x)$ which is not consistent. In other words the transformed object is no longer a chiral superfield.

The way to cure this problem is obvious: we have to promote the gauge parameter to a chiral superfield $\Lambda(y,\theta)$ so that

$$\Phi(y,\theta) \to e^{i\Lambda(y,\theta)}\Phi(y,\theta). \tag{4.36}$$

Now a chiral superfield correctly trasnforms into a chiral superfield (indeed, we have seen that any product of chiral superfields is a chiral superfield).

Expanding the gauge parameter chiral superfield, we have

$$\Lambda(y,\theta) = \alpha(y) + \theta\xi(y) + \theta^2 A(y).$$

Note that α must be complex, hence for the time being it looks like we are enlarging gauge invariance to a complexified gauge group.

In generalizing the gauge parameter to a chiral superfield, we see that the kinetic term for Φ is no longer invariant. Indeed

$$\bar{\Phi}\Phi \to \bar{\Phi}e^{-i\bar{\Lambda}}e^{i\Lambda}\Phi \neq \bar{\Phi}\Phi.$$

It would only work for $\Lambda = \alpha$ real. However we should not actually expect the free kinetic term to be invariant under a gauge symmetry. For a scalar field, $\partial_\mu\phi^*\partial^\mu\phi$ is *not* invariant under (4.35). One has instead to introduce a connection A_μ and a covariant derivative

$$D_\mu = \partial_\mu - iA_\mu$$

so that (4.35) is accompanied by

$$A_\mu \to A_\mu + \partial_\mu\alpha$$

and as a result

$$D_\mu\phi(x) \to e^{i\alpha(x)}D_\mu\phi(x).$$

Then $(D_\mu\phi)^* D^\mu\phi$ is invariant under gauge transformations.

Actually, following the same reasoning for superfields is even simpler. In order to render $\bar{\Phi}\Phi$ invariant, we introduce a superfield connection e^V between $\bar{\Phi}$ and Φ, so that the kinetic term becomes

$$\bar{\Phi} e^V \Phi.$$

It is invariant under (4.36) provided

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda}. \tag{4.37}$$

Note that the above transformation implies, for the hermitian conjugate superfield

$$e^{V^\dagger} \to e^{i\bar{\Lambda}} e^{V^\dagger} e^{-i\Lambda}.$$

Thus $V^\dagger$ transforms in the same way as V, and it is most economical to choose a real superfield $V = V^\dagger$ as the superfield connection.

Since Λ is a chiral superfield, it has $4+4$ degrees of freedom off-shell. It can then be used to reduce the $8+8$ degrees of freedom of $V = V^\dagger$ to a total of $4+4$ off-shell but gauge-fixed degrees of freedom, exactly as in the vector multiplet.

Let us take now for simplicity V to be a single (abelian) superfield. Then ordering of the (super)fields does not matter and the transformation law reads

$$V \to V - i(\Lambda - \bar{\Lambda})$$

or as a (gauge) variation

$$\delta V = -i(\Lambda - \bar{\Lambda}). \tag{4.38}$$

If we expand Λ as usual (see (4.23)), we obtain

$$\begin{aligned} \delta V &= -i(\alpha - \alpha^*) + \theta\sigma^\mu\bar{\theta}\partial_\mu(\alpha + \alpha^*) + \frac{i}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu(\alpha - \alpha^*) \\ &\quad -i\theta\xi + i\bar{\theta}\bar{\xi} + \frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\xi} \\ &\quad -i\theta^2 A + i\bar{\theta}^2 A^*. \end{aligned}$$

We can read the gauge transformations for every component:

$$\begin{aligned}
\delta\phi &= -i(\alpha-\alpha^*) \\
\delta\chi &= -i\xi \\
\delta m &= -iA \\
\delta A_\mu &= \partial_\mu(\alpha+\alpha^*) \\
\delta\lambda &= -\frac{i}{2}\sigma^\mu\partial_\mu\bar{\xi} \\
\delta d &= \frac{i}{2}\partial_\mu\partial^\mu(\alpha-\alpha^*).
\end{aligned}$$

We immediately observe that the above transformations are compatible with ϕ, A_μ and d being real. We can also perform a redefinition

$$\begin{aligned}
\lambda &\rightarrow \lambda' = \lambda + \frac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}, \\
d &\rightarrow d' = d + \frac{1}{2}\partial_\mu\partial^\mu\phi,
\end{aligned}$$

so that now in terms of these new component fields the transformations read

$$\begin{aligned}
\delta\lambda' &= 0 \\
\delta d' &= 0.
\end{aligned}$$

It is rather obvious that we can perform a (partial) gauge fixing as follows: We use Imα to set $\phi = 0$, ξ to set $\chi = 0$ and A to set $m = 0$. In other words, we set to zero all the component fields in V that transform by a shift without derivatives.

We are left with

$$V = \theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 d. \qquad (4.39)$$

This is the so-called *Wess-Zumino gauge*. The residual gauge symmetry is only the usual one acting on the vector A_μ and parametrized by the real part of α

$$\delta A_\mu = \partial_\mu(\alpha+\alpha^*).$$

This can be used to gauge away one last (off-shell) degree of freedom from A_μ. We are thus left with the exact field content of the previously discussed theory of the vector multiplet.

We can now derive the SUSY variations of the components of such a (gauge fixed) real superfield, and match them with those of the vector multiplet that we presented in the previous chapter. We start from

$$\delta_{SUSY} V = i(\epsilon Q + \bar{\epsilon}\bar{Q})V.$$

A subtlety concerns the fact that we will need to introduce compensating gauge transformations, since obviously the V in the Wess-Zumino (WZ) gauge (4.39) is *not* a proper superfield. Then new components will be generated by a supersymmetry transformation. A gauge transformation will be needed in order to bring back V in WZ-gauge. In particular note that if $\delta\chi$ is generated, the compensating gauge transformation will also generate a term proportional to

$$\frac{i}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\delta\bar{\chi} + c.c.$$

The SUSY variation reads

$$\begin{aligned}
\delta_{SUSY} V &= \left(\epsilon\partial - \bar{\epsilon}\bar{\partial} + i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})\partial_\mu\right) V \\
&= \epsilon\sigma^\mu\bar{\theta}A_\mu + 2i\epsilon\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\epsilon\lambda + \theta\epsilon\bar{\theta}^2 d \\
&\quad + \theta\sigma^\mu\bar{\epsilon}A_\mu + i\theta^2\bar{\epsilon}\bar{\lambda} - 2i\bar{\theta}\bar{\epsilon}\theta\lambda + \theta^2\bar{\theta}\bar{\epsilon}d \\
&\quad + i\theta\sigma^\mu\bar{\epsilon}\theta\sigma^\nu\bar{\theta}\partial_\mu A_\nu + \theta\sigma^\mu\bar{\epsilon}\bar{\theta}^2\theta\partial_\mu\lambda \\
&\quad - i\epsilon\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu A_\nu + \epsilon\sigma^\mu\bar{\theta}\theta^2\bar{\theta}\partial_\mu\bar{\lambda}.
\end{aligned}$$

Note that it is real, so that the reality condition indeed commutes with SUSY transformations.

From the by now familiar manipulation $\theta_\alpha\bar{\theta}_{\dot{\alpha}} = \frac{1}{2}\theta\sigma_\mu\bar{\theta}\sigma^\mu_{\alpha\dot{\alpha}}$, we have that, for instance

$$\epsilon\theta\bar{\theta}\bar{\lambda} = \tfrac{1}{2}\theta\sigma_\mu\bar{\theta}\epsilon\sigma^\mu\bar{\lambda}.$$

We thus derive

$$\delta_{SUSY} A_\mu = i\epsilon\sigma_\mu\bar{\lambda} - i\lambda\sigma_\mu\bar{\epsilon} = i\bar{\epsilon}\bar{\sigma}_\mu\lambda - i\bar{\lambda}\bar{\sigma}_\mu\epsilon, \tag{4.40}$$

which correctly reproduces the variation given in (3.16).

Another Fierz identity yields

$$\epsilon\sigma^\mu\bar{\theta}\theta^2\bar{\theta}\partial_\mu\bar{\lambda} = -\tfrac{1}{2}\bar{\theta}^2\epsilon\sigma^\mu\partial_\mu\bar{\lambda},$$

so that we further get

$$\delta_{SUSY} d = \bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\lambda - \epsilon\sigma^\mu\partial_\mu\bar{\lambda}, \tag{4.41}$$

again in agreement with (3.19).

As anticipated, in order to obtain the variation of λ we need to pay extra attention. First of all we have the identity

$$\epsilon\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu A_\nu = \tfrac{1}{2}\epsilon\sigma^\mu\bar{\sigma}^\nu\theta\bar{\theta}^2\partial_\mu A_\nu = \tfrac{1}{2}\bar{\theta}^2\theta\sigma^\nu\bar{\sigma}^\mu\epsilon\partial_\mu A_\nu,$$

so that

$$\delta_{SUSY}V = \bar{\theta}^2\theta\left(-\frac{i}{2}\sigma^\nu\bar{\sigma}^\mu\epsilon\partial_\mu A_\nu + \epsilon d\right) + \dots \tag{4.42}$$

However we also have

$$\delta_{SUSY}V = \theta\sigma^\mu\bar{\epsilon}A_\mu - \bar{\theta}\bar{\sigma}^\mu\epsilon A_\mu + \dots = \theta\delta_{SUSY}\chi + \bar{\theta}\delta_{SUSY}\bar{\chi} + \dots$$

from which we establish

$$\delta_{SUSY}\bar{\chi} = -\bar{\sigma}^\mu\epsilon A_\mu.$$

We thus need a compensating gauge transformation such that

$$\begin{aligned}\delta_{gauge}V &= \bar{\theta}\delta_{gauge}\bar{\chi} + \frac{i}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\delta_{gauge}\bar{\chi} + \dots \\ &= \bar{\theta}\bar{\sigma}^\mu\epsilon A_\mu + \frac{i}{2}\bar{\theta}^2\theta\sigma^\mu\bar{\sigma}^\nu\epsilon\partial_\mu A_\nu + \dots \end{aligned}\tag{4.43}$$

Combining the above gauge variation with (4.42), we restore V to the WZ gauge and get

$$\delta_{tot}V = \bar{\theta}^2\theta\left[\frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)\epsilon\partial_\mu A_\nu + \epsilon d\right] + \dots$$

and we eventually recover the variation (3.17)

$$\delta_{tot}\lambda = -\sigma^{\mu\nu}\epsilon F_{\mu\nu} + i\epsilon d. \tag{4.44}$$

We have thus recovered the supersymmetric variations discussed previously, confirming that the real superfield $V = V^\dagger$, in the WZ gauge, is the correct way of assembling in a manifestly supersymmetric fashion the fields of the vector supermultiplet.

4.3.2 The action and the gaugino chiral superfield

It is now our duty to write the action for the vector multiplet degrees of freedom in a manifestly SUSY invariant way.

We have the real superfield V at our disposal but it is easy to understand that it is not the best choice. For instance, it contains a "bare" A_μ, so it is not gauge covariant.

In other words, since V transforms as

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda},$$

it is impossible to build gauge invariant quantities by taking traces of products of e^V, for instance. What we need is an object which transforms as

$$\mathcal{V} \to e^{i\Lambda} \mathcal{V} e^{-i\Lambda},$$

i.e. *holomorphically*, so that for instance $\operatorname{tr} \mathcal{V}^2$ is gauge invariant.

Clearly, $\mathcal{V}$ must be a chiral superfield. Hence, we would like to build a chiral superfield out of the real superfield V.

By inspecting the SUSY variations of the components of V, we note that

$$\delta\lambda = \left(-\sigma^{\mu\nu} F_{\mu\nu} + id\right) \epsilon$$

is reminiscent of the variation of the lowest component of a chiral superfield,

$$\delta\phi = \sqrt{2}\psi\epsilon,$$

in that it only depends on ϵ. Then, λ should be the spin-$\frac{1}{2}$ lowest component of a chiral superfield. By substituting

$$\phi \to \lambda, \qquad \sqrt{2}\psi \to -\sigma^{\mu\nu} F_{\mu\nu} + id$$

we get

$$\mathcal{W}_\alpha = \lambda_\alpha + (-\sigma^{\mu\nu}{}_\alpha{}^\beta F_{\mu\nu} + id\delta_\alpha^\beta)\theta_\beta + \ldots \tag{4.45}$$

Inspecting now also the variations for d and A_μ, see (4.40) and (4.41), we see that there are two terms, proportional to ϵ and $\bar{\epsilon}$ respectively. Recall that the f-term appears in

$$\delta\psi = \sqrt{2} i \sigma^\mu \bar{\epsilon} \partial_\mu \phi + \sqrt{2}\epsilon f,$$

in the term proportional to ϵ. Hence, from (4.41) for instance, we expect that $f \propto \sigma^\mu \partial_\mu \bar{\lambda}$.

Instead of going through all the algebra, we now define $\mathcal{W}_\alpha$ directly from V as a superfield equation.

We know that the lowest component of $\mathcal{W}_\alpha$ is λ_α, the coefficient of $\bar{\theta}^2\theta^\alpha$ in V. Then we should be able to obtain it by applying the operator $\bar{D}^2 D_\alpha$ on V. Indeed

$$\bar{D}^2 D_\alpha(\bar{\theta}^2\theta\lambda) = \bar{\partial}^2\partial_\alpha(\bar{\theta}^2\theta\lambda) + \ldots = -4\lambda_\alpha + \ldots$$

Thus we define

$$\mathcal{W}_\alpha = -\frac{i}{4}\bar{D}^2 D_\alpha V. \tag{4.46}$$

It is a chiral superfield because $\bar{D}^3 = 0$. We will often refer to it as the *gaugino superfield*.

Analogously we define the anti-chiral superfield

$$\bar{\mathcal{W}}_{\dot{\alpha}} = -\frac{i}{4}D^2\bar{D}_{\dot{\alpha}}V = \bar{\lambda}_{\dot{\alpha}} + \ldots = (\mathcal{W}_\alpha)^*. \tag{4.47}$$

(Note that in the same way as $(\partial_\alpha)^* = -\bar{\partial}_{\dot{\alpha}}$, we also have that $(D_\alpha)^* = -\bar{D}_{\dot{\alpha}}$.)

It is best to compute $\mathcal{W}_\alpha$ in the WZ-gauge. We will later show that it is gauge covariant (and thus gauge invariant in the abelian case), so that the result of this computation extends to any gauge.

Since $\mathcal{W}_\alpha$ is a chiral superfield, let us start by writing all fields in terms of y^μ, that is we expand $x^\mu = y^\mu - i\theta\sigma^\mu\bar{\theta}$:

$$\begin{aligned} V &= \theta\sigma^\mu\bar{\theta}A_\mu - i\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu A_\nu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 d \\ &= \theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2(d - i\partial_\mu A^\mu), \end{aligned}$$

where all fields are intended as depending on y. As usual we recall that $\bar{D}_{\dot{\alpha}}\theta^\beta = 0$, $\bar{D}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}$, $\bar{D}_{\dot{\alpha}}y^\mu = 0$, and also $D_\alpha\theta^\beta = \delta^\beta_\alpha$, $D_\alpha\bar{\theta}^{\dot{\beta}} = 0$ and $D_\alpha y^\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$. We compute

$$\begin{aligned} D_\alpha V &= \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}A_\mu + 2i\theta_\alpha\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\lambda_\alpha + \theta_\alpha\bar{\theta}^2(d - i\partial_\mu A^\mu) \\ &\quad + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta\sigma^\nu\bar{\theta}\partial_\mu A_\nu - 2\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^2\bar{\theta}\partial_\mu\bar{\lambda}. \end{aligned}$$

Now, $\bar{D}^2$ will select only the terms with two $\bar{\theta}$s:

$$\begin{aligned} \bar{D}^2 D_\alpha V &= 4i\lambda_\alpha - 4\theta_\alpha(d - i\partial_\mu A^\mu) + 4i\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\partial_\mu A_\nu - 4\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\theta^2\partial_\mu\bar{\lambda}_{\dot{\beta}} \\ &= 4i\lambda_\alpha - 4\theta_\alpha(d - i\partial_\mu A^\mu) - 4i(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\theta_\beta\partial_\mu A_\nu - 4\theta^2\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} \\ &= 4i\lambda_\alpha - 4\theta_\alpha d - 4i\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta F_{\mu\nu} - 4\theta^2\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}, \end{aligned}$$

where we have used $\sigma^\mu \bar{\sigma}^\nu = 2\sigma^{\mu\nu} + \eta^{\mu\nu}$.

We thus have

$$\mathcal{W}_\alpha = \lambda_\alpha - \sigma^{\mu\nu}{}_\alpha{}^\beta \theta_\beta F_{\mu\nu} + i\theta_\alpha d + i\theta^2 \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}^{\dot{\alpha}}. \tag{4.48}$$

In the abelian case, the above expression is obviously gauge invariant. In the non-abelian case, we have to write a more general expression for the gaugino superfield

$$\mathcal{W}_\alpha = -\frac{i}{4}\bar{D}^2 \left(e^{-V} D_\alpha e^V\right). \tag{4.49}$$

Under a gauge transformation

$$\begin{aligned}
\mathcal{W}_\alpha \quad &\rightarrow \quad -\frac{i}{4}\bar{D}^2 \left[e^{i\Lambda} e^{-V} e^{-i\bar{\Lambda}} D_\alpha \left(e^{i\bar{\Lambda}} e^V e^{-i\Lambda}\right)\right] \\
&= \quad -\frac{i}{4}\bar{D}^2 \left[e^{i\Lambda} e^{-V} D_\alpha \left(e^V e^{-i\Lambda}\right)\right] \\
&= \quad -\frac{i}{4} e^{i\Lambda} \bar{D}^2 \left[e^{-V} \left(D_\alpha e^V e^{-i\Lambda} + e^V D_\alpha e^{-i\Lambda}\right)\right] \\
&= \quad -\frac{i}{4} e^{i\Lambda} \bar{D}^2 \left(e^{-V} D_\alpha e^V\right) e^{-i\Lambda} - \frac{i}{4} e^{i\Lambda} \bar{D}^2 D_\alpha e^{-i\Lambda} \\
&= \quad e^{i\Lambda} \mathcal{W}_\alpha e^{-i\Lambda} + \frac{i}{4} e^{i\Lambda} \bar{D}^{\dot{\beta}} \{D_\alpha, \bar{D}_{\dot{\beta}}\} e^{-i\Lambda} \\
&= \quad e^{i\Lambda} \mathcal{W}_\alpha e^{-i\Lambda},
\end{aligned}$$

where we have repeatedly used that $D_\alpha \bar{\Lambda} = 0$ and $\bar{D}_{\dot{\alpha}} \Lambda = 0$, and also that $[\bar{D}_{\dot{\gamma}}, \{D_\alpha, \bar{D}_{\dot{\beta}}\}] = 0$ in the last equality.

We have thus shown that $\mathcal{W}_\alpha$ is gauge covariant

$$\mathcal{W}_\alpha \rightarrow e^{i\Lambda} \mathcal{W}_\alpha e^{-i\Lambda}. \tag{4.50}$$

We obtain a Lorentz invariant object from $\mathcal{W}_\alpha$ by taking its square, which is also gauge covariant:

$$\mathcal{W}^\alpha \mathcal{W}_\alpha \rightarrow e^{i\Lambda} \mathcal{W}^\alpha \mathcal{W}_\alpha e^{-i\Lambda}.$$

Hence, we see that its trace is both Lorentz and gauge invariant, and is perfectly suited to appear in an action. It is a chiral superfield, and as a consequence

$$\int d^4x d^2\theta \mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha$$

is also manifestly SUSY invariant.

Let us do some power counting. First of all $[\mathcal{W}] = [\lambda] = M^{3/2}$. (As a side remark, note that of course $[V] = M^0$ since it appear in an exponential.) Thus $[\mathcal{W}^2] = M^3$ and $[\int d^2\theta \mathcal{W}^2] = M^4$ as it befits a Lagrangian density. It is easy to realize that this is actually the only option at all, at least for a renormalizable theory of course.

We then set out to compute $\mathcal{W}^\alpha \mathcal{W}_\alpha$, and for the present we stick to the abelian case. We are actually interested only in the θ^2 term:

$$\begin{aligned}\mathcal{W}^\alpha \mathcal{W}_\alpha &= 2i\lambda^\alpha \theta^2 \sigma^\mu_{\alpha\dot\alpha}\partial_\mu \bar\lambda^{\dot\alpha} + \theta^\beta({\sigma^{\mu\nu}}_\beta{}^\alpha F_{\mu\nu} + id\delta^\alpha_\beta)(-{\sigma^{\rho\sigma}}_\alpha{}^\gamma F_{\rho\sigma} + id\delta^\gamma_\alpha)\theta_\gamma + \dots \\ &= 2i\theta^2 \lambda \sigma^\mu \partial_\mu \bar\lambda - \theta\sigma^{\mu\nu}\sigma^{\rho\sigma}\theta F_{\mu\nu}F_{\rho\sigma} - \theta^2 d^2 + \dots \\ &= \theta^2\left(2i\lambda\sigma^\mu\partial_\mu\bar\lambda - \tfrac{1}{2}{\sigma^{\mu\nu}}_\beta{}^\alpha {\sigma^{\rho\sigma}}_\alpha{}^\beta F_{\mu\nu}F_{\rho\sigma} - d^2\right) + \dots,\end{aligned}$$

where note that $\epsilon^{\alpha\beta}{\sigma^{\mu\nu}}_\beta{}^\gamma\theta_\gamma = -\theta^\beta {\sigma^{\mu\nu}}_\beta{}^\alpha$.

Now recall the definition of the (anti)self-dual tensors

$$F^\pm_{\mu\nu} = F_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma},$$

which are in the $(1,0)$ and $(0,1)$ representations of $SU(2)\times SU(2)^*$. Actually, it is fairly easy to realize that it is precisely ${\sigma^{\mu\nu}}_\beta{}^\alpha F_{\mu\nu}$ and ${\bar\sigma^{\mu\nu\dot\beta}}{}_{\dot\alpha} F_{\mu\nu}$ that project $F_{\mu\nu}$ on these two irreducible representations. Hence, we know that there will be a relation such as

$${\sigma^{\mu\nu}}_\beta{}^\alpha {\sigma^{\rho\sigma}}_\alpha{}^\beta F_{\mu\nu}F_{\rho\sigma} \propto F^\pm_{\mu\nu}F^{\pm\mu\nu},$$

where we still have to fix the constant of proportionality and the choice of sign (since they are heavily convention-dependent). First, take

$$\begin{aligned}F^\pm_{\mu\nu}F^{\pm\mu\nu} &= \left(F_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right)\left(F^{\mu\nu} \mp \frac{i}{2}\epsilon^{\mu\nu\lambda\tau}F_{\lambda\tau}\right) \\ &= F_{\mu\nu}F^{\mu\nu} \mp i\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{1}{4}2(\delta^\lambda_\rho\delta^\tau_\sigma - \delta^\lambda_\sigma\delta^\tau_\rho)F^{\rho\sigma}F_{\lambda\tau} \\ &= (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} \mp i\epsilon^{\mu\nu\rho\sigma})F_{\mu\nu}F_{\rho\sigma}.\end{aligned}$$

We thus learn that

$${\sigma^{\mu\nu}}_\beta{}^\alpha {\sigma^{\rho\sigma}}_\alpha{}^\beta = a(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} \mp i\epsilon^{\mu\nu\rho\sigma}).$$

We now fix a and the relative sign by inserting specific values for the Lorentz indices. For instance:

$$\mathrm{tr}\,\sigma^{03}\sigma^{03} = \frac{1}{4}\mathrm{tr}\,\sigma^0\bar{\sigma}^3\sigma^0\bar{\sigma}^3 = \frac{1}{4}\mathrm{tr}\,\tau_3^2 = \frac{1}{2} \equiv a\eta^{00}\eta^{33} = -a,$$

(recall that τ_i are the Pauli matrices) so that $a = -\frac{1}{2}$, and

$$\begin{aligned}\mathrm{tr}\,\sigma^{03}\sigma^{12} &= \frac{1}{8}\mathrm{tr}\,\sigma^0\bar{\sigma}^3(\sigma^1\bar{\sigma}^2 - \sigma^2\bar{\sigma}^1) \\ &= -\frac{1}{8}\mathrm{tr}\,\tau_3[\tau_1,\tau_2] = -\frac{i}{4}\mathrm{tr}\,\tau_3^2 = -\frac{i}{2} \equiv \pm\frac{i}{2}\epsilon^{0312} = \pm\frac{i}{2}.\end{aligned}$$

It is then the lower sign which is the correct one.

Thus

$$\sigma^{\mu\nu}{}_\beta{}^\alpha\sigma^{\rho\sigma}{}_\alpha{}^\beta = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}.$$

As a result,

$$\sigma^{\mu\nu}{}_\beta{}^\alpha\sigma^{\rho\sigma}{}_\alpha{}^\beta F_{\mu\nu}F_{\rho\sigma} = -F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

and finally

$$\mathcal{W}^\alpha\mathcal{W}_\alpha = \theta^2\left(2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - d^2\right) + \ldots \qquad (4.51)$$

By simply taking the complex conjugate we also obtain

$$\bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} = \bar{\theta}^2\left(-2i\partial_\mu\lambda\sigma^\mu\bar{\lambda} + \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - d^2\right) + \ldots \qquad (4.52)$$

Eventually, the action is given by

$$\begin{aligned}S &= \int d^4x\left(-\frac{1}{4}\int d^2\theta\mathcal{W}^\alpha\mathcal{W}_\alpha - \frac{1}{4}\int d^2\bar{\theta}\bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}\right) \qquad (4.53)\\ &= \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}d^2 - \frac{i}{2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{i}{2}\partial_\mu\lambda\sigma^\mu\bar{\lambda}\right) \\ &= \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}d^2 - i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda\right),\end{aligned}$$

exactly what we had before, see (3.20).

Let us write the manifestly SUSY invariant action of a vector multiplet in full generality, that is we allow for a non-abelian gauge group and also for a parity-violating term in the action. The Lagrangian density is then

$$\mathcal{L} = -\frac{\tau}{2}\int d^2\theta \mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha - \frac{\tau^*}{2}\int d^2\bar{\theta} \mathrm{tr}\, \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}, \tag{4.54}$$

with conventionally

$$\tau = \frac{1}{g^2} - i\frac{\Theta}{8\pi^2}, \tag{4.55}$$

so that

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} - \frac{\Theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \ldots$$

4.3.3 Equations of motion for real superfields

In order to derive equations of motion from (4.53), we should vary it with respect to V. Then we have

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int d^4x \left(\int d^2\theta \mathcal{W}^\alpha \delta \mathcal{W}_\alpha + \int d^2\bar{\theta} \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\delta} \mathcal{W}^{\dot{\alpha}}\right) \\
&= \frac{i}{8}\int d^4x \left(\int d^2\theta \mathcal{W}^\alpha \bar{D}^2 D_\alpha \delta V + \int d^2\bar{\theta} \bar{\mathcal{W}}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} \delta V\right) \\
&= -\frac{i}{2}\int d^4x d^2\theta d^2\bar{\theta} \left(\mathcal{W}^\alpha D_\alpha \delta V + \bar{\mathcal{W}}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \delta V\right) \\
&= \frac{i}{2}\int d^4x d^2\theta d^2\bar{\theta} \left(D^\alpha \mathcal{W}_\alpha + \bar{D}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}\right) \delta V.
\end{aligned}$$

Thus, the equations of motion are

$$D^\alpha \mathcal{W}_\alpha = -\bar{D}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}, \tag{4.56}$$

or, since $(D^\alpha \mathcal{W}_\alpha)^* = -\bar{D}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}$, equivalently

$$\mathrm{Im} D^\alpha \mathcal{W}_\alpha = 0.$$

However, the superfield $\mathcal{W}_\alpha$ verifies a relation, due to its definition in terms of V. If we define

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \equiv v_{\alpha\dot{\alpha}},$$

then it is simple to see that

$$D^\alpha \bar{D}^2 D_\alpha = -D_\alpha \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} D^\alpha = \bar{D}_{\dot\alpha} D_\alpha \bar{D}^{\dot\alpha} D^\alpha - v_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} D^\alpha = \bar{D}^2 D^2 - 2v_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} D^\alpha,$$

and similarly

$$\bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} = -\bar{D}^{\dot\alpha} D^\alpha D_\alpha \bar{D}_{\dot\alpha} = \bar{D}^{\dot\alpha} D^\alpha \bar{D}_{\dot\alpha} D_\alpha - v_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} D^\alpha = \bar{D}^2 D^2 - 2v_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} D^\alpha$$

so that we have the identity

$$D^\alpha \bar{D}^2 D_\alpha V = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} V$$

which in terms of the gaugino superfield reads

$$D^\alpha \mathcal{W}_\alpha = \bar{D}_{\dot\alpha} \bar{\mathcal{W}}^{\dot\alpha} \tag{4.57}$$

or

$$\mathrm{Re} D^\alpha \mathcal{W}_\alpha = 0.$$

This is a consequence of the definition of $\mathcal{W}_\alpha$ and hence must be akin to Bianchi identities for the gauge field strength.

It is an obvious consequence of the above that imposing $D^\alpha \mathcal{W}_\alpha = 0$ will contemporarily set both the equations of motion and the Bianchi identities to zero.

Noticing again that $D^\alpha \theta_\beta = -\delta^\alpha_\beta$ and $D^\alpha y^\mu = -2i\bar\theta_{\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha}$, let us compute

$$\begin{aligned} D^\alpha \mathcal{W}_\alpha &= \sigma^{\mu\nu}{}_\alpha{}^\alpha F_{\mu\nu} - 2id + 2i\theta\sigma^\mu \partial_\mu \bar\lambda \\ &\quad -2i\bar\theta\bar\sigma^\mu \partial_\mu \lambda + 2i\bar\theta\bar\sigma^\mu \sigma^{\rho\sigma} \theta \partial_\mu F_{\rho\sigma} + 2\bar\theta\bar\sigma^\mu \theta \partial_\mu d + 2\theta^2 \bar\theta \bar\sigma^\mu \sigma^\nu \partial_\mu \partial_\nu \bar\lambda. \end{aligned}$$

The first term vanishes because the Lorentz generators are traceless. Further, we see that d, $\sigma^\mu \partial_\mu \bar\lambda$ and $\bar\sigma^\mu \partial_\mu \lambda$ all belong to the imaginary part of $D^\alpha \mathcal{W}_\alpha$ and hence are correctly set to zero when imposing the equations of motion of the superfield. This also implies that the last two terms vanish.

For the terms involving $F_{\rho\sigma}$ we see that

$$\begin{aligned} \mathrm{Im} D^\alpha \mathcal{W}_\alpha = 0 &\quad \Rightarrow \quad \bar\theta(\bar\sigma^\mu \sigma^{\rho\sigma} - \bar\sigma^{\rho\sigma} \bar\sigma^\mu) \theta \partial_\mu F_{\rho\sigma} = 0 \\ \mathrm{Re} D^\alpha \mathcal{W}_\alpha = 0 &\quad \Rightarrow \quad \bar\theta(\bar\sigma^\mu \sigma^{\rho\sigma} + \bar\sigma^{\rho\sigma} \bar\sigma^\mu) \theta \partial_\mu F_{\rho\sigma} = 0. \end{aligned}$$

Now, using the usual Fierz rearrangements, we have that

$$\begin{aligned} \bar\theta\bar\sigma^\mu \sigma^{\rho\sigma} \theta &= -\tfrac{1}{2} \theta\sigma_\nu \bar\theta \mathrm{tr}\, \sigma^\nu \bar\sigma^\mu \sigma^{\rho\sigma} \\ &= -\tfrac{1}{2} \theta\sigma_\nu \bar\theta \mathrm{tr}\, (2\sigma^{\nu\mu} \sigma^{\rho\sigma} + \eta^{\nu\mu} \sigma^{\rho\sigma}) \\ &= \theta\sigma_\nu \bar\theta \mathrm{tr}\, \sigma^{\mu\nu} \sigma^{\rho\sigma} \\ &= -\theta\sigma_\nu \bar\theta (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho} + i\epsilon^{\mu\nu\rho\sigma}). \end{aligned}$$

Eventually, we have

$$D^\alpha \mathcal{W}_\alpha = \theta\sigma_\mu\bar{\theta}\,(-4i\partial_\rho F^{\rho\mu} - 2\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}) + \dots,$$

so that as expected

$$\begin{aligned} \mathrm{Im} D^\alpha \mathcal{W}_\alpha = 0 \qquad &\Leftrightarrow \qquad \partial_\mu F^{\mu\nu} = 0 \\ \mathrm{Re} D^\alpha \mathcal{W}_\alpha = 0 \qquad &\Leftrightarrow \qquad \partial_{[\mu} F_{\rho\sigma]} = 0. \end{aligned}$$

We have thus correctly extracted from the equations of motion and Bianchi identities of the real superfield, the equations of motion and Bianchi identities of the vector multiplet component fields.

4.3.4 Fayet-Iliopoulos term

For completeness, we must point out here that there is yet another gauge invariant action term that can be built from V. It is just its d component, which is truly gauge invariant only in the abelian case of course (otherwise it transforms in the adjoint representation). This is called the *Fayet-Iliopoulos term*, and is written as

$$\mathcal{L}_{FI} = \xi \int d^2\theta d^2\bar{\theta} V = \xi d. \tag{4.58}$$

It is trivially SUSY invariant and Lorentz invariant. For a free vector model, it just shifts the on-shell value of d (and, as we will see later, most importantly also the vacuum energy). Though it may look like just an odd curiosity to be dismissed as irrelevant, we will see that it may play an important role in the context of models of spontaneous supersymmetry breaking.

Chapter 5

Supersymmetric gauge theories

Having now developped all the tools necessary to build any supersymmetric field theory, in this chapter we start studying physically relevant such theories. We will write the most general supersymmetric gauge theory coupled to matter (super)fields. One obvious application is to the minimal supersymmetric extension of the Standard Model of particle physics, also known as MSSM. We will also review in more detail some subsectors of it, such as Super-QCD (SQCD). In passing, we will introduce the important concept of the moduli space of vacua and discuss global symmetries, including R-symmetry which is specific to supersymmetric theories.

5.1 Most general SUSY gauge theory with matter

In the previous chapter, we have introduced and reviewed both chiral superfields and real superfields. Our aim now is to combine them into gauge theories with matter. What we really mean by "gauge theories with matter" is the sort of theories that we encounter in the Standard Model (SM), that is gauge bosons interacting with fermions, and possibly scalars, which in their turn interact through Yukawa or quartic couplings. Hence, the "gauge theory" part has to include gauge bosons, while the "matter" part usually includes fermions (we are all made of fermions after all..).

The first possibility that comes to mind is that the real superfield already includes both gauge bosons and fermions. Indeed, the fermionic partner of the gauge boson A_μ, which we call the *gaugino* λ, could be considered as "matter". It has however a strong restriction, which is the one of having to belong to the adjoint representation of the gauge group. This is, in particular, not the case for any matter field of the SM. Thus, if we want to have matter fields in different representations, we must fit them in additional chiral superfields. This is actually not a real surprise, since we have already seen how to couple chiral superfields to real superfields: we introduced the latter precisely by requiring gauge invariance of the chiral superfield kinetic term.

We can then write the most general manifestly SUSY invariant action with gauge and matter fields as

$$\begin{aligned} S &= \int d^4x \left\{ \int d^2\theta d^2\bar{\theta}\, \bar{\Phi} e^V \Phi - \frac{\tau}{2} \int d^2\theta \, \mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha - \frac{\tau^*}{2} \int d^2\bar{\theta}\, \mathrm{tr}\, \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}} \right. \\ &\qquad \left. + \int d^2\theta \, \mathrm{tr}\, W(\Phi) + \int d^2\bar{\theta}\, \mathrm{tr}\, \bar{W}(\bar{\Phi}) \right\}. \end{aligned} \tag{5.1}$$

Here we are taking the gauge group to be a generic one, typically non-abelian and not necessarily simple (hence, we will generically have several different gauge couplings). The superfield V is then really a hermitian matrix $V = V_a T_a$, with $T_a = T_a^\dagger$ the generators of the gauge group.

In the term $\bar{\Phi} e^V \Phi$ the generators are taken in the representation of the matter fields, which is generically reducible (often, quantum consistency does *not* allow the matter fields to be in a single irreducible representation). The generators T_a act on the matter superfields as

$$(T_a)^i{}_j \Phi^j,$$

where a runs over all the generators of the gauge group, while i and j run over all the components of the representation to which Φ belongs. Then $\bar{\Phi}$ is in the conjugate representation

$$\bar{\Phi}_i (T_a)^i{}_j.$$

We thus write $\bar{\Phi} e^V \Phi$ really as a shorthand for

$$\bar{\Phi}_i (e^V)^i{}_j \Phi^j.$$

Since we have that $V = V_a T_a$, we can also write $\mathcal{W}_\alpha = \mathcal{W}_{\alpha a} T_a$. It follows that

$$\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha = \mathcal{W}_a^\alpha \mathcal{W}_{\alpha b} \mathrm{tr}\, T_a T_b = \mathcal{W}_a^\alpha \mathcal{W}_{\alpha b} \tfrac{1}{2} \delta_{ab} = \tfrac{1}{2} \mathcal{W}_a^\alpha \mathcal{W}_{\alpha a}.$$

We have used the convention that

$$\operatorname{tr} T_a T_b = \tfrac{1}{2}\delta_{ab}$$

when T_a are the generators of the fundamental representation of the gauge group. In this way, the gauge kinetic terms can be rewritten as

$$\mathcal{L} = -\frac{\tau}{4}\int d^2\theta\, \mathcal{W}^\alpha_a \mathcal{W}_{\alpha a} - \frac{\tau^*}{4}\int d^2\bar{\theta}\, \mathcal{W}_{\dot{\alpha} a}\bar{\mathcal{W}}^{\dot{\alpha}}_a.$$

Some more attention must be paid to the matter superpotential. By $\operatorname{tr} W(\Phi)$ we mean that it must, of course, be gauge invariant. However note that gauge invariance cannot be achieved as in the kinetic term, since we cannot use $\bar{\Phi}$. For instance, in order to write a quadratic gauge invariant, the representation of the gauge group must be real (more precisely, self-complex conjugate). In other words, the reducible representation of Φ should split into irreducible representations plus their conjugates, and possibly some irreducible self-conjugate representations like the adjoint. (Of course, we have $SU(N)$ gauge groups in the back of our minds most of the time. With $SO(N)$ or $Sp(N)$ gauge groups the situation is simpler.)

Let us recall very briefly some basic facts about Lie algebras. Given a set of generators $T_a = T_a^\dagger$ satisfying the commutation relations

$$[T_a, T_b] = i f_{abc} T_c,$$

where the structure constants f_{abc} are real, it is easy to see that by taking the complex conjugate we obtain

$$[T_a^*, T_b^*] = -i f_{abc} T_c^*,$$

that is, the generators $-T_a^*$ also satisfy the same commutation relations. They are the generators of the conjugate representation. If $T_a = -T_a^*$ the representation is self-conjugate. For instance, taking the adjoint representation where

$$(T_a)_{bc} = i f_{abc}$$

we see that it satisfies $T_a^* = -T_a$.

Thus, to come back to the problem of writing gauge invariant superpotentials, we have just learned that if Φ is in the adjoint representation, that is it transforms as

$$\Phi \to e^{i\Lambda}\Phi e^{-i\Lambda},$$

then

$$\mathrm{tr}\,\Phi^2$$

is a gauge invariant term, as well actually as any term like $\mathrm{tr}\,\Phi^n$ more generally. If on the other hand Φ is in the fundamental representation, we must have another *chiral* superfield $\tilde{\Phi}$ in the anti-fundamental representation. They transform as

$$\Phi \to e^{i\Lambda}\Phi, \qquad \tilde{\Phi} \to \tilde{\Phi}e^{-i\Lambda},$$

so that

$$\tilde{\Phi}\Phi$$

is gauge invariant.

There can be other gauge invariants in some gauge groups, like for instance $\epsilon_{i_1 \ldots i_N}\Phi^{i_1} \ldots \Phi^{i_N}$ in $SU(N)$. However one must pay special attention to the statistics, since many such invariants could just vanish because of it.

5.1.1 An abelian example: SQED

We consider now a simple example of a SUSY gauge theory with matter. It is the one of the supersymmetric version of QED, usually denoted by SQED. More specifically, it is a SUSY version of an abelian $U(1)$ gauge theory coupled to charged fermions e^+ and e^-, the positron and the electron. In addition to the usual particle content of QED of photon, positron and electron, respectively associated to the fields A_μ, ψ and $\tilde{\psi}$ (note that we have split the conventional Dirac fermion of QED in its two Weyl components), we have also a fermionic field λ corresponding to the *photino*, and two scalar fields ϕ and $\tilde{\phi}$ which correspond to the *spositron* and *selectron*, the two bosonic superpartners of the positron and the electron.

We know that the "positron" and the "electron" transform as

$$\psi \to e^{i\alpha}\psi, \qquad \tilde{\psi} \to e^{-i\alpha}\tilde{\psi}.$$

We will thus consider two chiral superfields Φ and $\tilde{\Phi}$ such that the gauge transformations read

$$V \to V - i(\Lambda - \bar{\Lambda}),$$
$$\Phi \to e^{i\Lambda}\Phi, \qquad \tilde{\Phi} \to \tilde{\Phi}e^{-i\Lambda}.$$

Note that $\tilde{\Phi}$ is in the conjugate representation of Φ. Hence, the action of e^V on it will be replaced by $e^{-V^*} = e^{-V}$.

The kinetic part of the Lagrangian reads

$$\mathcal{L}_{\rm kin} = \int d^2\theta d^2\bar\theta \left(\bar\Phi e^V \Phi + \tilde\Phi e^{-V} \bar{\tilde\Phi}\right) - \frac{1}{4g^2}\int d^2\theta\, \mathcal{W}^\alpha \mathcal{W}_\alpha - \frac{1}{4g^2}\int d^2\bar\theta\, \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}. \tag{5.2}$$

The superpotential part, in order to be gauge invariant, must be a function of $\tilde\Phi\Phi$ (recall that the superpotential must be a holomorphic function of *chiral* superfields only). Now, since for the theory to be renormalizable the superpotential W must be at most of mass dimension 3, i.e. at most cubic in the matter superfields, the only term which is allowed is a mass term:

$$\mathcal{L}_W = \int d^2\theta\, m\tilde\Phi\Phi + \int d^2\bar\theta\, m^*\bar\Phi\bar{\tilde\Phi}. \tag{5.3}$$

Let us now compute the d-term of (5.2) in order to see explicitly the couplings between the gauge and matter sectors. We will work in the Wess-Zumino (WZ) gauge which we already used extensively in the previous chapter, where V starts at the $\theta\bar\theta$ order. Note that as a consequence, V^2 is just one term:

$$V^2 = \theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta A_\mu A_\nu = \tfrac{1}{2}\theta^2\bar\theta^2 A^\mu A_\mu. \tag{5.4}$$

Furthermore, the expansion in V stops here, indeed $V^3 = 0$ in the WZ-gauge. Thus, we have that

$$e^{\pm V} = 1 \pm V + \tfrac{1}{2}V^2. \qquad \text{(WZ gauge)}$$

The chiral superfield kinetic term in (5.2) thus expands as

$$\bar\Phi e^V\Phi = \bar\Phi\Phi + \bar\Phi V\Phi + \tfrac{1}{2}\bar\Phi V^2\Phi.$$

The $\bar\Phi\Phi$ term yields the kinetic Lagrangian of the free theory, as given for instance in (4.26). The other two terms are computed as follows. Note first of all that

$$\begin{aligned}\bar\Phi V\Phi \;&=\; (\phi^* + \sqrt{2}\bar\theta\bar\psi - i\theta\sigma^\mu\bar\theta\partial_\mu\phi^*)(\theta\sigma^\nu\bar\theta A_\nu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \tfrac{1}{2}\theta^2\bar\theta^2 d)\cdot \\ &\quad\cdot(\phi + \sqrt{2}\theta\psi + i\theta\sigma^\rho\bar\theta\partial_\rho\phi)\end{aligned}$$

Its d-term reads

$$\begin{aligned}
\bar{\Phi}V\Phi|_{\theta^2\bar{\theta}^2} &= \theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}(-i\phi\partial_\mu\phi^* A_\nu + i\phi^*\partial_\mu\phi A_\nu) + 2\theta\psi\bar{\theta}\bar{\psi}\theta\sigma^\mu\bar{\theta}A_\mu \\
&\quad + i\sqrt{2}\phi\bar{\theta}\bar{\psi}\theta^2\bar{\theta}\bar{\lambda} - i\sqrt{2}\phi^*\theta\psi\bar{\theta}^2\theta\lambda + \frac{1}{2}\phi^*\phi\theta^2\bar{\theta}^2 d \\
&= \theta^2\bar{\theta}^2\left\{\frac{i}{2}(\phi^* A^\mu\partial_\mu\phi - \phi A^\mu\partial_\mu\phi^*) + \frac{1}{2}\psi\sigma^\mu\bar{\psi}A_\mu \right. \\
&\qquad \left. - \frac{i}{\sqrt{2}}\phi\bar{\psi}\bar{\lambda} + \frac{i}{\sqrt{2}}\phi^*\psi\lambda + \frac{1}{2}\phi^*\phi d\right\}.
\end{aligned}$$

As for the last term, we have

$$\bar{\Phi}V^2\Phi = \theta^2\bar{\theta}^2\frac{1}{2}\phi^*\phi A^\mu A_\mu.$$

Putting all the above terms together, we obtain

$$\begin{aligned}
\int d^2\theta d^2\bar{\theta}\,\bar{\Phi}e^V\Phi &= \partial_\mu\phi^*\partial^\mu\phi + \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + f^* f \\
&\quad - \frac{i}{2}\partial_\mu\phi^* A^\mu\phi + \frac{i}{2}A_\mu\phi^*\partial^\mu\phi + \frac{1}{4}\phi^*\phi A_\mu A^\mu - \frac{1}{2}\bar{\psi}\bar{\sigma}^\mu\psi A_\mu \\
&\quad - \frac{i}{\sqrt{2}}\phi\bar{\psi}\bar{\lambda} + \frac{i}{\sqrt{2}}\phi^*\psi\lambda + \frac{1}{2}\phi^*\phi d \\
&= \left(\partial_\mu\phi^* + \frac{i}{2}A_\mu\phi^*\right)\left(\partial^\mu\phi - \frac{i}{2}A^\mu\phi\right) \\
&\quad + \frac{i}{2}\left(\partial_\mu\bar{\psi} + \frac{i}{2}A_\mu\bar{\psi}\right)\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\left(\partial_\mu\psi - \frac{i}{2}A_\mu\psi\right) \\
&\quad + f^* f + \frac{1}{2}\phi^*\phi d - \frac{i}{\sqrt{2}}\phi\bar{\psi}\bar{\lambda} + \frac{i}{\sqrt{2}}\phi^*\psi\lambda \\
&= D_\mu\phi^* D^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + f^* f + \frac{1}{2}\phi^*\phi d \\
&\quad - \frac{i}{\sqrt{2}}\phi\bar{\psi}\bar{\lambda} + \frac{i}{\sqrt{2}}\phi^*\psi\lambda. \qquad (5.5)
\end{aligned}$$

In the final expression we have introduced the gauge-covariant derivatives

$$D_\mu\phi = \partial_\mu\phi - \frac{i}{2}A_\mu\phi, \qquad D_\mu\phi^* = \partial_\mu\phi^* + \frac{i}{2}A_\mu\phi^*,$$

and similarly for ψ and $\bar{\psi}$.

In (5.5) we also note that, besides the kinetic terms for the scalars and the fermions, we have two purely bosonic terms which, as we will see, constitute the scalar potential, and two Yukawa-like terms (i.e. couplings between a scalar and a fermion bilinear).

In order to make the above action look more familiar, we can introduce the gauge coupling g (which we might want to call e in the case of QED) by redefining the real superfield by a rescaling

$$V \to 2gV$$

so that the gauge vector is similarly rescaled $A_\mu \to 2gA_\mu$ and the covariant derivative becomes more familiar

$$D_\mu\phi \to \partial_\mu\phi - igA_\mu\phi.$$

Note that rescaling the whole superfield implies that we need also to rescale the gaugino and the d-term, so that $\lambda \to 2g\lambda$ and $d \to 2gd$. The final matter and interaction Lagrangian becomes

$$\mathcal{L} = D_\mu\phi^* D^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + f^* f + g\phi^*\phi d - ig\sqrt{2}\phi\bar{\psi}\bar{\lambda} + ig\sqrt{2}\phi^*\psi\lambda. \quad (5.6)$$

The important thing to notice is that the Yukawa couplings between the selectron, electron and photino have their value exactly set to the one of the gauge coupling.

Note that under $V \to 2gV$ the kinetic term $\mathcal{W}^\alpha\mathcal{W}_\alpha + c.c.$ gets multiplied by $4g^2$. Hence, we need also to rescale $\tau \to \frac{1}{4}\tau$ in order to restore the usual normalization of the gauge sector kinetic terms. All in all we obtain for the full Lagrangian (5.2)–(5.3) in components:

$$\begin{aligned}
\mathcal{L} &= D_\mu\phi^* D^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + f^* f + D_\mu\tilde{\phi}^* D^\mu\tilde{\phi} - i\bar{\tilde{\psi}}\bar{\sigma}^\mu D_\mu\tilde{\psi} + \tilde{f}^*\tilde{f} \\
&\quad + gd(\phi^*\phi - \tilde{\phi}\tilde{\phi}^*) - ig\sqrt{2}(\phi\bar{\psi} - \tilde{\phi}\bar{\tilde{\psi}})\bar{\lambda} + ig\sqrt{2}(\phi^*\psi - \tilde{\phi}^*\tilde{\psi})\lambda \\
&\quad - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}d^2 \\
&\quad + m\phi\tilde{f} + m\tilde{\phi}f - m\psi\tilde{\psi} + m^*\phi^*\tilde{f}^* + m^*\tilde{\phi}^* f^* - m^*\bar{\psi}\bar{\tilde{\psi}}.
\end{aligned}$$

The covariant derivatives are defined as before for ϕ and ψ, while we have the opposite sign for the components of the charge conjugate superfield $\tilde{\Phi}$,

$$D_\mu\tilde{\phi} = \partial_\mu\tilde{\phi} + igA_\mu\tilde{\phi}$$

and similarly for $\tilde{\psi}$. Lastly, since the photino is neutral we have $D_\mu\lambda = \partial_\mu\lambda$.

Note that in the Lagrangian above, the terms $m\psi\tilde{\psi} + m\bar{\psi}\bar{\tilde{\psi}}$ actually constitute a Dirac mass for the electron/positron field as in ordinary QED.

We can now integrate out the auxiliary fields d and f, f^*. For the latter, the procedure is just the same as in the (ungauged) Wess-Zumino model. As for the d-terms, we have

$$\mathcal{L}_d = \frac{1}{2}d^2 + gd(\phi^*\phi - \tilde{\phi}\tilde{\phi}^*) \tag{5.7}$$

so that

$$\frac{\delta\mathcal{L}_d}{\delta d} = 0 \qquad \Leftrightarrow \qquad d = -g(\phi^*\phi - \tilde{\phi}\tilde{\phi}^*).$$

Reinstating in (5.7), we obtain

$$\mathcal{L}_d = -\frac{1}{2}g^2(\phi^*\phi - \tilde{\phi}\tilde{\phi}^*)^2.$$

Including also the part coming from the f-terms, we finally get for the potential:

$$\mathcal{V} = \frac{1}{2}g^2(\phi^*\phi - \tilde{\phi}\tilde{\phi}^*)^2 + |m|^2(\phi^*\phi + \tilde{\phi}\tilde{\phi}^*). \tag{5.8}$$

The most important feature is that there is a quartic potential for the scalars, and its coupling is given, because of supersymmetry, by the square of the gauge coupling. This fact of course is crucial for the cancellations that will occur also between radiative corrections in the gauge sector, much similarly to those happening for a theory of chiral superfields only: there the two couplings related to each other where the Yukawa and the quartic. Here, we have gauge couplings, Yukawa couplings (mixing matter fermions and gaugini) and quartic couplings all functions of the same g.

5.1.2 Non-abelian gauge group

We now generalize to a non-abelian gauge group. The only real difference will be in the gauge kinetic part, in going from $\mathcal{W}_\alpha = -\frac{i}{4}\bar{D}^2 D_\alpha V$ to

$$\mathcal{W}_\alpha = -\frac{i}{4}\bar{D}^2\left(e^{-V}D_\alpha e^V\right).$$

We have already shown above Eq. (4.50) that even in the full non-abelian case $\mathcal{W}_\alpha$ is gauge covariant,

$$\mathcal{W}_\alpha \to e^{i\Lambda}\mathcal{W}_\alpha e^{-i\Lambda}.$$

Thus, we already know that its expression for a non-abelian gauge group must be the one obtained from the abelian expression by the covariantization of all the derivatives inside it. Of course, the covariant derivative in this case should be intended in the adjoint representation.

Let us check this. In the WZ-gauge

$$\begin{aligned} e^{-V} D_\alpha e^V &= \left(1 - V + \tfrac{1}{2}V^2\right)\left(D_\alpha V + \tfrac{1}{2}D_\alpha V V + \tfrac{1}{2} V D_\alpha V\right) \\ &= D_\alpha V + \tfrac{1}{2}[D_\alpha V, V], \end{aligned}$$

where the simplifications occur because V starts at the order $\theta\bar\theta$ while D_α starts at the order $\bar\theta$.

We already know that

$$-\frac{i}{4}\bar D^2 D_\alpha V = \lambda_\alpha - \sigma^{\mu\nu}{}_\alpha{}^\beta \theta_\beta F_{\mu\nu} + i\theta_\alpha d + i\theta^2 \sigma^\mu_{\alpha\dot\alpha}\partial_\mu \bar\lambda^{\dot\alpha}.$$

We are left to compute the term with the commutator. As usual, we start with the expression we had where all fields are function of y^μ:

$$V = \theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \tfrac{1}{2}\theta^2\bar\theta^2(d - i\partial_\mu A^\mu).$$

Then we have

$$D_\alpha V = \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} A_\mu + 2i\theta_\alpha\bar\theta\bar\lambda + \ldots,$$

and therefore

$$\begin{aligned} [D_\alpha V, V] &= \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta}[A_\mu, A_\nu] + 2i\theta_\alpha\bar\theta_{\dot\alpha}\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}[\bar\lambda^{\dot\alpha}, A_\mu] \\ &\quad + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^2\bar\theta_{\dot\beta}[A_\mu, \bar\lambda^{\dot\beta}] \\ &= \frac{1}{2}\bar\theta^2(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta\theta_\beta[A_\mu, A_\nu] + \frac{i}{2}\theta^2\bar\theta^2\sigma^\mu_{\alpha\dot\alpha}[\bar\lambda^{\dot\alpha}, A_\mu] - \frac{i}{2}\theta^2\bar\theta^2\sigma^\mu_{\alpha\dot\alpha}[A_\mu, \bar\lambda^{\dot\alpha}] \\ &= \bar\theta^2\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta[A_\mu, A_\nu] - i\theta^2\bar\theta^2\sigma^\mu_{\alpha\dot\alpha}[A_\mu, \bar\lambda^{\dot\alpha}]. \end{aligned}$$

Finally, we have

$$-\frac{i}{8}\bar D^2[D_\alpha V, V] = \frac{i}{2}\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta[A_\mu, A_\nu] + \frac{1}{2}\theta^2\sigma^\mu_{\alpha\dot\alpha}[A_\mu, \bar\lambda^{\dot\alpha}].$$

Eventually, putting all terms together, we obtain

$$\begin{aligned} -\frac{i}{4}\bar D^2\left(e^{-V}D_\alpha e^V\right) &= \lambda_\alpha - \sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta\left(\partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i}{2}[A_\mu, A_\nu]\right) \\ &\quad + i\theta_\alpha d + i\theta^2\sigma^\mu_{\alpha\dot\alpha}\left(\partial_\mu\bar\lambda^{\dot\alpha} - \frac{i}{2}[A_\mu, \bar\lambda^{\dot\alpha}]\right). \end{aligned}$$

As before, we rescale $V \to 2gV$ so that

$$\mathcal{W}_\alpha = 2g\left(\lambda_\alpha - \sigma^{\mu\nu}{}_\alpha{}^\beta \theta_\beta F_{\mu\nu} + i\theta_\alpha d + i\theta^2 \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar\lambda^{\dot\alpha}\right), \tag{5.9}$$

where we have now

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \\ D_\mu \bar\lambda &= \partial_\mu \bar\lambda - ig[A_\mu, \bar\lambda]. \end{aligned}$$

Note for completeness that we also have

$$D_\mu \lambda = \partial_\mu \lambda - ig[A_\mu, \lambda].$$

(Recall we then have to take $\tau \to \frac{1}{4}\tau$ or $\mathcal{W}_\alpha \to \frac{1}{2}\mathcal{W}_\alpha$ to restore the usual normalization of the gauge kinetic terms.)

5.1.3 Most general action in components

To close this section, we can now write the most general action for a gauge theory with a generically non-abelian gauge group, coupled to matter fields in an arbitrary (reducible) representation. Writing it further in components will highlight the couplings that it has to feature because of supersymmetry.

The manifestly supersymmetric Lagrangian is

$$\begin{aligned} \mathcal{L} &= \int d^2\theta d^2\bar\theta \sum_r \bar\Phi_r e^{2gV_a T^r_a} \Phi_r + \int d^2\theta \,\mathrm{tr}\, W(\Phi_r) + \int d^2\bar\theta \,\mathrm{tr}\, \bar W(\bar\Phi_r) \\ &\quad -\frac{\tau}{8} \int d^2\theta \,\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha - \frac{\tau^*}{8} \int d^2\bar\theta \,\mathrm{tr}\, \bar{\mathcal{W}}_{\dot\alpha} \bar{\mathcal{W}}^{\dot\alpha}. \end{aligned} \tag{5.10}$$

Here the index r labels the irreducible representations into which the matter fields are decomposed. Note that $\bar\Phi_r$ is in the ρ_r^* complex conjugate representation. For every representation, we will label its components by $i = 1 \ldots \mathrm{dim}\rho_r$ and use the convention of summing over repeated indices.

Then the same Lagrangian in components reads

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_r \left[D_\mu \phi^*_{ri} D^\mu \phi^i_r - i\bar{\psi}_{ri}\bar{\sigma}^\mu (D_\mu \psi_r)^i + f^*_{ri} f^i_r \right. \\
& \left. -ig\sqrt{2}\bar{\psi}_{ri}\bar{\lambda}_a {T^{(r)}_a}^i{}_j \phi^j_r + ig\sqrt{2}\phi^*_{ri}\lambda_a {T^{(r)}_a}^i{}_j \psi^j_r + g\phi^*_{ri} d_a {T^{(r)}_a}^i{}_j \phi^j_r \right] \\
& + \left[\sum_r \left(\frac{\partial W}{\partial \Phi_r} \right)_i f^i_r - \frac{1}{2} \sum_{r,s} \left(\frac{\partial^2 W}{\partial \Phi_r \partial \Phi_s} \right)_{ij'} \psi^i_r \psi^{j'}_s + c.c. \right] \qquad (5.11) \\
& + \sum_a \left(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2} d_a d_a - i\bar{\lambda}_a \bar{\sigma}^\mu D_\mu \lambda_a - \frac{\Theta}{64\pi^2} g^2 \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \right).
\end{aligned}
$$

Note the two kinds of Yukawa couplings, i.e. interactions involving two fermions and a scalar. The ones involving the gaugino have a coupling given by $g\sqrt{2}$, while the ones involving only matter fermions have arbitrary couplings, given by $\partial^3 W$ for a renormalizable superpotential. The scalar potential on the other hand is determined by the terms involving the auxiliary fields f and d. In the next section we write it in its most general form.

5.2 Scalar potential and moduli space of vacua

The goal of this section is to analyze, through the scalar potential, the structure of the supersymmetric vacua of a generic SUSY gauge theory. We will see instantly how this comes about. Starting from (5.11), we can solve for the d- and f-terms, so as to find the scalar potential. The variations of the Lagrangian are

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta d_a} &= d_a + g \sum_r \phi^*_{ri} {T^{(r)}_a}^i{}_j \phi^j_r, \\
\frac{\delta \mathcal{L}}{\delta f^i_r} &= f^*_{ri} + \left(\frac{\partial W}{\partial \Phi_r}(\phi_s) \right)_i.
\end{aligned}
$$

Thus we have the following equations which fix the auxiliary fields in terms of the scalar fields:

$$
\begin{aligned}
d_a &= -g \sum_r \phi^*_{ri} {T^{(r)}_a}^i{}_j \phi^j_r, \qquad (5.12) \\
f^*_{ri} &= -\left(\frac{\partial W}{\partial \Phi_r}(\phi_s) \right)_i. \qquad (5.13)
\end{aligned}
$$

Note that, because of gauge invariance of the superpotential $W(\Phi_r)$, its first derivative $\frac{\partial W}{\partial \Phi_r}$ is in the conjugate representation of Φ_r, i.e. the same as $\bar{\Phi}_r$.

The scalar potential is then given by

$$\begin{aligned}
\mathcal{V}(\phi_r, \phi_r^*) &= -\mathcal{L}_{d,f,f^*} \\
&= -\sum_r \left[f_{ri}^* f_r^i + \left(\frac{\partial W}{\partial \Phi_r}\right)_i f_r^i + \left(\frac{\partial \bar{W}}{\partial \bar{\Phi}_r}\right)^i f_{ri}^* \right] \\
&\quad -\sum_a \left(\frac{1}{2} d_a d_a + g d_a \sum_r \phi_{ri}^* T_a^{(r)i}{}_j \phi_r^j \right) \\
&= \sum_r \left(\frac{\partial W}{\partial \Phi_r}\right)_i \left(\frac{\partial \bar{W}}{\partial \bar{\Phi}_r}\right)^i + \frac{1}{2} g^2 \sum_a \left(\sum_r \phi_{ri}^* T_a^{(r)i}{}_j \phi_r^j \right)^2 .
\end{aligned}$$

There is actually a much simpler expression for the scalar potential. It is simply

$$\mathcal{V}_{\text{scalar}} = \sum_r f_{ri}^* f_r^i + \frac{1}{2} \sum_a d_a d_a, \tag{5.14}$$

where it is understood that the auxiliary fields are solved in terms of the scalar fields as in (5.12)–(5.13).

From both expressions above, we immediately see that

$$\mathcal{V}_{\text{scalar}} \geq 0,$$

and that the equality is satisfied only when all auxiliary fields vanish, i.e.

$$\mathcal{V}_{\text{scalar}} = 0 \qquad \Leftrightarrow \qquad f_r^i = 0, \quad d_a = 0 \qquad \forall\, r, i, a. \tag{5.15}$$

This fact is extremely relevant when one is looking for *classical vacua* of a given field theory. Even when one is set to quantize the theory, a classical vacuum is usually a good starting point. Such a vacuum is generally chosen to be Poincaré invariant. Hence, we must set to zero all the fields which are not Lorentz scalars, and the scalar vacuum expectation values (VEVs) must be constant throughout all spacetime. The only scalars in the general theory described here are ϕ_r, f_r and d_a. However, if we want the vacuum to also preserve supersymmetry, we need to set $f_r^i = 0$ and $d_a = 0$. Indeed, we recall that non-zero VEVs such as $f \neq 0$ or $d \neq 0$ break SUSY because $\delta\psi \propto f\epsilon$ and $\delta\lambda \propto d\epsilon$, see (4.24) and (4.44), while in a SUSY vacuum the variations of *all* fields must vanish.

Another way to see this is to recall what was said at the level of the SUSY algebra. The energy of a SUSY invariant state must be zero because essentially $E \propto QQ^\dagger$ and $Q, Q^\dagger|0\rangle = 0$ imply that $E|0\rangle = 0$.

All in all, we see that a supersymmetric ground state must have zero energy and vanishing auxiliary fields. From the point of view of the scalar potential (5.14), the fact that it is positive semi-definite implies that setting f and d to zero is sufficient to find a global minimum.

To sum up, we have a SUSY vacuum if and only if there exists a configuration of ϕ_r^i such that

$$\left(\frac{\partial W}{\partial \Phi_r}(\phi_s)\right)_i = 0 \qquad\qquad \forall\, r, i \tag{5.16}$$

and

$$\sum_r \phi^*_{ri} T_a^{(r)\,i}{}_j \phi^j_r = 0 \qquad\qquad \forall\, a. \tag{5.17}$$

The first set of equations (5.16), often referred to as *F-flatness conditions*, consists of $\sum_r \dim\rho_r$ complex conditions, while the second set of equations (5.17), often referred to as *D-flatness conditions*, consists of $\dim G$ real conditions, where G is the gauge group. If there is no solution to both of these sets of equations together, then the theory does not have a SUSY vacuum, i.e. it breaks supersymmetry spontaneously.

Of course, if these conditions are satisfied with non-zero VEVs for a scalar ϕ_r in a non-trivial representation of G, this will induce some spontaneous breaking of the gauge symmetry. However this is irrelevant as far as the supersymmetry of the vacuum is concerned, since SUSY and gauge symmetry commute.

Let us make now some trivial remarks first.

- In a gauge theory without matter chiral superfields, there are no conditions since there are simply no scalars in the theory.
- In a gauge theory with $W(\Phi) = 0$ (in particular, all matter is massless), only the conditions $d = 0$ are non-trivial.
- Conversely, in a non-gauge theory only the conditions $f = 0$ are imposed.

We can elaborate a bit further on the last situation. There are as many conditions $f_r^i = 0$ than there are scalar fields ϕ_r^i. Hence, for a generic superpotential, i.e. a generic function of the ϕ_r^i, there is always a solution to those complex equations. (Non-generic situations can arise because of global symmetries, as we will see later on.) Actually, this is true even when gauge interactions are turned on. Indeed, it may seem that the two sets of relations (5.16)–(5.17), being more numerous than the fields ϕ_r^i, overconstrain the scalar VEVs and have no solutions but in very special cases. However, gauge invariance of the superpotential $W(\Phi)$ implies there are dim$G$ redundant f-term equations among (5.16):

$$\begin{aligned}
\delta_{gauge} W(\Phi_r) = 0 \quad &\Leftrightarrow \quad \sum_r \left(\frac{\partial W}{\partial \Phi_r}\right)_i \delta_{gauge} \Phi_r^i = 0 \\
&\Leftrightarrow \quad \sum_r \left(\frac{\partial W}{\partial \Phi_r}\right)_i {T_a^{(r)}}^i{}_j \Phi_r^j = 0 \qquad \forall\, a \\
&\Leftrightarrow \quad f_{ri}^* {T_a^{(r)}}^i{}_j \Phi_r^j = 0 \qquad \forall\, a.
\end{aligned}$$

Note that the number of redundant equations among the f-terms is the same as the number of d-term equations.

Let us exemplify these considerations with our simple theory of SQED. The conditions for a SUSY vacuum read

$$\begin{aligned}
f = 0 \quad &\Leftrightarrow \quad m\tilde{\phi} = 0, \\
\tilde{f} = 0 \quad &\Leftrightarrow \quad m\phi = 0, \\
d = 0 \quad &\Leftrightarrow \quad \phi^*\phi - \tilde{\phi}\tilde{\phi}^* = 0.
\end{aligned}$$

Clearly, if $m \neq 0$, the two f-conditions imply the d-condition, and the only vacuum is the one in which $\phi = 0 = \tilde{\phi}$.

If $m = 0$ we are left with only the last equation, the d-condition. Being it one real condition on two complex constants (the components of ϕ and $\tilde{\phi}$), it necessarily leaves some fields undetermined. Essentially, it implies that $|\phi|^2 = |\tilde{\phi}|^2$. Thus, if we rewrite the scalar fields as $\phi = \rho e^{i\varphi}$ and $\tilde{\phi} = \tilde{\rho} e^{i\tilde{\varphi}}$, the condition reads

$$d = 0 \qquad \Leftrightarrow \qquad \rho = \tilde{\rho}.$$

We seem to have a manifold of vacua parametrized by ρ, φ and $\tilde{\varphi}$. However we have not yet taken into account the gauge transformations. They act by the shifts

$$\varphi \to \varphi + \alpha, \qquad\qquad \tilde{\varphi} \to \tilde{\varphi} - \alpha,$$

with α the real part of the lowest component of Λ. We can then use the gauge symmetry to set $\varphi = \tilde{\varphi}$. After doing that, we have that the scalars satisfy $\phi = \tilde{\phi}$ and the gauge symmetry is completely fixed (up to a $\mathbb{Z}_2$ discrete subgroup, which corresponds to taking $\alpha = \pi$; it shifts $\varphi \to \varphi + \pi$, $\tilde{\varphi} \to \tilde{\varphi} - \pi$ while keeping $\phi = \tilde{\phi}$).

We thus observe that we are left with a one-complex dimensional manifold of physically different vacua, parametrized by ϕ. The constant values that the scalar fields take in the vacuum are referred to as *moduli*, and hence such a manifold of vacua is called *moduli space*.

There is an alternative description of the moduli space, which is gauge invariant. If, as in the previous description, we have $\phi = \tilde{\phi} \neq 0$, then it follows that $\phi\tilde{\phi} \neq 0$. The latter combination obviously parametrizes the complex plane $\mathbb{C}$. Gauge transformations change the gauge fixing condition but leave $\phi\tilde{\phi}$ invariant. Therefore, all points parametrized by different values of $\phi\tilde{\phi}$ are physically inequivalent.

5.2.1 General characterization of the moduli space

The notion of moduli space can be defined in all generality. Let us first concentrate on a gauge theory with vanishing superpotential,

$$W(\Phi) = 0.$$

Then the moduli space is defined only by the $\dim G$ real conditions $d_a = 0$. These conditions will determine some relations among the scalar fields ϕ_r^i. However, not all different configurations satisfying $d_a = 0$ are physically inequivalent. Those which are mapped to each other by a gauge transformation (i.e. those belonging to the same gauge orbit) must be considered as physically equivalent. We should thus mod out the space of configurations such that $d_a = 0$ by the equivalence classes induced by gauge transformations. As a result, we reduce the field space by another $\dim G$ real parameters. If we denote the moduli space of a gauge theory by $\mathcal{M}$, we can summarize the above by writing

$$\mathcal{M} = \{\phi_r^i : d_a = 0\}/G. \tag{5.18}$$

At this point, we can suspect that we should be able to impose the D-flatness conditions and mod out by the action of the gauge group at the same time. Indeed, as shown in the example of SQED, the $d_a = 0$ conditions usually fix a number of constraints on the values of $|\phi|$, while modding out by G eliminates the same number of phases.

Note first of all that, as we had already remarked, if there is a superpotential, it has to be invariant under *complexified* gauge transformations. Indeed, since $W(\Phi)$ only depends on chiral superfields, the statement that it is gauge invariant means that under a gauge transformation, all dependence on Λ must disappear, for any Λ, in particular for a generically complex lowest component. In other words, we act on the fields with the complexified gauge group $G_{\mathbb{C}}$

$$\phi \to e^{i\lambda}\phi$$

with $\lambda \in \mathbb{C}$ instead of $\alpha \in \mathbb{R}$ previously. Since W is a holomorphic function, gauge invariance cannot be achieved using fields transforming with $e^{(i\alpha)^*}$.

Let us now go back to the situation where $W = 0$ and see if complexified gauge transformations can be nevertheless relevant. Consider the function

$$F = \sum_i \phi_i^* \phi^i,$$

where for simplicity we have bundled together all the fields in a single (reducible) representation. Consider also a generator t of the gauge group G, which can be taken for definiteness to lie in the Cartan subalgebra of G (by the action of G itself any generator can be brought to satisfy this condition). Under a gauge transformation along this generator we have

$$\phi^i \to (e^{i\lambda t})^i{}_j \phi^j = e^{i\lambda q_i}\phi^i,$$

with q^i a positive or negative charge (t being in the Cartan subalgebra, its action is diagonal on the ϕ^i).

Under a real gauge transformation, the function F is invariant. Under a complex gauge transformation $\lambda = i\beta$, with $\beta \in \mathbb{R}$, on the other hand, we have

$$F \to F(\beta) = \sum_i e^{-2\beta q_i}\phi_i^*\phi^i.$$

Hence, the function F varies along the orbit of $G_{\mathbb{C}}$. If all the charges q^i are of the same sign, then $F \to 0$ for $\beta \to \infty$ or $\beta \to -\infty$. This means that we are on the same complexified gauge orbit of the origin. However, for a generic choice of ϕ^i, this is never going to be the case, at least in any theory that makes sense at the quantum level.

Thus, if there exist q^is of different sign, then $F \to \infty$ for both $\beta \to \pm\infty$, starting of course from a generic configuration of ϕ^i. In turn, this implies that there must exist a minimum of F as a function of β. It is found by solving

$$\frac{\partial F}{\partial \beta} = 0 \qquad \Rightarrow \qquad \sum_i e^{-2\beta q_i} \phi_i^* q_i \phi^i = 0.$$

In other words, we can gauge transform ϕ^i to a new value given by

$$\phi'^i = e^{-\beta q_i} \phi^i$$

so that at the extremum we satisfy the condition

$$\sum_i \phi'^*_i q_i \phi'^i = 0.$$

This means that the expectation values ϕ' satisfy the D-flatness conditions. Hence, at the minimum of F along the $G_{\mathbb{C}}$ orbit, the conditions $d_a = 0$ are satisfied. It can be shown that this minimum is moreover unique (from the fact that $\partial^2 F/\partial \beta^2 > 0$). Since $G_{\mathbb{C}}$ orbits cover all of field space, it follows that all solutions of $d_a = 0$ are covered in this way. That is, by using complex gauge tranformations, we can bring any configuration of ϕ^i to another of ϕ'^i that satisfies the D-flatness conditions.

Modding out by complexified gauge transformations is thus equivalent to enforcing the equations $d_a = 0$ and then modding out by real gauge transformations. Indeed, in every complex equivalence class there is one real equivalence class consisting of solution of $d_a = 0$. An equivalent description of the moduli space is thus

$$\mathcal{M} = \{\phi_r^i\}/G_{\mathbb{C}}. \tag{5.19}$$

A completely gauge invariant characterization of $\mathcal{M}$ can be achieved by considering gauge invariant monomials composed of the ϕ_r^i. Since we are now considering complex gauge transformations, the chiral and anti-chiral monomials must be separately gauge invariant (for instance, as we have just seen above, $\phi^*\phi$ is no longer gauge invariant). The moduli space $\mathcal{M}$ can be parametrized by holomorphic (chiral) gauge invariants, possibly subject to algebraic conditions among them. More precisely, the chiral gauge invariants have the structure of a ring, called the *chiral ring*, with a number of invariants being the generators of the chiral ring. This number (modulo the possible relations among the generators) defines the dimension of the moduli space.

As an example, we go back to massless SQED. We see that under $U(1)_{\mathbb{C}}$ the scalars transform as

$$\phi \to z\phi, \qquad \tilde{\phi} \to z^{-1}\tilde{\phi},$$

and we can always set $\phi = \tilde{\phi}$ (with a left over $\mathbb{Z}_2$ gauge symmetry for $z = -1$). The gauge invariant description of $\mathcal{M}$ is given by the chiral invariant $\tilde{\phi}\phi$. Hence we determine that for massless SQED $\mathcal{M} \cong \mathbb{C}$.

As a last remark on moduli space, we note that it is straightforward to revert to a theory with a non-trivial superpotential $W(\Phi_r) \neq 0$. Indeed, the F-flatness conditions $f_r = 0$ are already covariant under $G_{\mathbb{C}}$, and hence the moduli space just becomes

$$\mathcal{M}_{W\neq 0} = \{\phi_r^i : f_r^i = 0\}/G_{\mathbb{C}}. \tag{5.20}$$

5.3 The example of SQCD

We discuss now a specific example of a non-abelian gauge theory with matter, which is the supersymmetric version of QCD, thus called Super-QCD or SQCD for short. It consists of a $SU(N_c)$ gauge group, where N_c is the number of colors, and N_f flavors, that is N_f chiral superfields Q_i^a, with $a = 1\dots N_c$ and $i = 1\dots N_f$, in the fundamental representation $\mathbf{N_c}$ of $SU(N_c)$, together with another N_f chiral superfields $\tilde{Q}_a^{\tilde{\imath}}$, with $\tilde{\imath} = 1\dots N_f$, in the anti-fundamental representation $\bar{\mathbf{N}}_\mathbf{c}$ of $SU(N_c)$.

Note that the generators of $SU(N_c)$ in the fundamental and anti-fundamental representations are related by

$$(T_A^{(\mathbf{N_c})})^a{}_b = -(T_A^{(\bar{\mathbf{N}}_\mathbf{c})})^a{}_b, \qquad A = 1\dots N_c^2 - 1.$$

The Lagrangian for SQCD is then simply

$$\begin{aligned}\mathcal{L} \;=\; & \int d^2\theta d^2\bar{\theta} \left(\sum_i \bar{Q}^i e^{2gV} Q_i + \sum_{\tilde{\imath}} \tilde{Q}^{\tilde{\imath}} e^{-2gV} \bar{\tilde{Q}}_{\tilde{\imath}} \right) \\ & -\frac{\tau}{8}\int d^2\theta \,\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha - \frac{\tau^*}{8}\int d^2\bar{\theta}\,\mathrm{tr}\,\bar{\mathcal{W}}_{\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}} \\ & + \int d^2\theta \; m^i{}_{\tilde{\imath}} \tilde{Q}_a^{\tilde{\imath}} Q_i^a + c.c.\end{aligned} \tag{5.21}$$

The quadratic superpotential in the last line is the only renormalizable one we can write. In the following, we will often consider the limit of massless SQCD, that is we will set $m^i{}_{\tilde{\imath}} = 0$.

It is instructive to consider the global symmetries of the theory above. There is a large group of such symmetries. Indeed, when acting separately on the indices i and $\tilde{\imath}$ of Q and $\tilde{Q}$, we see that independent unitary transformations leave the kinetic term invariant. This is the end of the story if the mass term is absent, while in its presence the symmetry group is smaller, as we will review shortly. Hence we have the full group $U(N_f)_Q \times U(N_f)_{\tilde{Q}}$ as global symmetry of massless SQCD. It is usually written as

$$SU(N_f)_Q \times SU(N_f)_{\tilde{Q}} \times U(1)_B \times U(1)_A, \tag{5.22}$$

where the action of the *baryonic* $U(1)_B$ is given by

$$\begin{array}{rcl} Q_i & \to & e^{i\alpha} Q_i, \\ \tilde{Q}^{\tilde{\imath}} & \to & e^{-i\alpha} \tilde{Q}^{\tilde{\imath}}, \end{array}$$

and the action of the *axial* $U(1)_A$ by

$$\begin{array}{rcl} Q_i & \to & e^{i\alpha} Q_i, \\ \tilde{Q}^{\tilde{\imath}} & \to & e^{i\alpha} \tilde{Q}^{\tilde{\imath}}. \end{array}$$

The names “baryonic” and “axial” are given in reference to the similar global symmetries existing in QCD.

5.3.1 R-symmetry

In SUSY theories, there is yet another global symmetry, intimately related to SUSY itself. Consider for instance a pure SUSY gauge theory, i.e. without matter chiral superfields. From a non-SUSY point of view, it is a sort of QCD theory where the matter is a fermion in the adjoint representation. The action is clearly invariant under the global $U(1)$ which rotates $\lambda \to e^{i\alpha}\lambda$ and keeps, obviously, A_μ fixed.

However, from the point of view of the real superfield, this $U(1)$ rotates differently the different components of V, and thus of $\mathcal{W}_\alpha$. This is in contrast to the previous global symmetries that acted on the whole chiral superfields. For instance if

$$Q = q + \sqrt{2}\theta\psi_q + \theta^2 f_q,$$

then the action of, say, $U(1)_A$ on the components is

$$q \to e^{i\alpha} q, \qquad \psi_q \to e^{i\alpha}\psi_q, \qquad f_q \to e^{i\alpha} f_q.$$

On the other hand, the gaugino superfield that we can schematically write as $\mathcal{W} = \lambda + \theta F_{\mu\nu} + \dots$ will have a definite transformation law $\mathcal{W}_\alpha \to e^{i\alpha}\mathcal{W}_\alpha$ under the $U(1)$ rotating the gaugino only if we postulate that under this $U(1)$ also the superspace coordinates θ transform non-trivially,

$$\theta \to e^{i\alpha}\theta.$$

Note that V is left invariant.

Such a global symmetry is called an *R-symmetry*. Is the gauge kinetic part of the Lagrangian invariant under the $U(1)_R$ R-symmetry? Note first that if $\theta \to e^{i\alpha}\theta$, then for the differential the opposite is true, $d\theta \to e^{-i\alpha} d\theta$, because of the relation $\int d^2\theta\, \theta^2 = 1$. We then observe that

$$\int d^2\theta \,\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha \to e^{-2i\alpha} \int d^2\theta\, e^{2i\alpha} \mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha = \int d^2\theta \,\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha,$$

that is, R-symmetry is a symmetry of the gauge Lagrangian.

As for the chiral matter superfields, we can assign them an arbitrary overall R-charge, as the latter can be anyway changed at will by redefining the R-symmetry by linear combinations involving $U(1)_B$ and $U(1)_A$. If we assign them a unit R-charge, often denoted by $R(Q) = 1$ and $R(\tilde{Q}) = 1$, we see that as superfields they transform under $U(1)_R$ as

$$\begin{aligned} Q &\to e^{i\alpha} Q, \\ \tilde{Q} &\to e^{i\alpha}\tilde{Q}. \end{aligned}$$

It looks very much like the axial symmetry $U(1)_A$, however we have to pay attention to the different transformation laws of the components, for instance for Q we have

$$q \to e^{i\alpha} q, \qquad \psi_q \to \psi_q, \qquad f_q \to e^{-i\alpha} f_q,$$

and similarly for $\tilde{Q}$.

We have to note here that, as in QCD, a linear combination of $U(1)_A$ and $U(1)_R$ (as we have just seen, the latter is also "axial" in the QCD sense, as far as the fermions are concerned) will be anomalous, i.e. the path integral measure is actually not invariant under it.

To conclude this discussion on global symmetries, it is easy to see that a mass term as written in the last line of (5.21) preserves $U(1)_B$ and the $U(1)_R$ as given above (note that a superpotential has to have R-charge $R(W) = 2$ in order to preserve R-symmetry), while it breaks $U(1)_A$ and $SU(N_f)_Q \times SU(N_f)_{\tilde{Q}}$. The latter is completely broken generically, but a subgroup can survive for specific choices of the mass matrix. For instance, if $m^i{}_{\tilde{\imath}} = \delta^i_{\tilde{\imath}}$, then $SU(N_f)_Q \times SU(N_f)_{\tilde{Q}}$ is broken down to $SU(N_f)_{diag}$, i.e. the diagonal $SU(N_f)$ which acts simultaneously on Q and $\tilde{Q}$ (in the fundamental and anti-fundamental representation respectively).

5.3.2 The classical vacua of SQCD

We can consider the classical vacua of SQCD, and we stress on "classical" because at the quantum level the story is different, much interesting, but very much beyond the scope of this chapter (and these notes).

In presence of mass terms, the f-terms impose

$$m^i{}_{\tilde{\imath}} \tilde{Q}^{\tilde{\imath}}_a = 0, \qquad\qquad m^i{}_{\tilde{\imath}} Q^a_i = 0.$$

If we assume that the matrix m is invertible, then we have to set $Q^a_i = 0 = \tilde{Q}^{\tilde{\imath}}_a$. There is only one vacuum at the origin.

It is clearly more interesting to consider the case of massless SQCD, where $m = 0$. The D-flatness conditions read

$$\sum_i \bar{Q}^i_a (T_A)^a{}_b Q^b_i - \sum_{\tilde{\imath}} \tilde{Q}^{\tilde{\imath}}_a (T_A)^a{}_b \bar{\tilde{Q}}^b_{\tilde{\imath}} = 0, \qquad (5.23)$$

for $A = 1 \dots N_c^2 - 1$. The generators $(T_A)^a{}_b$ are traceless hermitian matrices. We can then label the index A by the N_c^2 couples of indices cd (with one redundancy):

$$(T^d_c)^a{}_b = \delta^a_c \delta^d_b - \frac{1}{N_c} \delta^a_b \delta^d_c.$$

Indeed these generators are traceless, $(T^d_c)^a{}_a = \delta^d_c - \delta^d_c = 0$. We can rewrite then the D-flatness conditions as

$$\sum_i \bar{Q}^i_a Q^b_i - \sum_{\tilde{\imath}} \tilde{Q}^{\tilde{\imath}}_a \bar{\tilde{Q}}^b_{\tilde{\imath}} = \frac{1}{N_c} \delta^b_a c, \qquad (5.24)$$

where the constant c is given by

$$c = \sum_i \bar{Q}^i_a Q^a_i - \sum_{\tilde{\imath}} \tilde{Q}^{\tilde{\imath}}_a \bar{\tilde{Q}}^a_{\tilde{\imath}}.$$

One can then solve for the moduli space, but things change slightly as we change the number of flavors, according to whether $N_f < N_c$ or $N_f \geq N_c$. Let us just consider here the case with $N_f < N_c$ where there is a simpler description of the moduli space $\mathcal{M}$.

The only gauge invariants that one can write are

$$M_i^{\tilde{\imath}} = \tilde{Q}_a^{\tilde{\imath}} Q_i^a,$$

which are usually referred to as *mesonic* operators, though as far as the lowest components of the superfields are concerned, they really are *squark* bilinears (and not quark bilinears as are the real-life mesons of QCD). It is only these squark bilinear "mesons" that can acquire a VEV $\langle M_i^{\tilde{\imath}} \rangle \neq 0$ in a SUSY vacuum.

Since all of these operators are independent, and the matrix $M_i^{\tilde{\imath}}$ can be of maximal rank (because $N_c > N_f$), it results that they parametrize a moduli space of complex dimension N_f^2.

Now, for generic VEVs of $M_i^{\tilde{\imath}}$, also the squarks Q_i^a are generic, and hence we are specifying N_f different directions in $SU(N_c)$. The remaining gauge symmetry keeping those directions fixed is then $SU(N_c - N_f)$.

By the Brout-Englert-Higgs mechanism, the gauge bosons in the broken part of the $SU(N_c)$ gauge group acquire a mass by "eating" a scalar, to be found among the lowest components $q_i, \tilde{q}^{\tilde{\imath}}$ of the quark superfields. Actually, since a massive vector supermultiplet contains also a massive scalar (in addition to the longitudinal polarization of the vector), one complex scalar is eaten and becomes massive for every broken gauge generator. More generally, we had already seen early on that a massive vector supermultiplet is the result of the composition of a massless vector multiplet with a (massless) scalar multiplet.

The number of broken generators is

$$N_c^2 - 1 - (N_c - N_f)^2 + 1 = 2N_cN_f - N_f^2$$

and hence the same number of chiral superfields must become massive. We are left with

$$2N_cN_f - (2N_cN_f - N_f^2) = N_f^2$$

massless chiral superfields, whose lowest components are exactly the ones parametrizing the moduli space.

Note that the global symmetries are broken by a generic VEV $\langle M_i^{\tilde{\imath}} \rangle \neq 0$ to the subgroup consisting only of $U(1)_B$. For specific choices of the VEVs we can still have a larger global symmetry which survives, as for instance for $\langle M_i^{\tilde{i}} \rangle = M\delta_i^{\tilde{i}}$ where we still have $SU(N_f)_{diag}$ unbroken.

In the former, generic case, *all* massless modes but one are Goldstone bosons. When a larger symmetry group is preserved, as in the latter example, fewer of the massless modes are Goldstone bosons. Thus a moduli space is generically larger than the space parametrized by Goldstone bosons. This was to be expected since it is obvious from the above considerations that the moduli space of SQCD is non-compact, while the space parametrized by Goldstone bosons is compact, being it associated to broken compact symmetries.

5.4 A brief look at the MSSM

We close this chapter with a very brief description of the Minimal Supersymmetric Standard Model, usually referred to as MSSM. It is indeed a specific case of a gauge theory with matter like the ones considered in this chapter. Here we will only review its field content and the tree level supersymmetric couplings.

The key word in MSSM is "minimal". The MSSM is directly obtained from the Standard Model (SM) by introducing SUSY partners to every particle in the SM spectrum. There is one exception which is the Higgs sector which, for reasons that we will explain below, we need to extend a bit further with a second doublet.

Let us start with the gauge sector of the SM. We will have real superfields with gauge group

$$SU(3) \times SU(2)_L \times U(1)_Y .$$

The fermionic superpartners to the usual gauge bosons are the gluinos, Winos and Bino. (One usually talks about Winos and Bino because in real world those are expected to have masses larger than the electroweak scale. Hence by the time we have a Z boson and a photon the fermionic superpartners are long gone and it makes no sense of speaking of a Zino and a photino.)

As for the fermionic sector of the SM, we associate to each fermion a chiral superfield. It results that there will be scalar partners for all the fermions, they are collectively called *sfermions* and more specifically squarks and sleptons.

We have then the following chiral superfields, with charges under the gauge group

	$SU(3)\times SU(2)_L\times U(1)_Y$
Q_i	$(\mathbf{3},\mathbf{2},+\frac{1}{6})$
$\tilde{U}_i$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})$
$\tilde{D}_i$	$(\bar{\mathbf{3}},\mathbf{1},+\frac{1}{3})$
L_i	$(\mathbf{1},\mathbf{2},-\frac{1}{2})$
$\tilde{e}_i$	$(\mathbf{1},\mathbf{1},+1)$

(5.25)

The index $i=1,2,3$ is the family index, and since we have written all chiral superfields, their fermionic component is always a left-handed Weyl fermion.

Note that each family is a complex, reducible representation of the gauge group. The fact that it is complex means that supersymmetric mass terms cannot be written, just as one cannot write mass terms for the fermions in the SM (before electroweak symmetry breaking of course!). The components of the chiral superfields read, for example

$$\begin{aligned} Q_i &= \hat{q}_i+\sqrt{2}\theta q_i+\ldots, \\ \tilde{U}_i &= \hat{u}_i+\sqrt{2}\theta\tilde{u}_i+\ldots, \end{aligned}$$

where $\hat{q}_i$ and $\hat{u}_i$ are squarks, q_i are the left-handed quarks of the SM, while $\tilde{\bar{u}}_i$ are the right-handed quarks of the SM, more usually written as $\bar{u}_i$. All is exactly similar for the (s)leptons.

We now turn to the Higgs sector in order to write the supersymmetric version of the Yukawa couplings of the SM. The Higgs field will also be part of a chiral superfield in the **2** of $SU(2)_L$, and it will be the lowest component. The Yukawa couplings will directly descend from the superpotential. However, we immediately realize that in order to be able to write all the needed couplings in a holomorphic way, we need *two* Higgs doublets, with opposite $U(1)_Y$ charges:

$$W=\lambda^e_{ij}HL_i\tilde{e}_j+\lambda^d_{ij}HQ_i\tilde{D}_j+\lambda^u_{ij}\tilde{H}Q_i\tilde{U}_j,$$

with the charges given by

	$SU(3)\times SU(2)_L\times U(1)_Y$
H	$(\mathbf{1},\mathbf{2},-\frac{1}{2})$
$\tilde{H}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})$

(5.26)

In the SM we could use H^* instead of $\tilde{H}$ but here using the anti-chiral superfield $\bar{H}$ is not allowed by SUSY.

Of course, H and $\tilde{H}$ also have fermionic partners, the higgsinos. These fermions have the same gauge charges of the superfields to which they belong, and so we see that we need a pair of charge-conjugate Higgs superfields anyway otherwise both $U(1)_Y$ and $SU(2)_L$ would have anomalies at the quantum level. (Indeed it can be proven that an $SU(2)$ gauge theory with an odd number of fermions in the **2** is not consistent at the quantum level.)

Within the Higgs sector, the only renormalizable coupling we can write is

$$W' = \mu H\tilde{H}.$$

Note again that we can only write such a SUSY mass term for the Higgs if we have two doublets.

There is also a quartic coupling involving the Higgs scalars in the MSSM Lagrangian due to the d-terms, its coupling being therefore proportional to the gauge couplings. However notice that, after solving for the f- and d-terms, we have a Higgs potential given by

$$\mathcal{V}_{H,\tilde{H}} \propto |\mu|^2 \left(|H|^2 + |\tilde{H}|^2\right) + g^2 \left(|H|^2 - |\tilde{H}|^2\right)^2.$$

Thus $H = 0 = \tilde{H}$ is *always* a stable solution.

In order to break $SU(2)_L \times U(1)_Y$ to $U(1)_{em}$ one is forced to introduce SUSY breaking terms that make the squared mass for H and $\tilde{H}$ negative. Thus electroweak symmetry breaking and SUSY breaking are intimately related.

One last remark on the MSSM is that there are a lot of potentially dangerous terms that one could add to W, allowing processes violating baryon and lepton numbers (and eventually leading to fast proton decay), for instance terms such as $\tilde{D}_i Q_j L_k$, $L_i L_j \tilde{e}_k$ or $\tilde{D}_i \tilde{D}_j \tilde{U}_k$. These terms are avoided if we assign R-charges to superfields in a way that SM particles all have vanishing R-charge:

$$R(Q) = R(\tilde{U}) = \cdots = 1, \qquad R(H) = R(\tilde{H}) = 0,$$

so that for instance $R(q) = 0$ and the R-charge of the scalar Higgs which becomes the SM one is also zero. The continuous $U(1)_R$ is broken, for instance by the μ-term, but a discrete Z_2 subgroup survives, called *R-parity*, and it is enough to prevent the dangerous terms in W to appear.

Chapter 6

Radiative corrections and non-renormalization theorems

In this chapter we finally address the question that was the main motivation in considering supersymmetric field theories: the claim they have a "softer" behavior under renormalization, i.e. there are cancellations among quantum radiative corrections. This is crucial for instance to ensure the protection of hierachies in the MSSM.

One way to proceed would be to take a SUSY action in component fields, compute radiative corrections as in an ordinary field theory, and find that upon summing different contributions, miraculous cancellations occur. Those will clearly be due to the fact that we have identified several different couplings, such as the quartic scalar ϕ^4 couplings and $\phi\psi\psi$ Yukawas, or gauge couplings and $\phi\psi\lambda$ Yukawas.

Another way to proceed is to take profit of the manifestly SUSY formalism of superspace and superfields. The idea here is to compute directly radiative corrections to superfield propagators and vertices. The cancellations will be automatic in this formalism. It is the latter route that we will employ.

6.1 Superfield propagators

For definiteness (and simplicity), we will consider only the theory of a chiral superfield, i.e. the Wess-Zumino model.

In order to write the propagators of a theory, one has to consider only the part of the action that is quadratic in the (super)fields

$$S = \int d^4x \left(\int d^2\theta d^2\bar{\theta}\; \bar{\Phi}\Phi + \tfrac{1}{2} \int d^2\theta\; m\Phi^2 + \tfrac{1}{2} \int d^2\bar{\theta}\; m\bar{\Phi}^2 \right), \tag{6.1}$$

where we have taken the mass to be real, $m^* = m$ for simplicity. Then, it is a matter of taking the "inverse" of this quadratic kinetic term. All these notions can be clearly summarized in the path integral approach to quantum field theory, and we will review the relevant steps in the process of deriving the superpropagators below.

We will derive the superpropagators in two different ways, in order to foster confidence in the final results. Firstly, we will make use of the known propagators of the component fields. Secondly, we will formulate the path integral directly in terms of the superfields.

6.1.1 Formulation with component fields

The action (6.1) in components reads

$$S = \int d^4x \left(\partial_\mu\phi\partial^\mu\phi^* - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + ff^* + m\phi f + m\phi^* f^* - \tfrac{1}{2}m\psi\psi - \tfrac{1}{2}m\bar{\psi}\bar{\psi} \right). \tag{6.2}$$

It can be recast using matrices that summarize both the bosonic and the fermionic kinetic terms

$$\begin{aligned} S \;=\; \int d^4x \Big\{ & \begin{pmatrix} \phi^* & f \end{pmatrix} \begin{pmatrix} -\Box & m \\ m & 1 \end{pmatrix} \begin{pmatrix} \phi \\ f^* \end{pmatrix} \\ & + \tfrac{1}{2} \begin{pmatrix} \bar{\psi} & \psi \end{pmatrix} \begin{pmatrix} -i\bar{\sigma}^\mu\partial_\mu & -m \\ -m & -i\sigma^\mu\partial_\mu \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \Big\}. \end{aligned} \tag{6.3}$$

Then, the various propagators are the entries in the inverse matrices of the ones appearing in the expression above.

Let us recall that the Green function for a scalar (Klein-Gordon) field is defined as

$$(-\Box_x - m^2)\Delta(x, x') = i\delta^4(x - x').$$

It is usually represented simply as

$$\Delta(x, x') \equiv \frac{i}{-\Box - m^2}\delta^4(x - x').$$

Most notably (and possibly more familiar), its Fourier transform then reads

$$\tilde{\Delta}(p) = \frac{i}{p^2 - m^2}.$$

When more fields are present, with non-trivial kinetic matrices, the prescription to compute propagators from the path integral is the following.

Given that the generating functional for Green functions is

$$Z[J, J^*] = \int [\mathcal{D}\phi] e^{i\int(\phi K \phi^* + J\phi + J^*\phi^*)}, \tag{6.4}$$

where we have written schematically $S = \int \phi K \phi^*$, the Green functions are evaluated as

$$\begin{aligned} \langle 0|\phi(x)\phi^*(x')|0\rangle &= \frac{1}{Z}\int [\mathcal{D}\phi]\phi(x)\phi^*(x') e^{i\int(\phi K\phi^* + J\phi + J^*\phi^*)}\Big|_{J=J^*=0} \\ &= \frac{1}{Z}\frac{\delta}{\delta i J(x)}\frac{\delta}{\delta i J^*(x')} Z\Big|_{J=J^*=0}. \end{aligned} \tag{6.5}$$

The propagators (or two-point functions) are thus easily obtained by just performing the Gaussian integral (6.4)

$$Z[J, J^*] = \mathcal{N} e^{-i\int J K^{-1} J^*} \tag{6.6}$$

($\mathcal{N}$ is the normalization constant coming from the fluctuation determinant), so that

$$\langle 0|\phi(x)\phi^*(x')|0\rangle = iK^{-1}. \tag{6.7}$$

We thus have to invert the kinetic matrices. For the bosonic one, we have

$$K_B = \begin{pmatrix} -\Box & m \\ m & 1 \end{pmatrix} \qquad \Rightarrow \qquad K_B^{-1} = \frac{1}{-\Box - m^2}\begin{pmatrix} 1 & -m \\ -m & -\Box \end{pmatrix}, \tag{6.8}$$

while for the fermionic one we have

$$K_F = \begin{pmatrix} -i\bar{\sigma}^\mu\partial_\mu & -m \\ -m & -i\sigma^\mu\partial_\mu \end{pmatrix} \Rightarrow K_F^{-1} = \frac{1}{-\Box - m^2}\begin{pmatrix} -i\sigma^\mu\partial_\mu & m \\ m & -i\bar{\sigma}^\mu\partial_\mu \end{pmatrix}. \tag{6.9}$$

From the entries of the inverse matrices above we can read the two-point functions, or propagators, of all the component fields. These are:

$$\langle 0|\phi(x)\phi^*(x')|0\rangle = \frac{i}{-\Box - m^2}\delta^4(x-x'), \quad (6.10)$$

$$\langle 0|\phi(x)f(x')|0\rangle = \frac{-im}{-\Box - m^2}\delta^4(x-x'), \quad (6.11)$$

$$\langle 0|\phi^*(x)f^*(x')|0\rangle = \frac{-im}{-\Box - m^2}\delta^4(x-x'), \quad (6.12)$$

$$\langle 0|f(x)f^*(x')|0\rangle = \frac{-i\Box}{-\Box - m^2}\delta^4(x-x'), \quad (6.13)$$

$$\langle 0|\bar\psi^{\dot\alpha}(x)\psi^\alpha(x')|0\rangle = \frac{\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu}{-\Box - m^2}\delta^4(x-x'), \quad (6.14)$$

$$\langle 0|\psi_\alpha(x)\bar\psi_{\dot\alpha}(x')|0\rangle = \frac{\sigma^\mu_{\alpha\dot\alpha}\partial_\mu}{-\Box - m^2}\delta^4(x-x'), \quad (6.15)$$

$$\langle 0|\psi_\alpha(x)\psi^\beta(x')|0\rangle = \frac{im\delta_\alpha^\beta}{-\Box - m^2}\delta^4(x-x'), \quad (6.16)$$

$$\langle 0|\bar\psi^{\dot\alpha}(x)\bar\psi_{\dot\beta}(x')|0\rangle = \frac{im\delta^{\dot\alpha}_{\dot\beta}}{-\Box - m^2}\delta^4(x-x'). \quad (6.17)$$

Notice in passing that, since we are using Weyl fermions, we have several different fermionic propagators. In particular, note that the chiral ones (the last two) are actually vanishing for massless Weyl fermions.

Armed with the above two-point functions, we can now compute superfield correlators, just by decomposing them:

$$\begin{aligned}
\langle 0|\Phi(y,\theta)\Phi(y',\theta')|0\rangle &= \langle 0|(\phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 f(y)) \cdot \\
&\qquad \cdot(\phi(y') + \sqrt{2}\theta'\psi(y') + {\theta'}^2 f(y'))|0\rangle \\
&= {\theta'}^2\langle 0|\phi(y)f(y')|0\rangle + \theta^2\langle 0|f(y)\phi(y')|0\rangle \\
&\qquad +2\theta^\alpha\theta'_\beta\langle 0|\psi_\alpha(y)\psi^\beta(y')|0\rangle \\
&= \left[({\theta'}^2 + \theta^2)(-im) + 2\theta\theta' im\right]\frac{1}{-\Box - m^2}\delta^4(y-y') \\
&= -im(\theta-\theta')^2\frac{1}{-\Box - m^2}\delta^4(y-y'). \quad (6.18)
\end{aligned}$$

Now notice first of all that $(\theta-\theta')^2$ can be rewritten as $\delta^2(\theta-\theta')$. Indeed

$$\int d^2\theta F(\theta)(\theta-\theta')^2 = \int d^2\theta(a+\theta b+\theta^2 c)(\theta^2-2\theta\theta'+{\theta'}^2) = a+\theta' b+{\theta'}^2 c = F(\theta'),$$

which is the correct behavior for a Dirac delta-function (on G-odd variables).

A second point to be noticed in (6.18) is that the spacetime delta-function has actually the *chiral* coordinates as argument. One way to obtain a more familiar Dirac delta is to recall that the Taylor expansion can be recast simply as the exponential of a derivative. For instance

$$F(y) = F(x + i\theta\sigma\bar{\theta}) = e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu} F(x).$$

Hence the propagator (6.18) can be rewritten as

$$\langle 0|\Phi(x,\theta,\bar{\theta})\Phi(x',\theta',\bar{\theta}')|0\rangle = -ime^{i(\theta\sigma^\mu\bar{\theta}-\theta'\sigma^\mu\bar{\theta}')\partial_\mu}\frac{1}{-\Box - m^2}\delta^2(\theta-\theta')\delta^4(x-x'). \tag{6.19}$$

There is however another, possibly more interesting, way to rewrite the above propagator. Let us first notice the identity

$$\bar{D}^2(\bar{\theta}-\bar{\theta}')^2 = -4.$$

Indeed $\bar{D}_{\dot{\alpha}}$ acts trivially on $\bar{\theta}'$. We can then perform the following chain of identities

$$\begin{aligned}
(\theta-\theta')^2 F(y-y') &= -\frac{1}{4}(\theta-\theta')^2[\bar{D}^2(\bar{\theta}-\bar{\theta}')^2]F(y-y') \\
&= -\frac{1}{4}\bar{D}^2[(\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2 F(y-y')] \\
&= -\frac{1}{4}\bar{D}^2[(\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2 F(x-x')].
\end{aligned}$$

In the second equality we have used that $\bar{D}\theta = 0 = \bar{D}y$, while in the third equality we have used that

$$y - y' = x - x' + i\theta\sigma\bar{\theta} - i\theta'\sigma\bar{\theta}' = x - x' + i(\theta-\theta')\sigma\bar{\theta} + i\theta'\sigma(\bar{\theta}-\bar{\theta}')$$

and the fact that $(\theta-\theta')^3 = 0 = (\bar{\theta}-\bar{\theta}')^3$. Using the above, we can rewrite (6.18) as

$$\langle 0|\Phi(x,\theta,\bar{\theta})\Phi(x',\theta',\bar{\theta}')|0\rangle = \frac{im}{4}\bar{D}^2\frac{1}{-\Box - m^2}\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}')\delta^4(x-x'). \tag{6.20}$$

Writing it in this way, it is also obvious that it is a propagator for chiral superfields, since acting with $\bar{D}$ it necessarily gives a vanishing result because $\bar{D}^3 = 0$.

Similarly, we also get

$$\langle 0|\bar{\Phi}(x,\theta,\bar{\theta})\bar{\Phi}(x',\theta',\bar{\theta}')|0\rangle = \frac{im}{4}D^2\frac{1}{-\Box - m^2}\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}')\delta^4(x-x'). \tag{6.21}$$

We can now compute the remaining propagator

$$\begin{aligned}
\langle 0|\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x',\theta',\bar{\theta}')|0\rangle &= \langle 0|\Phi(y,\theta)\bar{\Phi}(\bar{y}',\bar{\theta}')|0\rangle \\
&= \langle 0|(\phi(y)+\sqrt{2}\theta\psi(y)+\theta^2 f(y))\cdot \\
&\quad \cdot(\phi^*(\bar{y}')+\sqrt{2}\bar{\theta}'\bar{\psi}(\bar{y}')+\bar{\theta}'^2 f^*(\bar{y}'))|0\rangle \\
&= \left[1+2\theta^\alpha\bar{\theta}'^{\dot{\alpha}}(-i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu)+\theta^2\bar{\theta}'^2(-\Box)\right]\cdot \\
&\quad \cdot\frac{i}{-\Box-m^2}\delta^4(y-\bar{y}') \\
&= e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}\frac{i}{-\Box-m^2}\delta^4(y-\bar{y}') \qquad (6.22)\\
&= e^{i(\theta\sigma^\mu\bar{\theta}+\theta'\sigma^\mu\bar{\theta}'-2\theta\sigma^\mu\bar{\theta}')\partial_\mu}\frac{i}{-\Box-m^2}\delta^4(x-x').
\end{aligned}$$

We can perform a similar trick to the one used for the chiral propagator. We write

$$\begin{aligned}
e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}F(y-\bar{y}') &= -\frac{1}{4}[\bar{D}^2(\bar{\theta}-\bar{\theta}')^2]e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}F(y-\bar{y}') \\
&= -\frac{1}{4}\bar{D}^2\left[(\bar{\theta}-\bar{\theta}')^2 e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}F(y-\bar{y}')\right] \\
&= \frac{1}{16}\bar{D}^2\left[(D'^2(\theta-\theta')^2)(\bar{\theta}-\bar{\theta}')^2 e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}F(y-\bar{y}')\right] \\
&= \frac{1}{16}\bar{D}^2 D'^2\left[\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}')e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}F(y-\bar{y}')\right] \\
&= \frac{1}{16}\bar{D}^2 D'^2\left[\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}')F(x-x')\right],
\end{aligned}$$

where we have used that

$$\begin{aligned}
e^{-2i\theta\sigma^\mu\bar{\theta}'\partial_\mu}F(y-\bar{y}') &= F(x-x'+i\theta\sigma\bar{\theta}+i\theta'\sigma\bar{\theta}'-2i\theta\sigma\bar{\theta}') \\
&= F(x-x'+i\theta\sigma(\bar{\theta}-\bar{\theta}')-i(\theta-\theta')\sigma\bar{\theta}').
\end{aligned}$$

Eventually, the propagator (6.22) becomes

$$\langle 0|\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x',\theta',\bar{\theta}')|0\rangle = \frac{1}{16}\bar{D}^2 D'^2\frac{i}{-\Box-m^2}\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}')\delta^4(x-x'). \tag{6.23}$$

Again, one can read directly from the propagator that the superfield in x is chiral while the superfield in x' is anti-chiral.

Thus, we have obtained in (6.20), (6.21) and (6.23) all the propagators for the theory of a chiral superfield. We now turn to derive the same propagators in a formulation where even the path integral is expressed in terms of superfields.

6.1.2 Formulation with superfields

It is instructive to rederive the above propagators directly from the action written in superspace. However attention must be paid to the fact that parts of the action are integrated over just half of superspace.

We thus now write the action for a chiral superfield, coupled to sources J and $\bar{J}$ which are respectively chiral and anti-chiral superfields themselves:

$$\bar{D}_{\dot{\alpha}} J = 0, \qquad D_\alpha \bar{J} = 0.$$

The action is

$$S = \int d^4x \left[\int d^2\theta d^2\bar{\theta}\; \bar{\Phi}\Phi + \int d^2\theta (\tfrac{1}{2}m\Phi^2 + J\Phi) + \int d^2\bar{\theta}(\tfrac{1}{2}m\bar{\Phi}^2 + \bar{J}\bar{\Phi}) \right]. \tag{6.24}$$

In order to convert it to an expression involving only integration over all of superspace, we take profit of the following identity:

$$\begin{aligned}
\bar{D}^2 D^2 \Phi &= \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}} D_\beta D_\alpha \Phi \\
&= \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} (-\bar{D}_{\dot{\alpha}} D_\beta \bar{D}_{\dot{\beta}} D_\alpha \Phi + 2i\sigma^\mu_{\beta\dot{\beta}} \bar{D}_{\dot{\alpha}} D_\alpha \partial_\mu \Phi) \\
&= \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} (-2i\bar{D}_{\dot{\alpha}} D_\beta \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \Phi - 4\sigma^\mu_{\beta\dot{\beta}} \sigma^\nu_{\alpha\dot{\alpha}} \partial_\mu \partial_\nu \Phi) \\
&= \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} (4\sigma^\mu_{\alpha\dot{\beta}} \sigma^\nu_{\beta\dot{\alpha}} - 4\sigma^\mu_{\beta\dot{\beta}} \sigma^\nu_{\alpha\dot{\alpha}}) \partial_\mu \partial_\nu \Phi \\
&= -8(\sigma^\mu \bar{\sigma}^\nu)_\alpha{}^\alpha \partial_\mu \partial_\nu \Phi \\
&= -16\Box\Phi.
\end{aligned}$$

We can thus write, allowing for a (temporarily) non-local expression

$$\Phi = -\frac{1}{16}\bar{D}^2 D^2 \frac{1}{\Box}\Phi.$$

Now we can use the familiar substitution $\bar{D}^2 = -4\int d^2\bar{\theta}$ to eventually write

$$S = \int d^4x d^2\theta d^2\bar{\theta}\left[\bar{\Phi}\Phi + \frac{1}{8}m\Phi\frac{D^2}{\Box}\Phi + \frac{1}{8}m\bar{\Phi}\frac{\bar{D}^2}{\Box}\bar{\Phi} + \frac{1}{4}\Phi\frac{D^2}{\Box}J + \frac{1}{4}\bar{\Phi}\frac{\bar{D}^2}{\Box}\bar{J}\right]. \tag{6.25}$$

Using matrices, this is

$$S = \int d^4x d^2\theta d^2\bar{\theta}\left[\frac{1}{2}\begin{pmatrix}\Phi & \bar{\Phi}\end{pmatrix}\begin{pmatrix}\frac{mD^2}{4\Box} & 1 \\ 1 & \frac{m\bar{D}^2}{4\Box}\end{pmatrix}\begin{pmatrix}\Phi \\ \bar{\Phi}\end{pmatrix} + \begin{pmatrix}\Phi & \bar{\Phi}\end{pmatrix}\begin{pmatrix}\frac{D^2}{4\Box}J \\ \frac{\bar{D}^2}{4\Box}\bar{J}\end{pmatrix}\right]. \tag{6.26}$$

The kinetic matrix and its inverse are thus

$$K = \begin{pmatrix}\frac{mD^2}{4\Box} & 1 \\ 1 & \frac{m\bar{D}^2}{4\Box}\end{pmatrix} \quad\Rightarrow\quad K^{-1} = \frac{\Box}{-\Box - m^2}\begin{pmatrix}\frac{m\bar{D}^2}{4\Box} & -1 \\ -1 & \frac{mD^2}{4\Box}\end{pmatrix}. \tag{6.27}$$

We have used for instance that $\frac{m^2 D^2\bar{D}^2}{16\Box^2} = \frac{-m^2}{\Box}$ on anti-chiral superfields.

Hence, if we define

$$e^{iW(J,\bar{J})} = \int [\mathcal{D}\Phi] e^{iS(\Phi,\bar{\Phi},J,\bar{J})},$$

we obtain by performing the Gaussian integration

$$\begin{aligned}
W &= -\int d^4x d^2\theta d^2\bar{\theta}\,\frac{1}{2}\begin{pmatrix}\frac{D^2}{4\Box}J & \frac{\bar{D}^2}{4\Box}\bar{J}\end{pmatrix}\frac{\Box}{-\Box - m^2}\begin{pmatrix}\frac{m\bar{D}^2}{4\Box} & -1 \\ -1 & \frac{mD^2}{4\Box}\end{pmatrix}\begin{pmatrix}\frac{D^2}{4\Box}J \\ \frac{\bar{D}^2}{4\Box}\bar{J}\end{pmatrix} \\
&= -\int d^4x d^2\theta d^2\bar{\theta}\,\frac{1}{2}\begin{pmatrix}\frac{D^2}{4\Box}J & \frac{\bar{D}^2}{4\Box}\bar{J}\end{pmatrix}\frac{1}{-\Box - m^2}\begin{pmatrix}-mJ - \frac{1}{4}\bar{D}^2\bar{J} \\ -\frac{1}{4}D^2J - m\bar{J}\end{pmatrix} \\
&= \int d^4x d^2\theta d^2\bar{\theta}\left(-J\frac{1}{-\Box - m^2}\bar{J} + \frac{1}{2}mJ\frac{1}{-\Box - m^2}\frac{D^2}{4\Box}J\right. \\
&\qquad\qquad \left. + \frac{1}{2}m\bar{J}\frac{1}{-\Box - m^2}\frac{\bar{D}^2}{4\Box}\bar{J}\right). \qquad (6.28) \\
&= \int d^4x\left[\int d^2\theta d^2\bar{\theta}\left(-J\frac{1}{-\Box - m^2}\bar{J}\right)\right. \\
&\qquad \left. + \int d^2\theta\,\frac{1}{2}mJ\frac{1}{-\Box - m^2}J + \int d^2\bar{\theta}\,\frac{1}{2}m\bar{J}\frac{1}{-\Box - m^2}\bar{J}\right]. \qquad (6.29)
\end{aligned}$$

We can find the correlation functions for the superfields by taking functional derivatives of iW with respect to iJ and $i\bar{J}$. However note that there is a slight subtlety in taking a functional derivative of a chiral superfield. A good definition is

$$\frac{\delta}{\delta\Phi(y',\theta')}\Phi(y,\theta)=\delta^2(\theta-\theta')\delta^4(y-y')=-\frac{1}{4}\bar{D}^2\delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta')\delta^4(x-x'). \tag{6.30}$$

Indeed, applying this rule, we have

$$\begin{aligned}\frac{\delta}{\delta J'}\int d^4xd^2\theta\ \Phi J &= -\frac{1}{4}\int d^4xd^2\theta\ \Phi(y,\theta)\bar{D}^2\delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta')\delta^4(x-x')\\ &= \int d^4xd^2\theta d^2\bar\theta\ \Phi(y,\theta)\delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta')\delta^4(x-x')\\ &= \Phi(y',\theta').\end{aligned}$$

Using the shorthand $\Phi'\equiv\Phi(y',\theta')$, we then compute the propagators. The chiral propagator is

$$\begin{aligned}\langle 0|\Phi\Phi'|0\rangle &= \frac{\delta}{\delta iJ}\frac{\delta}{\delta iJ'}e^{iW}\\ &= -i\frac{\delta}{\delta iJ}m\frac{1}{-\Box-m^2}iJ'\\ &= \frac{im}{4}\frac{1}{-\Box-m^2}\bar{D}^2\delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta')\delta^4(x-x'),\end{aligned}$$

exactly reproducing (6.20). As for the chiral-to-anti-chiral propagator, we get

$$\begin{aligned}\langle 0|\Phi\bar\Phi'|0\rangle &= \frac{\delta}{\delta iJ}\frac{\delta}{\delta i\bar{J}'}e^{iW}\\ &= i\frac{\delta}{\delta iJ}\left(\frac{1}{-\Box-m^2}(-\frac{1}{4}D'^2)iJ'\right)\\ &= \frac{i}{16}\frac{1}{-\Box-m^2}\bar{D}^2D'^2\delta^2(\theta-\theta')\delta^2(\bar\theta-\bar\theta')\delta^4(x-x'),\end{aligned}$$

again in exact agreement with (6.23).

Before going on to compute the radiative corrections, let us perform the Fourier transform on the spacetime coordinates. (Note that we will not attempt here to also perform a Fourier transform in the G-odd coordinates, though in some instances this could be of interest.)

In momentum space, the propagators read:

$$\langle 0|\Phi(p,\theta,\bar{\theta})\Phi(-p,\theta',\bar{\theta}')|0\rangle = \frac{im}{4}\frac{1}{p^2-m^2}\bar{D}^2\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}'), \tag{6.31}$$

$$\langle 0|\Phi(p,\theta,\bar{\theta})\bar{\Phi}(-p,\theta',\bar{\theta}')|0\rangle = \frac{i}{16}\frac{1}{p^2-m^2}\bar{D}^2 D'^2\delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}'), \tag{6.32}$$

where now of course in D and $\bar{D}$ we have replaced ∂_μ by ip_μ.

6.2 Some radiative corrections

We are now able to compute quantum radiative corrections, which appear as Feynman diagrams including loops. Before establishing a theorem that will constrain such corrections, we will consider radiative corrections to some selected quantities to get a feeling of how supersymmetry and superspace techniques facilitate such computations.

Let us first compute the corrections to the propagators themselves.

A diagram with a loop inserted inside a $\langle\Phi\Phi\rangle$ propagator involves two cubic vertices of the type

$$\int d^4x d^2\theta\, \frac{\lambda}{3}\Phi^3(y,\theta). \tag{6.33}$$

Note that both vertices should be *chiral*, in order to tie correctly with the chiral superfields at the two ends of the (corrected) propagator.

As usual in perturbative quantum field theory, for instance from the path integral formulation, one has to bring down the vertices in the correlation function, and integrate over the momentum running in the loop. Going to the Fourier transform, we obtain for the corrected propagator

$$\frac{1}{2}\int d^4x e^{ip(x-x')}\langle 0|\Phi(x,\theta,\bar{\theta})\Phi(x',\theta',\bar{\theta}')i\int d^4x_1 d^2\theta_1\,\frac{\lambda}{3}\Phi^3(x_1,\theta_1,\bar{\theta}_1)\cdot$$

$$\cdot i\int d^4x_2 d^2\theta_2\,\frac{\lambda}{3}\Phi^3(x_2,\theta_2,\bar{\theta}_2)|0\rangle =$$

$$= -2\lambda^2\int d^4k d^2\theta_1 d^2\theta_2 \frac{im}{4}\frac{1}{p^2-m^2}\bar{D}^2\delta^2(\theta-\theta_1)\delta^2(\bar{\theta}-\bar{\theta}_1)\cdot$$

$$\cdot\frac{im}{4}\frac{1}{k^2-m^2}\bar{D}_1^2\delta^2(\theta_1-\theta_2)\delta^2(\bar{\theta}_1-\bar{\theta}_2)\frac{im}{4}\frac{1}{(p-k)^2-m^2}\bar{D}_1^2\delta^2(\theta_1-\theta_2)\delta^2(\bar{\theta}_1-\bar{\theta}_2)\cdot$$

$$\cdot\frac{im}{4}\frac{1}{p^2-m^2}\bar{D}'^2\delta^2(\theta_2-\theta')\delta^2(\bar{\theta}_2-\bar{\theta}').$$

We can concentrate on the central part (which actually corresponds to the amputated diagram, i.e. all that we need to compute the corrections). Recall that $\delta^2(\theta_1-\theta_2)$ remains as a factor in the internal propagators, and in the expression above it will appear twice. Hence, since $(\theta_1-\theta_2)^4=0$ because of the G-odd nature of the superspace coordinates, the whole expression vanishes and *there are no corrections to the* $\langle\Phi\Phi\rangle$ *propagator*.

This result amounts to stating that the mass parameter appearing in the superpotential is not renormalized. As we have just seen, this is automatic in superfield language. In components, the same result would be seen as a cancellation between two types of one-loop corrections to the scalar propagator: a seagull diagram with a scalar loop and a quartic vertex, against a diagram with a fermionic loop and two Yukawa vertices. The cancellation occurs because of the relation between the couplings of the two kinds of vertices.

A completely similar vanishing result is obtained for the one-loop correction to the $\langle\bar{\Phi}\bar{\Phi}\rangle$ propagator.

On the other hand, the $\langle\Phi\bar{\Phi}\rangle$ propagator is a different story. Let us consider a one-loop diagram similar to the one considered previously for the $\langle\Phi\Phi\rangle$ propagator, however now one vertex is chiral while the other is anti-chiral. As a consequence, the internal propagators are also of $\langle\Phi\bar{\Phi}\rangle$ type. Let us compute the correction, focusing on the amputated diagram where we have dropped the two external propagators. After bringing down the vertices and going to Fourier space, we obtain a contribution proportional to

$$2\lambda\lambda^*\int d^4k d^2\theta_1 d^2\bar{\theta}_2\frac{i}{16}\frac{1}{k^2-m^2}\bar{D}_1^2D_2^2\delta_{12}\frac{i}{16}\frac{1}{(p-k)^2-m^2}\bar{D}_1^2D_2^2\delta_{12}=$$

$$=-2\lambda\lambda^*\int d^4k d^2\theta_1 d^2\bar{\theta}_2\frac{1}{k^2-m^2}\frac{1}{(p-k)^2-m^2}e^{-(\theta_1\sigma^\mu\bar{\theta}_1+\theta_2\sigma^\mu\bar{\theta}_2-2\theta_1\sigma^\mu\bar{\theta}_2)p_\mu},$$

where in the first line we have used the shorthand $\delta_{12}=\delta^2(\theta_1-\theta_2)\delta^2(\bar{\theta}_1-\bar{\theta}_2)$ and in the second line we have used the expression appearing in (6.22). It is easy to convince oneself that the above expression will contain a non-vanishing, logarithmically divergent term proportional to p^2 (for instance, by bringing down twice the last term in the exponential). This divergence will be compensated by a counterterm which contributes to the wave function renormalization of the superfield Φ. Indeed, it was expected that such wave function renormalization would take place. In components, we would expect indeed from, e.g., the action for the scalar multiplet (3.15) non trivial wave function renormalization Z_ϕ and Z_ψ. What supersymmetry tells us is that there is a single function (of the couplings) that renormalizes all the fields $Z_\phi = Z_\psi \neq 1$.

Let us consider other graphs that yield vanshing contributions. For instance, take a "tadpole" graph with only one external Φ line, and a loop attached to it. The only vertex is chiral, and the internal line in the loop is necessarily of $\langle\Phi\Phi\rangle$ type. However it is a two-point function evaluated at the same point, in particular at $\theta = \theta'$, $\bar\theta = \bar\theta'$. Since $\delta^2(\theta-\theta) = (\theta-\theta)^2 = 0$, the internal propagator is

$$\langle 0|\Phi(p,\theta,\bar\theta)\Phi(-p,\theta,\bar\theta)|0\rangle = 0,$$

it vanishes identically, and so does any tadpole diagram.

Next, we can consider the (one particle irreducible) one-loop correction to the cubic vertex. Since all external lines are chiral, all three vertices are also chiral, and then all internal propagators are $\langle\Phi\Phi\rangle$ too. The correction to the amputated graph is proportional to

$$4\lambda^3 \int d^4k d^2\theta_1 d^2\theta_2 d^2\theta_3 \left(\frac{im}{4}\right)^3 \frac{1}{k^2-m^2}\frac{1}{(p-k)^2-m^2}\frac{1}{(k-q)^2-m^2}\cdot$$
$$\cdot \bar D_1^2\delta_{12}\bar D_1^2\delta_{13}\bar D_2^2\delta_{23}.$$

It is now a simple matter to see that

$$\bar D_1^2\delta_{12}\bar D_1^2\delta_{13}\bar D_2^2\delta_{23} \propto (\theta_1-\theta_2)^2(\theta_1-\theta_3)^2(\theta_2-\theta_3)^2 = (\theta_1-\theta_2)^4(\theta_2-\theta_3)^2 = 0.$$

This means that there are no radiative corrections to the Φ^3 vertex. It implies that there is no renormalization (at least at one-loop) of the coupling constant λ. In other words we have that $Z_\lambda = 1$, in the same way as we also had $Z_m = 1$. Of course the wave function renormalization affects the normalization of the vertices and of the physical mass, but supersymmetry imposes that there is no extra "intrinsic" renormalization of the couplings appearing in the superpotential.

As a last example, we consider quantum loop corrections to the vacuum energy (which of course is vanishing classically). At one loop they all vanish for the same reason as tadpoles (they involve a two-point function evaluated at the same point). At two-loop level we have one potentially non-trivial diagram, with two vertices, one chiral and one anti-chiral, and three $\langle\Phi\bar{\Phi}\rangle$ propagators between them. We now integrate over two different loop momenta. The contribution is proportional to

$$4\lambda\lambda^* \int d^4p d^4k d^2\theta_1 d^2\bar{\theta}_2 \left(\frac{i}{16}\right)^3 \frac{1}{p^2-m^2}\frac{1}{k^2-m^2}\frac{1}{(p+k)^2-m^2}\cdot$$

$$\cdot \bar{D}_p^2 D_p^2 \delta_{12} \bar{D}_k^2 D_k^2 \delta_{12} \bar{D}_{-p-k}^2 D_{-p-k}^2 \delta_{12} =$$

$$= 4\lambda\lambda^* \left(\frac{i}{16}\right)^3 \int d^4p d^4k d^2\theta_1 d^2\bar{\theta}_2 \frac{1}{p^2-m^2}\frac{1}{k^2-m^2}\frac{1}{(p+k)^2-m^2}\cdot$$

$$\cdot e^{-(\theta_1\sigma^\mu\bar{\theta}_1+\theta_2\sigma^\mu\bar{\theta}_2-2\theta_1\sigma^\mu\bar{\theta}_2)(p_\mu+k_\mu-p_\mu-k_\mu)} =$$

$$= 4\lambda\lambda^* \left(\frac{i}{16}\right)^3 \int d^4p d^4k \frac{1}{p^2-m^2}\frac{1}{k^2-m^2}\frac{1}{(p+k)^2-m^2} \int d^2\theta_1 d^2\bar{\theta}_2 \; 1 = 0.$$

Thus we see that the vacuum energy is still zero at two-loops. This is reassuring since a non-zero vacuum energy would mean that supersymmetry itself is broken by perturbative quantum corrections. Let us now try to systematize these results, also to see if supersymmetry is powerful enough to prevent corrections to some quantities at all orders in the perturbative expansion.

6.3 General results on renormalization

In order to derive systematic results on the renormalization of SUSY theories, we need to develop some more formalism, eventually leading to (super)Feynman rules that will help in considering generic diagrams contributing to radiative corrections.

Let us first recall the theoretical framework in which Feynman rules are usually derived. The trick is to define correlators in the interacting theory in terms of correlators of the free theory, as follows (below, we will be schematical and call ϕ a representative field):

$$\begin{aligned}
\langle\phi\phi\ldots\rangle_{\text{int}} &= \int [\mathcal{D}\phi](\phi\phi\ldots)e^{iS_0+iS_{\text{int}}(\phi)} \\
&= \int [\mathcal{D}\phi](\phi\phi\ldots e^{iS_{\text{int}}(\phi)})e^{iS_0} \\
&= \langle\phi\phi\ldots e^{iS_{\text{int}}(\phi)}\rangle_{\text{free}},
\end{aligned}$$

where S_0 is the (quadratic) action of the free theory.

We now recall that in the free theory, where we define

$$e^{iW_0(J)} = \int [\mathcal{D}\phi] e^{iS_0(\phi,J)},$$

the correlation functions are evaluated as

$$\begin{aligned} \langle \phi \ldots \rangle_{\text{free}} &= \int [\mathcal{D}\phi](\phi \ldots) e^{iS_0(\phi)} \\ &= \left(\frac{\delta}{\delta i J} \ldots \right) \int [\mathcal{D}\phi] e^{iS_0(\phi,J)} \\ &= \left(\frac{\delta}{\delta i J} \ldots \right) e^{iW_0(J)}. \end{aligned}$$

Then, we can define the functional W (the generating function for the connected Green functions) for the interacting theory as

$$\begin{aligned} e^{iW_{\text{int}}(J)} &= \int [\mathcal{D}\phi] e^{iS_0(\phi,J)+iS_{\text{int}}(\phi)} = \int [\mathcal{D}\phi] e^{iS_{\text{int}}(\phi)} e^{iS_0(\phi,J)} \\ &= \langle e^{iS_{\text{int}}(\phi)} \rangle_{\text{free}} = e^{iS_{\text{int}}(\frac{\delta}{\delta i J})} e^{iW_0(J)}. \end{aligned}$$

The effective action $\Gamma(\phi_{\text{cl}})$, i.e. the generating function of the amputated one particle irreducible Green functions, is then obtained by taking the Legendre transform of $W_{\text{int}}(J)$, where we substitute $\phi_{\text{cl}} = \frac{\delta W}{\delta J}$.

In practice, we will compute the correction to a free correlator at nth order in the couplings by evaluating

$$\langle \phi \ldots (iS_{\text{int}}(\phi))^n \rangle_{\text{free}} = \left(\frac{\delta}{\delta i J} \ldots \right) \left(iS_{\text{int}}(\frac{\delta}{\delta i J}) \right)^n e^{iW_0(J)}. \tag{6.34}$$

When computing contributions to the effective action $\Gamma(\phi_{\text{cl}})$, we must insert classical fields ϕ_{cl} where the amputated external propagators used to be in the expression above, and then integrate over the position of such insertions (or, in Fourier space, integrate over the external momenta).

Having summarized the general framework for perturbative computations in quantum field theories, we now apply it to our SUSY field theories.

6.3.1 Super-Feyman rules

In order to derive Feynman rules for the SUSY field theory of the Wess-Zumino model, we need to see how does the cubic vertex (6.33) enter in the loop graphs.

In particular, referring to (6.34), we will need to evaluate

$$
\begin{aligned}
iS_{\text{int}}(\frac{\delta}{\delta iJ})iJ_1iJ_2iJ_3 &= 2i\int d^4xd^2\theta_4\,\lambda\left(-\frac{\bar{D}_1^2}{4}\right)\delta_{14}\left(-\frac{\bar{D}_2^2}{4}\right)\delta_{24}\left(-\frac{\bar{D}_3^2}{4}\right)\delta_{34} \\
&= 2i\int d^4xd^2\theta_4d^2\bar{\theta}_4\,\lambda\left(-\frac{\bar{D}_1^2}{4}\right)\delta_{14}\left(-\frac{\bar{D}_2^2}{4}\right)\delta_{24}\delta_{34}.
\end{aligned}
$$

We have labeled the fields by their spacetime point, we remind that δ_{12} is a Grassmann Dirac function on all superspace coordinates, and in the last equality we have traded a $\bar{D}^2$ for an integral $\int d^2\bar{\theta}$.

When computing amputated corrections, we should make sure that we drop a complete propagator for every external line. Such a propagator always has a $\bar{D}^2\delta$ corresponding to it. (Eventually, we must replace each external propagator with a classical superfield.) Then, we learn that the $\bar{D}^2$ that we have eliminated in the expression above has to be the one corresponding to an internal line. As a consequence, the rule must be that for every vertex there is a $\bar{D}^2$ acting on all but one of the internal lines.

For stating the Feynman rules, it is thus more convenient to associate the action of $\bar{D}^2$ (or D^2 for anti-chiral vertices) to the vertices themselves rather than to (internal) propagators. We use then the Grisaru-Rocek-Siegel propagators, which are the ones appearing "naked" in $W_0(J)$, see (6.28):

$$
\langle\Phi_1\Phi_2\rangle_{\text{GRS}} = \frac{im}{p^2-m^2}\frac{\bar{D}_1^2}{4p^2}\delta_{12}, \tag{6.35}
$$

$$
\langle\Phi_1\bar{\Phi}_2\rangle_{\text{GRS}} = \frac{i}{p^2-m^2}\delta_{12}. \tag{6.36}
$$

Then, in order to compute terms in the effective action:

- For every chiral vertex we associate a factor of $2i\lambda$ and a $-\frac{\bar{D}^2}{4}$ acting on all but one internal (GRS) propagators attached to the vertex.

- For an anti-chiral vertex, we associate a $2i\lambda^*$ and distribute $-\frac{D^2}{4}$ on all but one internal propagators attached to it.

- For both kind of vertices, we integrate over *all* of superspace $\int d^2\theta d^2\bar{\theta}$.
- We multiply chiral or antichiral superfields where every external line used to be, and integrate over the external momenta.
- Finally, we integrate over all loop momenta, and take care of the combinatorial factors.

(As usual, it is very important to remind oneself of the path integral from which the Feynman rules were derived in order to get all factors and signs right in any specific correlator...)

Let us stress that the important point in the Feynman rules above is really that we integrate over all of superspace at every vertex. The reason is that this will allow us to integrate by parts any SUSY covariant derivative at any vertex, thus making it possible to shift the action of $\bar{D}^2$ and/or D^2 from one propagator to another.

6.3.2 Non-renormalization theorem

With this Feynman rules, we can now prove a general theorem on renormalization. Given the rules above, any contribution to the effective action (we are considering amputated one particle irreducible graphs) will take the following form.

There will be $\int d^2\theta d^2\bar{\theta}$ integrations at every vertex, and all propagators have a $(\bar{D}^2)^k(D^2)^l\delta_{12}$ or a $(D^2)^k(\bar{D}^2)^l\delta_{12}$ factor, where both k and l are either 0 or 1.

The fact that we have written the vertices with a $\int d^2\theta d^2\bar{\theta}$ integration allows us, as advertised previously, to perform integrations by parts and move the D^2 and $\bar{D}^2$ from one propagator to another (of course, distributing them according to the Leibniz rule).

Suppose now that we choose one specific loop in the graph. By (Grassmann) partial integrations on every successive vertex along the loop, we can remove all the D^2 and $\bar{D}^2$ to only one of the propagators composing the loop, let us say for definiteness the one between the first two vertices. Noting that we can lower the number of SUSY-covariant derivatives by using expressions such as $D^2\bar{D}^2D^2 \propto \Box D^2$, we will eventually arrive at an expression like

$$\int d^4\theta_1 d^4\theta_2 \dots d^4\theta_n \; \delta_{23}\delta_{34}\dots\delta_{n1}(D^2)^k(\bar{D}^2)^l\delta_{12} = \int d^4\theta_1 d^4\theta_2 \; \delta_{12}(D^2)^k(\bar{D}^2)^l\delta_{12}.$$

Such an expression is non vanishing if and only if $k = l = 1$. Indeed, if either k or l is larger than 1 then e.g. $D^4 = 0$, while if either $k = 0$ or $l = 0$ we will have e.g. $(\theta_1 - \theta_2)^4 = 0$.

Thus in the only case where we have a non-vanishing result, the contribution will read

$$\int d^4\theta_1 d^4\theta_2\ \delta_{12} D^2 \bar{D}^2 \delta_{12} = \int d^4\theta_1 d^4\theta_2\ \delta_{12}(\dots) = \int d^4\theta_1\ (\dots).$$

The end result of this procedure has been to effectively contract the loop to one of its vertices. Importantly, attached to this last vertex we still have a full superspace integral $\int d^4\theta_1 \equiv \int d^2\theta_1 d^2\bar{\theta}_1$. The integrand is a function of the momenta, both loop and external (which will generically enter at every vertex, and appear in the denominator of the propagators), of θ_1 and $\bar{\theta}_1$, and of all the other Grassmann variables associated to the other vertices.

We can then repeat the same argument on another remaining loop of the effective graph obtained in the previous step. The procedure will go on exactly as before. Eventually, if the graph does not vanish, we will end up necessarily with a contribution given by an expression like

$$\int d^2\theta d^2\bar{\theta} \int d^4p_1 \dots d^4p_k d^4k_1 \dots d^4k_n\ \tilde{F}_1(p_1, \theta, \bar{\theta}) \dots \tilde{F}_k(p_k, \theta, \bar{\theta}) \cdot$$
$$\cdot \tilde{G}(p_1, \dots, p_k; k_1, \dots, k_n),$$

where the graph had originally n loops with their associated momenta k_i, while the functions $\tilde{F}_i$ are generic (Fourier transforms of) superfields depending on each external momentum p_l. The function $\tilde{G}$ encodes all the remaining momentum dependence coming from the (bosonic) denominators of the propagators.

Going back to x-space, we see that the typical contribution to the effective action will be

$$\int d^4x_1 \dots d^4x_k \int d^2\theta d^2\bar{\theta}\ F_1(x_1, \theta, \bar{\theta}) \dots F_k(x_k, \theta, \bar{\theta}) G(x_1, \dots, x_k).$$

This expression can of course be non-local in space-time, but (oddly enough) it is local in the G-odd variables, and it is necessarily integrated over all of superspace. Hence the effective action contains corrections to the d-terms, but not to the f-terms. This is why there is wave-function renormalization

$$\Gamma \propto \delta_\Phi \int d^4x d^2\theta d^2\bar{\theta}\ \Phi\bar{\Phi},$$

but no renormalization of m and λ, which would imply terms like

$$\delta_m \int d^4x d^2\theta \; \Phi^2 \qquad \text{and} \qquad \delta_\lambda \int d^4x d^2\theta \; \Phi^3.$$

Also, the vacuum energy cannot receive corrections, since that would imply a contribution to Γ with no external superfields, that is with all $F_i = 1$, but then the Grassmann integral gives trivially zero. The vacuum energy then stays zero at any order in perturbation theory.

This latter result is of course very important, it means that supersymmetry cannot be broken by perturbative quantum effects. As we will see in the next chapter, the only option to break supersymmetry spontaneously will be either at tree level, or due to non-perturbative effects.

It is a simple exercise to revisit the previous examples of radiative corrections using the Feynman rules of the present section.

The one-loop correction to the term with two external chiral superfields Φ in the effective action will be computed by writing the same diagram, putting a D^2 on both internal $\langle\Phi\Phi\rangle$ lines, and also a $\bar{D}^2$ on each line from both vertices. Integrating by parts, we end up (being careful to which Grassmann coordinates the supercovariant derivatives act upon) with an expression containing $\int d^2\theta d^2\bar{\theta}\Phi(x,\theta)^2$ which gives just a boundary term in the effective action. An exactly similar argument can be given for all one-loop diagrams with only chiral external legs, thus proving that there are no corrections to the superpotential in the quantum effective action.

Instead, the one-loop correction to the term with Φ and $\bar{\Phi}$ external legs can be evaluated by putting a $\bar{D}^2$ on one line from the chiral vertex, and a D^2 on the same line (or the other, and then integrating by parts) from the anti-chiral vertex. Here we end up with a contribution to the effective action proportional to $\int d^2\theta d^2\bar{\theta}\Phi(x,\theta)\bar{\Phi}(x,\bar{\theta})$, which is just the familiar wave-function renormalization.

Finally, the two loop vacuum energy diagram is evaluated by distributing a $\bar{D}^2$ and a D^2 on two of the three lines. Then one of the two loops can be eliminated, leaving a one-loop vacuum diagram with a propagator with $\bar{D}^2 D^2$ acting on it. This leads to $\int d^2\theta d^2\bar{\theta} 1 = 0$ as before.

More details and other examples can be found in M. T. Grisaru, W. Siegel and M. Rocek, "Improved Methods For Supergraphs," Nucl. Phys. B **159** (1979) 429, and in chapter 6 of the book/review S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace, or one thousand and one lessons in supersymmetry," Front. Phys. **58** (1983) 1 [arXiv:hep-th/0108200].

6.4 Renormalization of the real superfield

All the above discussion and theorems concerned the theory of a single chiral superfield. The generalization to a theory with many different chiral superfields is obvious and straightforward. However extending the formalism to real (vector) superfields is more complicated, and we will not review it here.

Let us only remark the following. Corrections to the quantum effective action must be written as manifestly SUSY *and* gauge invariant terms. Hence any term $\delta\mathcal{L}$ will have to be either a d-term or an f-term. In particular, the renormalization of the gauge coupling is related to the wave-function renormalization of the fields in V. This is not a real surprise as this is already the case in non-supersymmetric gauge theories after all.

To see this in more detail, couple first V to matter fields Φ in some representation. Then the d-term

$$\int d^2\theta d^2\bar{\theta}\ \bar{\Phi}e^{2gV}\Phi$$

gets corrections, which however must be of the same form as the kinetic term itself, so that

$$\mathcal{L}_{\text{eff}} \propto Z_\Phi \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}e^{2gV}\Phi,$$

where $Z_\Phi = 1 + \delta_\Phi$ and δ_Φ is really the coefficient extracted from the loop computation.

It is important to note that we *cannot* have a generic

$$\mathcal{L}_{\text{eff}} \propto Z_\Phi \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}e^{2Z_g g Z_V^{1/2} V}\Phi,$$

because that would mean that we would have generated from loop diagrams a correction term like

$$\delta\mathcal{L} \propto \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}e^{2gV}V\Phi,$$

which is not gauge invariant.

In other words, since we can rescale g into V, we really need to have only one independent renormalization function, which we will choose to be Z_V. This is purely a consequence of gauge symmetry, and not of supersymmetry. Supersymmetry just tells us that the gaugino undergoes the same wave-function renormalization as the gauge field.

A similar argument, which applies also to pure (non-abelian) gauge theories is as follows. The gaugino superfield $\mathcal{W}_\alpha$ will be renormalized by the same function as the real superfield V. This implies that the non-abelian field strength will be renormalized as

$$F_{\mu\nu} \to Z_V^{1/2} F_{\mu\nu}.$$

Now, $F_{\mu\nu}$ contains a quadratic term $g[A_\mu, A_\nu]$ which has to be renormalized in the same way as the term linear in A_μ, otherwise, again, that would mean non-gauge invariant corrections in the effective action. We thus obtain again that

$$Z_g Z_V^{1/2} = 1.$$

Note that $Z_V \neq 1$ means that there are corrections

$$\delta\mathcal{L} \propto \delta_V \int d^2\theta \, \mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha + c.c.$$

This seems to contradict the theorem that the terms generated in the effective action should all be of the form $\int d^2\theta d^2\bar{\theta}$ (to be honest, we proved the theorem only in the presence of chiral superfields but it is indeed true also when real superfields are present).

However recall that

$$\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha \propto \bar{D}^2 \mathrm{tr}\, \mathcal{W}^\alpha e^{-V} D_\alpha e^V,$$

and thus we can rewrite

$$\delta\mathcal{L} \propto \delta_V \int d^2\theta d^2\bar{\theta} \, \mathrm{tr}\, \mathcal{W}^\alpha e^{-V} D_\alpha e^V + c.c.$$

which gives a SUSY and gauge invariant term in the effective action (after integrating over spacetime). The truth is that, as we had already noticed, the gauge kinetic term is only a "fake" f-term since it can be rewritten as a perfectly fine local d-term.

Hence, $Z_g \neq 1$ and the gauge coupling in SUSY gauge theories indeed runs. This is a welcome feature, and it was expected since, for instance, in $SU(N)$ super-Yang-Mills theory it is quite easy to compute the one-loop beta function in component fields (it is just like QCD with one adjoint Weyl fermion), finding

$$\beta(g) \equiv \frac{dg}{d\log\mu} = -\frac{3N}{16\pi^2} g^3.$$

For completeness, we list here the beta function for $SU(N_c)$ SQCD with N_f flavors. Extracting it from known computations involving the component fields is also quite straightforward, though one needs to know the contribution to the beta function of QCD from charged scalars. [This information can be retreived e.g. in chapter 16 of M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," Addison-Wesley (1995).] The one-loop result is

$$\beta(g) = -\frac{1}{16\pi^2}(3N_c - N_f)g^3. \tag{6.37}$$

We see that there is a large range of values of N_f such that the theory is asymptotically free ($\beta < 0$).

6.5 Holomorphy and non-renormalization

As a last comment on renormalization properties of SUSY field theories, let us argue for the non-renormalization of the superpotential $W(\Phi)$ in a completely different way. This approach is mainly due to Seiberg.

Let us consider again a Wess-Zumino model of one chiral superfield Φ with

$$W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3. \tag{6.38}$$

Recall that we have symmetries acting on the fields. Here we have essentially two $U(1)$s: one rotating each physical component of Φ. At the level of the superfield, we have an R-symmetry and an axial-like symmetry:

$$\begin{array}{lllll} U(1)_R & : & \phi \to \phi, & \psi \to e^{-i\alpha}\psi & (\theta \to e^{i\alpha}\theta) \\ U(1)_\Phi & : & \phi \to e^{i\alpha}\phi, & \psi \to e^{i\alpha}\psi. & \end{array}$$

The superpotential breaks explicitly both of these symmetries.

Let us now remark that also the coupling constants m and λ enter holomorphically in the superpotential W. We could thus promote them to background (moduli) chiral superfields, so that the physical couplings are actually the vacuum expectation values of these fields whose dynamics is frozen by some other means (e.g., their kinetic term is suppressed). Now, it becomes possible to assign charges under the global symmetries to these superfields so that the superpotential is invariant, i.e. it should have R-charge 2 and Φ charge 0. If we keep calling the background superfields by the constants they are related to, the transformation laws that we need to impose are the following:

$$\begin{aligned} U(1)_R &: \quad m \to e^{2i\alpha} m, \qquad \lambda \to e^{2i\alpha}\lambda \\ U(1)_\Phi &: \quad m \to e^{-2i\alpha} m, \qquad \lambda \to e^{-3i\alpha}\lambda. \end{aligned}$$

Now since $W(\Phi, m, \lambda)$ preserves both $U(1)$s (which are not anomalous since there are no gauge symmetries around), then the effective superpotential W_{eff} (that is, the part of the effective action that cannot be written as a d-term) must also preserve them, and because supersymmetry is also preserved W_{eff} must be a holomorphic function of Φ, m and λ.

These requirements imply that

$$W_{\text{eff}} = \frac{1}{2} m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right), \tag{6.39}$$

where $\frac{\lambda\Phi}{m}$ is the only chargeless holomorphic combination of the superfields, and $f(z)$ is a holomorphic function.

Once we assume that the background superfields have frozen to their VEVs, we can now analyze various limits in order to constrain the form of $f(z)$. When $\lambda = 0$ the theory is free, and we should find no corrections to the classical action. Hence

$$W_{\text{eff}}(\lambda = 0) = W_{\text{tree}}(\lambda = 0) = \frac{1}{2} m\Phi^2.$$

This implies that $f(0) = 1$.

If we Taylor expand f so that

$$f(z) = 1 + a_1 z + a_2 z^2 + \ldots,$$

with a_i some numerical coefficients, we get for the effective superpotential

$$W_{\text{eff}} = \frac{1}{2} m\Phi^2 + \frac{1}{2} a_1 \lambda \Phi^3 + \frac{1}{2} a_2 \frac{\lambda^2}{m} \Phi^4 + \mathcal{O}\left(\frac{\lambda^3}{m^2}\right). \tag{6.40}$$

We can then scale both λ and m to zero in such a way that $m \propto \lambda^2 \to 0$. The first two terms scale to zero, while the third is constant and the higher order ones diverge. This is clearly impossible for a theory with $W_{\text{tree}} \to 0$. Hence we understand that $a_2 = a_3 = \cdots = 0$ and $f = 1 + a_1 t$.

Another way to see this is the following. The coefficient a_n multiplies a term like $\frac{\lambda^n}{m^{n-1}}\Phi^{n+2}$. Such a term must arise from a graph with n chiral vertices and $n+2$ external lines. It is easy to see that only tree-level graphs can accomodate that, and they are not one-particle irreducible. Hence they cannot contribute to the effective action and such corrections cannot arise, i.e. all their coefficient must vanish.

Finally, the $\mathcal{O}(\lambda)$ term in W_{eff} cannot be anything else than the vertex in W_{tree}. Thus, we have proven that

$$W_{\text{eff}} = W_{\text{tree}}.$$

This is entirely equivalent to the perturbative non-renormalization theorem of the superpotential W_{tree}.

Actually, in the present context we can even extend this result to non-perturbative effects. If their contribution is to be holomorphic in λ, then such terms will be proportional to, say, $e^{-1/\lambda}$. However if λ approaches 0 from the negative real axis, the non-perturbative contribution would explode, which makes no sense for a vanishingly small coupling. Hence holomorphy rules out also non-perturbative corrections to the effective superpotential, at least in theories of chiral superfields only.

Let us mention as a last remark that when (asymptotically free) gauge theories are also in the game, there is a class of non-perturbative corrections that are possible, and indeed can be shown to arise in many cases. The reason is that because of dimensional transmutation, one can trade the running gauge coupling g for a *holomorphic* scale which is defined in terms the complexified coupling τ given in (4.55) by

$$\Lambda = \mu e^{-\frac{8\pi^2}{b_0}\tau(\mu)} = \mu e^{-\frac{8\pi^2}{b_0 g^2(\mu)}+i\frac{\Theta}{b_0}}. \tag{6.41}$$

It is this Λ that can be considered as a background chiral superfield as in the argument above, and thus holomorphy does not prevent it from entering in the effective superpotential W_{eff}. Furthermore, Λ goes to zero when the gauge coupling is sent to zero. The perturbative non-renormalization theorem is not contradicted because corrections proportional to some (necessarily positive) power of Λ are of course non-perturbative in the gauge coupling g.

Chapter 7

Supersymmetry breaking

Supersymmetry breaking is an issue that must be addressed since in the real world supersymmetry *is* broken. Indeed, there is no degeneracy between boson and fermion masses. More precisely, since we established that the minimally supersymmetric version of the Standard Model is the MSSM, it is then an experimental fact that the masses of all the superpartners are not lighter than the Electro-weak scale, otherwise they would have been already observed in the previous generation of accelerators.

Yet, for supersymmetry to be really helpful in curing the hierarchy problem, it must be restored at some intermediate scale between $M_{EW} \sim 100$ GeV and $M_{GUT} \sim 10^{16}$ GeV. Usually, it is expected that SUSY is restored close to M_{EW}, so as to maintain (and possibly explain) the hierarchy between M_{EW} and M_{GUT}.

How do we cook up then a theory which is supersymmetric at high energies but is not supersymmetric at lower energies? Supersymmetry will be broken by some dimensionful quantity M_{SUSY}, so that for $E > M_{SUSY}$ the theory behaves supersymmetrically, while for $E < M_{SUSY}$ it will not (e.g., there will be mass splittings among superpartners).

There are two ways in which we can introduce such a dimensionful breaking:

i) *Spontaneous SUSY breaking:* The theory as a whole preserves SUSY, but its vacuum does not. Namely, there will be a field acquiring a VEV $\langle\varphi\rangle \sim M_{SUSY}$ such that SUSY is broken.

ii) *Explicit (soft) SUSY breaking:* The Lagrangian of the theory contains terms which do not preserve SUSY. However it is assumed that are present only those terms (dubbed *soft*) that preserve the high energy behavior of SUSY (i.e. the UV cancellations). These soft terms always correspond to dimensionful parameters.

We will actually see that it is possible to treat case *ii*) as a specific limit of case *i*).

In the rest of this chapter we concentrate on spontaneous SUSY breaking, which is also the most natural option, since in some sense it requires the theory itself to predict that SUSY is broken.

7.1 Spontaneous SUSY breaking

We have seen at the very beginning, in Chapter 2, that the first consequence of the SUSY algebra is that a SUSY ground state must have vanishing energy. Indeed, to repeat the argument, the superalgebra

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

implies that, for any state and thus also for a ground state $|\Omega\rangle$

$$\begin{aligned} \delta^{\alpha\dot\alpha}\langle\Omega|\{Q_\alpha, \bar{Q}_{\dot\alpha}\}|\Omega\rangle &= \sum_\alpha \|Q_\alpha|\Omega\rangle\|^2 + \sum_{\dot\alpha} \|\bar{Q}_{\dot\alpha}|\Omega\rangle\|^2 \geq 0 \\ &= 4E\||\Omega\rangle\|^2, \end{aligned}$$

so that $E = 0$ if and only if $Q_\alpha|\Omega\rangle = 0 = \bar{Q}_{\dot\alpha}|\Omega\rangle$. What is most important here, is that it is also true that as soon as $E > 0$, then we know that there must exist an α or an $\dot\alpha$ such that $Q_\alpha|\Omega\rangle \neq 0$ or $\bar{Q}_{\dot\alpha}|\Omega\rangle \neq 0$. In other words, the vacuum $|\Omega\rangle$ is *not* SUSY invariant. Hence, SUSY is broken spontaneously.

This means that the vacuum energy E is an order parameter for broken SUSY: whenever $E > 0$ SUSY is broken, while if $E = 0$ SUSY is preserved. This observation is in line with the fact that spontaneous breaking of global symmetries is always associated to an order parameter. Below, we turn to the Lagrangian version of this consequence of broken SUSY.

7.1.1 Vacuum energy and f- and d-terms

We can restate the above observation in more detail and at the classical level considering the most general SUSY theory with gauge fields and matter. The

Lagrangian is

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar{\theta} \sum_r \bar{\Phi}_r e^{2gV_a T_a^{(r)}} \Phi_r + \int d^2\theta\, W(\Phi_r) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}_r) \\ &\quad -\frac{\tau}{8} \int d^2\theta \,\mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha - \frac{\tau^*}{8} \int d^2\bar{\theta} \,\mathrm{tr}\, \bar{\mathcal{W}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}.\end{aligned}$$

The only contribution to the vacuum energy, at the classical level at least, comes from the scalar potential:

$$\begin{aligned}\mathcal{V}(\phi_r, \phi_r^*) &= \sum_r \left(\frac{\partial W}{\partial \phi_r}\right)_i \left(\frac{\partial \bar{W}}{\partial \bar{\phi}_r}\right)^i + \frac{1}{2} g^2 \sum_a \left(\sum_r \phi^*_{ri} T_a^{(r)i}{}_j \phi_r^j\right)^2 \\ &\equiv \sum_r f^*_{ri} f^i_r + \frac{1}{2} \sum_a d_a d_a,\end{aligned}$$

where we write derivatives with respect to the scalar fields, to denote the fact that the expression is a function of them. We have $\mathcal{V} = 0$, and thus vanishing classical energy, if and only if $f_r = 0$ and $d_a = 0$ for all values of r and a.

Indeed, consider a vacuum state of such a theory. In order for the vacuum to preserve Poincaré symmetry, the only fields which can acquire a VEV must be Lorentz scalars. Moreover the VEV must be constant throughout spacetime. Consider now the SUSY variations of the fermions in such a vacuum. (The ones for the bosons are automatically vanishing since the fermions are set to zero.) For the matter fields we have

$$\delta \psi_r^i = \sqrt{2} \epsilon f_r^i,$$

while in the gauge sector we have

$$\delta \lambda_a = i \epsilon d_a.$$

Hence a vacuum configuration is SUSY invariant, i.e. the variations of all the fields are zero, if and only if $f_r = 0$ and $d_a = 0$, which as we have just seen implies that the potential vanishes, $\mathcal{V} = 0$.

We have thus demonstrated that $\mathcal{V}$ is an order parameter for SUSY breaking, playing at the classical level the role that E was playing at the quantum level.

7.1.2 The SUSY breaking vacuum and the goldstino

For a SUSY preserving vacuum, we can content ourselves with solving the "first order" conditions $f = 0$ and $d = 0$. Since the potential is positive semi-definite, $\mathcal{V} \geq 0$, if we find a state with $\mathcal{V} = 0$ we are ensured that it is a global minimum of $\mathcal{V}$, in particular it is a stationary point.

For a vacuum which is not SUSY, there are no short-cuts and we must find a stationary point of the potential $\mathcal{V}$ by computing its first derivative and setting it to zero:

$$d\mathcal{V} = 0 \qquad \Leftrightarrow \qquad \frac{\partial \mathcal{V}}{\partial \phi^i_r} = 0, \quad \forall r, i. \tag{7.1}$$

For simplicity, let us suppress from now on the index r of the representation, we can just group together all the irreducible representations into a single reducible one.

Then, given the potential

$$\mathcal{V} = \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \bar{\phi}_i} + \frac{1}{2} g^2 \sum_a \left(\phi^*_i T_a{}^i{}_j \phi^j \right)^2,$$

we obtain for its first derivative

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial \phi^i} &= \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \frac{\partial \bar{W}}{\partial \bar{\phi}_j} + g^2 \sum_a \left(\phi^*_j T_a{}^j{}_k \phi^k \right) \phi^*_l T_a{}^l{}_i \\ &= -\frac{\partial^2 W}{\partial \phi^i \partial \phi^j} f^j - g \sum_a \phi^*_l T_a{}^l{}_i d_a, \end{aligned}$$

where we have used (5.12) and (5.13). We thus have the condition for an extremum of the potential:

$$d\mathcal{V} = 0 \qquad \Leftrightarrow \qquad \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} f^j + g \phi^*_l T_a{}^l{}_i d_a = 0, \tag{7.2}$$

where now it is intended that there is a sum also over the repeated a indices.

Of course, one should now check that the extremum that has been found is really a minimum, at least a local one. For instance, one should compute the mass matrix for the scalars around that minimum, and check that all the eigenvalues m^2 are positive. We will show an example of such a procedure for an example in the following.

By considering instead the mass matrix for the fermions, there is a nice consequence of the existence of such an extremum that can be extracted from the above expression for $d\mathcal{V} = 0$.

Consider the mass matrix of the fermions of the theory, including the Yukawa-like interactions where we suppose that the scalars are frozen at their extremum value:

$$\begin{aligned} \mathcal{L}_{\text{mass,fermions}} &= ig\sqrt{2}\phi_i^* {T_a}^i{}_j \lambda_a \psi^j - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\psi^i\psi^j + c.c. \qquad (7.3) \\ &= -\frac{1}{2}\begin{pmatrix} \psi^i & \lambda_a \end{pmatrix}\begin{pmatrix} \frac{\partial^2 W}{\partial\phi^i\partial\phi^j} & -\frac{ig}{\sqrt{2}}\phi_l^* {T_b}^l{}_i \\ -\frac{ig}{\sqrt{2}}\phi_k^* {T_a}^k{}_j & 0 \end{pmatrix}\begin{pmatrix} \psi^j \\ \lambda_b \end{pmatrix} + c.c. \end{aligned}$$

Before jumping to any conclusion, we need the following last bit of information. The superpotential is gauge invariant, which means that under a gauge transformation

$$\delta_\alpha \phi^i = \alpha_a {T_a}^i{}_j \phi^j$$

it should stay invariant, $\delta_\alpha W = 0$. This translates into

$$0 = \frac{\partial W}{\partial \phi^i}\delta_\alpha\phi^i = -f_i^* \alpha_a {T_a}^i{}_j \phi^j,$$

so that invariance of W under any gauge transformation imposes a relation between the f-terms:

$$\phi_i^* {T_a}^i{}_j f^j = 0. \qquad (7.4)$$

This relation is always true, in particular for values of the scalar fields that correspond to an extremum of the potential. Hence, summing together the conditions (7.2) and (7.4), we see that at an extremum $d\mathcal{V} = 0$ the fermionic mass matrix

$$\begin{pmatrix} \frac{\partial^2 W}{\partial\phi^i\partial\phi^j} & -\frac{ig}{\sqrt{2}}\phi_l^* {T_a}^l{}_j \\ -\frac{ig}{\sqrt{2}}\phi_k^* {T_b}^k{}_i & 0 \end{pmatrix}$$

has at least one zero eigenvalue, with eigenvector given by

$$\begin{pmatrix} f^i \\ i\sqrt{2}d_a \end{pmatrix}. \qquad (7.5)$$

This eigenvector is non-trivial only in a SUSY breaking vacuum (i.e. when f^i and d_a are not all zero). Hence we have proven the *Goldstone theorem* for broken supersymmetry: a vacuum spontaneously breaking SUSY necessarily possesses a massless fermionic particle in the spectrum. This particle is usually called the Goldstone fermion of broken SUSY, or more familiarly the *goldstino*.

We thus see that supersymmetry in this respect is much similar to any other continuous global symmetry. Indeed, the Goldstone theorem dictates that for any spontaneously broken (bosonic) global symmetry there must be a massless boson in the spectrum, the Goldstone boson.

We should point out that there is also a more general proof of the Goldstone theorem for broken SUSY, which is not based on an explicit classical Lagrangian, but on the properties of the supercurrent. This more general proof establishes the presence of a goldstino even if SUSY is broken in a strongly coupled phase of a given theory, where classical arguments may not apply.

7.1.3 The supertrace theorem

Let us now restrict to a theory of chiral superfields only, in order to avoid too involved formulas. We would like to consider the mass spectrum in a generic vacuum, where supersymmetry is possibly broken.

We have already seen in the previous subsection that the fermionic mass matrix is given by

$$\mathcal{L}^f_{\text{mass}} = -\tfrac{1}{2}\psi^i \mathcal{M}^f_{ij}\psi^j + c.c.,$$

where

$$\mathcal{M}^f_{ij} = \frac{\partial^2 W}{\partial\phi^i\partial\phi^j} \equiv \partial_i\partial_j W. \tag{7.6}$$

The physical squared masses of the fermions (i.e. the real poles in the tree-level fermionic propagators) will be given by the eigenvalues of

$$\mathcal{M}^f_{ij}\mathcal{M}^{f*jk} \equiv (\mathcal{M}^2_f)_i^k = \partial_i\partial_j W \partial_{\bar{j}}\partial_{\bar{k}}\bar{W},$$

where we have defined $\partial_{\bar{\imath}} \equiv \frac{\partial}{\partial\bar{\phi}_i}$ and the sum over the index j is intended. The full squared mass matrix over all of the fermionic degrees of freedom (of both chiralities) is thus

$$\mathcal{M}^2_f = \begin{pmatrix} \partial_i\partial_j W\partial_{\bar{j}}\partial_{\bar{k}}\bar{W} & 0 \\ 0 & \partial_{\bar{\imath}}\partial_{\bar{j}}\bar{W}\partial_j\partial_k W \end{pmatrix}. \tag{7.7}$$

Consider now the scalar mass matrix. It is obtained by expanding the scalar potential to quadratic order around the vacuum values of the scalars. We get:

$$\begin{aligned}\mathcal{L}^b_{\text{mass}} &= -\partial_i W \partial_{\bar{\imath}} \bar{W} \\ &= -(\partial_i W + \partial_i\partial_j W \phi^j + \tfrac{1}{2}\partial_i\partial_j\partial_k W \phi^j \phi^k) \cdot \\ &\qquad \cdot(\partial_{\bar{\imath}} \bar{W} + \partial_{\bar{\imath}}\partial_{\bar{l}} \bar{W} \phi^*_l + \tfrac{1}{2}\partial_{\bar{\imath}}\partial_{\bar{l}}\partial_{\bar{m}} \bar{W} \phi^*_l \phi^*_m).\end{aligned}$$

In the above expression, we define the fields ϕ^i as their fluctuation around the minimal value, while the derivatives of the superpotential are evaluated at the vacuum values. Note that for a renormalizable superpotential the expansion indeed stops at quadratic order, while in a more generic situation it can go on, but since we are interested only in quadratic terms we do not need the higher order terms anyway.

Supposing that we expand around an extremum, and neglecting the constant vacuum energy term, the quadratic mass terms are hence given by

$$\mathcal{L}^b_{\text{mass}} = -\partial_i\partial_j W \partial_{\bar{\imath}}\partial_{\bar{k}} \bar{W} \phi^j \phi^*_k - \tfrac{1}{2}\partial_i\partial_j\partial_k W \partial_{\bar{\imath}} \bar{W} \phi^j \phi^k - \tfrac{1}{2}\partial_{\bar{\imath}}\partial_{\bar{\jmath}}\partial_{\bar{k}} \bar{W} \partial_i W \phi^*_j \phi^*_k. \tag{7.8}$$

Eventually, we have

$$\mathcal{L}^b_{\text{mass}} = -\tfrac{1}{2}\begin{pmatrix} \phi^*_i & \phi^i \end{pmatrix} (\mathcal{M}^2_b)_{ij} \begin{pmatrix} \phi^j \\ \phi^*_j \end{pmatrix},$$

with

$$(\mathcal{M}^2_b)_{ij} = \begin{pmatrix} \partial_{\bar{\imath}}\partial_{\bar{k}} \bar{W} \partial_j\partial_k W & -\partial_{\bar{\imath}}\partial_{\bar{\jmath}}\partial_{\bar{k}} \bar{W} f^*_k \\ -\partial_i\partial_j\partial_k W f^k & \partial_i\partial_k W \partial_{\bar{\jmath}}\partial_{\bar{k}} \bar{W} \end{pmatrix}. \tag{7.9}$$

There are two immediate consequence that one can gather by comparing (7.7) and (7.9). First of all, there will be mass splittings in the eigenvalues of $\mathcal{M}^2_f$ and $\mathcal{M}^2_b$ if and only if $f \neq 0$. Indeed, otherwise the two matrices are identical, in agreement with the fact that when SUSY is unbroken the spectrum of fermions and bosons is degenerate.

However also in a broken SUSY vacuum there is a relation between the two mass spectra. We can compute the sum of the eigenvalues by taking the trace of the two mass matrices. For the fermionic one, which is block diagonal, we get

$$\operatorname{tr} \mathcal{M}^2_f = 2\sum_{i\in f} m_i^2 = 2\partial_i\partial_j W \partial_{\bar{\imath}}\partial_{\bar{\jmath}} \bar{W},$$

(the factor of 2 in front of the sum corresponds to the helicities of each fermion), and for the bosonic one we obtain

$$\mathrm{tr}\,\mathcal{M}_b^2 = \sum_{i\in b} m_i^2 = 2\partial_i\partial_j W \partial_{\bar{i}}\partial_{\bar{j}}\bar{W},$$

which is the same since the two matrices only differ by off-diagonal elements.

The important consequence is that the *supertrace*, that is the trace weighted according to the statistics, vanishes:

$$\mathrm{Str}\mathcal{M}^2 = \sum_{i\in b} m_i^2 - 2\sum_{i\in f} m_i^2 = 0. \tag{7.10}$$

This important result generalizes straightforwardly to theories with gauge fields.

It means that when SUSY is broken in such a classical fashion, the masses of the previously degenerate superpartners are split around an average value, which is related to the unbroken (SUSY) mass.

This fact, which is a consequence of the theory being supersymmetric, is unsatisfactory because it leads to strong constraints on the spectrum of superpartners, which are not met phenomenologically. For instance, for every quark there should be at least a real component of the squark superpartner which would be lighter: this is clearly not observed.

Does this mean that spontaneous SUSY breaking is not viable for the MSSM? In reality, the “supertrace theorem” just means that SUSY is not spontaneously broken at tree level (i.e. classically) within the MSSM itself. Indeed, a way out for this problem is to assume that SUSY breaking is not a tree level phenomenon, or that it happens in a separate, hidden sector. In both cases, or even better in a combination of these two possibilities, SUSY breaking is transmitted to the MSSM by quantum corrections. At the quantum level, there are effective corrections to the kinetic terms (e.g. wave function renormalization) such that the supertrace theorem can be violated, and sizable masses for all the superpartners can be generated. We will come back briefly on this issue when discussing the soft terms.

7.2 Examples of SUSY breaking theories

The supertrace theorem notwithstanding, and for the reasons just mentioned, it is still relevant to review models of tree level spontaneous SUSY breaking.

We will discuss below the two archetypal models, one of d-term breaking and the other of f-term breaking.

7.2.1 d-term breaking: Fayet-Iliopoulos model

The Fayet-Iliopoulos model is a very simple model, where one essentially plays the f-terms and the d-terms against each other.

In its simplest incarnation, it is the SQED model that we had already considered previously, with a mass term for the quark superfields and the addition of the FI term (4.58). All in all the model is described by

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar{\theta}\left(\bar{\Phi}e^{2gV}\Phi + \tilde{\Phi}e^{-2gV}\bar{\tilde{\Phi}}\right) + \int d^2\theta\, m\Phi\tilde{\Phi} + \int d^2\bar{\theta}\, m^*\bar{\Phi}\bar{\tilde{\Phi}} \\ &\quad -\frac{1}{4}\int d^2\theta\, \mathcal{W}^\alpha\mathcal{W}_\alpha - \frac{1}{4}\int d^2\bar{\theta}\, \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} + \int d^2\theta d^2\bar{\theta}\, 2\xi gV. \qquad (7.11)\end{aligned}$$

Recall that the last term is gauge invariant because V is an abelian real superfield, and that ξ is a real parameter.

The potential is given by

$$-\mathcal{V} = gd(\phi^*\phi - \tilde{\phi}\tilde{\phi}^*) + ff^* + \tilde{f}\tilde{f}^* + (m\phi\tilde{f} + m\tilde{\phi}f + c.c.) + \tfrac{1}{2}d^2 + g\xi d.$$

Solving for the auxiliary fields d, f and $\tilde{f}$, we have

$$\begin{aligned} f^* &= -m\tilde{\phi} \\ \tilde{f}^* &= -m\phi \\ d &= -g(\phi^*\phi - \tilde{\phi}\tilde{\phi}^* + \xi), \end{aligned} \qquad (7.12)$$

so that the potential for the scalars becomes

$$\mathcal{V} = \tfrac{1}{2}g^2(|\phi|^2 - |\tilde{\phi}|^2 + \xi)^2 + |m|^2(|\phi|^2 + |\tilde{\phi}|^2). \qquad (7.13)$$

It is clear that we will not find SUSY vacua. Indeed, $f = 0 = \tilde{f}$ implies that the scalars themselves must vanish, $\phi = 0 = \tilde{\phi}$, while, when $\xi \neq 0$, the condition $d = 0$ would imply that either ϕ or $\tilde{\phi}$ must be non zero. It is then impossible to set to zero all the right hand sides of (7.12), and SUSY is broken.

To find the non SUSY vacuum, we extremize the potential $\mathcal{V}$:

$$\begin{aligned} \partial_\phi\mathcal{V} &= g^2(|\phi|^2 - |\tilde{\phi}|^2 + \xi)\phi^* + |m|^2\phi^*, \\ \partial_{\tilde{\phi}}\mathcal{V} &= g^2(|\tilde{\phi}|^2 - |\phi|^2 - \xi)\tilde{\phi}^* + |m|^2\tilde{\phi}^*. \end{aligned}$$

We see that we have a vacuum at $\phi = 0 = \tilde{\phi}$. Assuming without loss of generality that $\xi > 0$, we also have a vacuum for $\phi = 0$ and $|\tilde{\phi}|^2 = \xi - \frac{|m|^2}{g^2}$, which exists only when $|m|^2 < g^2\xi$. When this inequality is satisfied, it is a simple matter to show that the vacuum with a non trivial scalar VEV is the stable one, while the other at the origin becomes unstable.

When the stable vacuum is at the origin, we have $\phi = 0 = \tilde{\phi}$ which implies that

$$f = 0 = \tilde{f} \qquad \text{and} \qquad d = -g\xi \neq 0,$$

the vacuum energy being

$$\mathcal{V} = \tfrac{1}{2}g^2\xi^2.$$

This is pure d-term SUSY breaking, and it is rather easy to see that the goldstino in this case is just the photino λ, the only massless fermion around in this vacuum.

When the stable vacuum is the one with $\phi = 0$ and $|\tilde{\phi}|^2 = \xi - \frac{|m|^2}{g^2}$, we see that both $f \neq 0$ and $d \neq 0$. This situation becomes one of pure f-term SUSY breaking only in the limit $\xi \gg \frac{|m|^2}{g^2}$. Then the role of the goldstino is essentially played by ψ. (Indeed, $\tilde{\psi}$ and λ become massive due to the super-Brout-Englert-Higgs mechanism: $\tilde{\phi} \simeq \sqrt{\xi}$ breaks the $U(1)$ gauge symmetry and through the Yukawa couplings gives a fermionic Dirac mass term $\mathcal{L}_m^f = -ig\sqrt{2\xi}\tilde{\psi}\lambda$, while ψ remains massless.)

7.2.2 f-term breaking: O'Raifeartaigh model

The model first proposed by O'Raifeartaigh is composed of chiral superfields only. The idea here is to have a superpotential W such that one cannot set to zero all the f-terms. Of course this means that W is not generic.

A superpotential that works is the following. There must be three different chiral superfields:

$$W = hX(\Phi_1^2 - \mu^2) + m\Phi_1\Phi_2. \tag{7.14}$$

If we call X (by a slight abuse of notation), ϕ_1 and ϕ_2 the lowest components of the respective superfields, then the f-term equations give the following conditions

$$-f_X^* = h(\phi_1^2 - \mu^2), \tag{7.15}$$

$$-f_1^* = 2hX\phi_1 + m\phi_2, \tag{7.16}$$

$$-f_2^* = m\phi_1. \tag{7.17}$$

It is evident that one cannot set $f_X = 0$ and $f_2 = 0$ simultaneously, as soon as $\mu \neq 0$. Let us look then at the scalar potential

$$\mathcal{V} = |h|^2|\phi_1^2 - \mu^2|^2 + |2hX\phi_1 + m\phi_2|^2 + |m|^2|\phi_1|^2. \tag{7.18}$$

It will have an extremum $d\mathcal{V} = 0$ if

$$\begin{aligned} 2h^*\phi_1^*(2hX\phi_1 + m\phi_2) &= 0, \\ 2|h|^2(\phi_1^2 - \mu^2)\phi_1^* + 2h^*X^*(2hX\phi_1 + m\phi_2) + |m|^2\phi_1 &= 0, \\ m^*(2hX\phi_1 + m\phi_2) &= 0. \end{aligned}$$

From the third equation we see that we can solve for ϕ_2:

$$\phi_2 = -\frac{2h}{m}X\phi_1. \tag{7.19}$$

Then the first equation is also satisfied, and we are left with the second one, simplified as

$$2|h|^2|\phi_1|^2\phi_1 - 2|h|^2\mu^2\phi_1^* + |m|^2\phi_1 = 0. \tag{7.20}$$

Note that by rotations of X, Φ_1 and Φ_2 we can set for simplicity all constants to be real and positive.

It is evident that (7.20) has always the solution $\phi_1 = 0$. At this vacuum, we have also $\phi_2 = 0$ while X is undetermined.

Otherwise, if $\phi_1 \neq 0$, since we have chosen all constants to be real and positive, we see that also ϕ_1 has to be real, and has to satisfy

$$2h^2\phi_1^2 = 2h^2\mu^2 - m^2.$$

Thus this alternative vacuum exists only when $2h^2\mu^2 \geq m^2$. When this inequality is satisfied, the latter vacuum is the stable one. Otherwise, the only vacuum is the one at the origin $\phi_1 = \phi_2 = 0$. From now on we set ourselves in this vacuum, and suppose that it is stable.

The f-terms are

$$f_1 = f_2 = 0, \qquad\qquad f_X^* = h\mu^2,$$

so that the vacuum energy is given by

$$\mathcal{V}_{\text{vac}} = |h\mu^2|^2.$$

What is puzzling about this vacuum is that there is actually a complex plane of vacua, parameterized by $\langle X\rangle$. This is a usual fact for SUSY vacua, but it is rather unusual for SUSY breaking vacua. In non-SUSY theories, such degeneracies are expected to be lifted by quantum corrections.

Actually, the whole X supermultiplet is massless in the non-SUSY ground states. The fermionic component, ψ_X, is the goldstino, and then it has a good reason to be massless. The same can be said about the phase of the complex scalar X, since as we will argue below it corresponds to the Goldstone boson for broken R-symmetry. However the modulus $|X|$ has no specific reason at all to stay massless. Presumably, it should acquire a mass because of quantum corrections in the non-SUSY vacuum.

Before considering this issue, and actually in order to do so, we first compute the spectrum of masses in a vacuum parameterized by the VEV of X.

We start by considering the fermionic mass matrix, computed thus in the vacuum with $\phi_1 = \phi_2 = 0$ and X arbitrary:

$$\mathcal{L}^f_{\rm mass} = -\frac{1}{2}\begin{pmatrix} \psi_X & \psi_1 & \psi_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2hX & m \\ 0 & m & 0 \end{pmatrix} \begin{pmatrix} \psi_X \\ \psi_1 \\ \psi_2 \end{pmatrix} + c.c., \tag{7.21}$$

so that

$$\mathcal{M}^2_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4|hX|^2+|m|^2 & 2hXm^* \\ 0 & 2h^*X^*m & |m|^2 \end{pmatrix}. \tag{7.22}$$

There is a vanishing eigenvalue (for ψ_X), while the other two are given by

$$\begin{aligned} m^2_{f,\pm} &= |m|^2 + 2|hX|^2 \pm \sqrt{4|hXm|^2+4|hX|^4} \\ &= \left(|hX| \pm \sqrt{|m|^2+|hX|^2}\right)^2. \end{aligned} \tag{7.23}$$

As for the bosons, we use the general expression (7.9), and obtain

$$\mathcal{L}^b_{\rm mass} = -\tfrac{1}{2} v^\dagger \mathcal{M}^2_b v, \tag{7.24}$$

with

$$v^\dagger = \begin{pmatrix} X & \phi_1 & \phi_2 & X^* & \phi_1^* & \phi_2^* \end{pmatrix}, \tag{7.25}$$

and

$$\mathcal{M}_b^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4|hX|^2 + |m|^2 & 2hXm^* & 0 & -2|h|^2\mu^{*2} & 0 \\ 0 & 2h^*X^*m & |m|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2|h|^2\mu^2 & 0 & 0 & 4|hX|^2 + |m|^2 & 2h^*X^*m \\ 0 & 0 & 0 & 0 & 2hXm^* & |m|^2 \end{pmatrix}. \tag{7.26}$$

As is clear from the above, the complex scalar X is classically massless. The 4 other eigenvalues are given by

$$m^2_{b,\pm\pm'} = |m|^2 + 2|hX|^2 \pm' |h\mu|^2 \pm \sqrt{4|hXm|^2 + 4|hX|^4 \pm' 4|h^2\mu X|^2 + |h\mu|^4}. \tag{7.27}$$

In the expression above we have distinguished two independent choices of sign by $\pm$ and $\pm'$, which make up a total of 4 different eigenvalues.

We see that for $\mu = 0$ (i.e. when SUSY is *not* broken) then the second choice of sign becomes trivial, and we have $m^2_{b,\pm} = m^2_{f,\pm}$. Otherwise, as soon as $\mu \neq 0$, there will be mass splittings. Of course, we can also check that $\sum m_b^2 = \sum m_f^2$. Lastly, one can see that for $|m|^2 < 2|h\mu|^2$, we have one negative eigenvalue, $m^2_{b,--} < 0$, and the vacuum at the origin is no longer stable.

We will now see that the spectrum is crucial in computing the effective potential that will eventually lift the flat direction along X.

7.3 Quantum corrections to the vacuum energy

We now wish to evaluate the quantum corrections to the vacuum energy. Since SUSY is broken, these are no longer required to vanish by the theorem discussed in the previous chapter.

This exercise is particularly interesting in models such as the one that we have just discussed above, where the vacua are parameterized by a modulus X. In such models, we expect to find a result which depends on X, and thus manifests itself as a potential for the light scalar X, which then acquires a mass so that the moduli space is lifted. Such moduli that are lifted after taking into account quantum corrections are called *pseudomoduli*. Actually, in a broken SUSY vacuum all massless scalar which are not Goldstone bosons are pseudomoduli and potentially acquire a mass by quantum corrections.

We thus compute the one-loop vacuum energy, by just assuming that in the loop are circulating all bosons and fermions with their specified masses. Recall that the vacuum energy density for a free (scalar) field of mass m is given, in Euclidean space, by

$$\begin{aligned} Z(0) &\equiv e^{-\int d^4x\mathcal{E}} \equiv \int [\mathcal{D}\phi] e^{-S_E(\phi)} = \int [\mathcal{D}\phi] e^{-\int d^4x \phi(-\Box + m^2)\phi} \\ &= \left[\det(-\Box + m^2)\right]^{-1/2} = e^{-\frac{1}{2}\mathrm{tr}\,\log(-\Box+m^2)}. \end{aligned}$$

Going to Fourier space, this gives a vacuum energy density

$$\mathcal{E} = \frac{1}{2}\int^\Lambda d^4p\ \log(p^2+m^2). \tag{7.28}$$

It is obviously divergent, and we have inserted an explicit UV cut-off Λ. It is however of interest to extract its dependence on m^2:

$$\begin{aligned} \frac{\partial}{\partial m^2}\mathcal{E} &= \frac{1}{2}\int^\Lambda d^4p\,\frac{1}{p^2+m^2} = \frac{1}{2}\int^\infty d^4p \int_{\frac{1}{\Lambda^2}}^\infty dt\ e^{-t(p^2+m^2)} \\ &= \frac{1}{2}\int_{\frac{1}{\Lambda^2}}^\infty dt\ e^{-tm^2}\int d^4p e^{-tp^2} \propto \int_{\frac{1}{\Lambda^2}}^\infty dt\ e^{-tm^2}\frac{1}{t^2} \end{aligned}$$

In the second equality we have simply traded a way to implement the cut-off with another which is more convenient for this computation, while in the last equality we have discarded numerical factors coming from the 4 Gaussian integrals (but we keep track of the signs).

We thus obtain

$$\frac{\partial}{\partial m^2}\mathcal{E} \propto m^2\int_{\frac{m^2}{\Lambda^2}}^\infty ds\frac{1}{s^2}e^{-s}. \tag{7.29}$$

We now evaluate the remaining integral, recalling that we are interested in taking eventually the limit $\Lambda \to \infty$. We have

$$\begin{aligned} \int_{\frac{m^2}{\Lambda^2}}^\infty ds\frac{1}{s^2}e^{-s} &= \left| -\frac{1}{s}e^{-s} - (\log s)e^{-s}\right|_{\frac{m^2}{\Lambda^2}}^\infty - \int_{\frac{m^2}{\Lambda^2}}^\infty ds\ (\log s)e^{-s} \\ &= \frac{\Lambda^2}{m^2} + \log\frac{m^2}{\Lambda^2} + \text{finite}. \end{aligned}$$

Hence we have

$$\frac{\partial}{\partial m^2}\mathcal{E} \propto \Lambda^2 + m^2 \log \frac{m^2}{\Lambda^2} + m^2\gamma,$$

where γ is a finite constant. This eventually implies

$$\mathcal{E} \propto \Lambda^4 + m^2\Lambda^2 + m^4 \log \frac{m^2}{\Lambda^2}, \qquad (7.30)$$

where Λ^4 is the integration constant, we have suppressed all terms which vanish when $\Lambda \to \infty$, and we have redefined Λ in such a way as to have no terms with just m^4.

Now supersymmetry comes finally into play. When summing over all the particles in the spectrum, recall that there is a minus sign for the fermions (it comes from the fact that the Grassmann integration brings down a positive power of the determinant). Then the term proportional to Λ^4 cancels because

$$\mathrm{tr}_B 1 - \mathrm{tr}_F 1 = 0,$$

i.e. the number of bosonic modes is equal to the number of fermionic modes.

The term proportional to Λ^2 cancels too because of the supertrace theorem

$$\mathrm{tr}_B m^2 - \mathrm{tr}_F m^2 = 0.$$

Hence we are left with

$$\mathcal{V}_{\text{eff}} \propto (\mathrm{tr}_B - \mathrm{tr}_F) m^4 \log \frac{m^2}{\Lambda^2}. \qquad (7.31)$$

This formula, known as the Coleman-Weinberg effective potential, thus gives a non-trivial one-loop correction to the vacuum energy in a SUSY breaking vacuum. (It is obvious that in a SUSY vacuum, the above expression vanishes because of the degeneracy of the spectrum.) Let us evaluate it further for the example that we discussed previously.

7.3.1 The case of the O'Raifeartaigh model

We should now just plug in (7.31) the squared mass eigenvalues that we obtained before for the O'Raifeartaigh model, see (7.23) and (7.27).

As we have just noted, it is clear that since when $\mu = 0$, the bosonic and fermionic masses are degenerate, $m_f^2 = m_b^2$, then we have that $\mathcal{V}_{\text{eff}}(\mu = 0) = 0$

as expected, according to the non-renormalization theorem. There should then be an overall power of $|\mu|$ in front of the effective potential $\mathcal{V}_{\text{eff}}$.

Note also that all the squared mass eigenvalues depend on X only through $|X|^2$. Hence we must have

$$\mathcal{V}_{\text{eff}} = \mathcal{V}_{\text{eff}}(|X|^2).$$

We see that the one-loop vacuum energy becomes an effective potential for the pseudomodulus X. From the expression above, it is clear that $X = 0$ is an extremum. We still have to check that in the expansion

$$\mathcal{V}_{\text{eff}} = \mathcal{V}_{\text{eff}}^{(0)} + \mathcal{V}_{\text{eff}}^{(1)}|X|^2 + \ldots$$

the zero order term $\mathcal{V}_{\text{eff}}^{(0)}$ is finite, and that $\mathcal{V}_{\text{eff}}^{(1)} > 0$ for stability of the extremum at the origin.

Let us compute $\mathcal{V}_{\text{eff}}^{(0)}$:

$$\begin{aligned}
\mathcal{V}_{\text{eff}}^{(0)} &= (|m|^2 + 2|h\mu|^2)^2 \log \frac{|m|^2 + 2|h\mu|^2}{\Lambda^2} + |m|^4 \log \frac{|m|^2}{\Lambda^2} \\
&\quad + (|m|^2 - 2|h\mu|^2)^2 \log \frac{|m|^2 - 2|h\mu|^2}{\Lambda^2} + |m|^4 \log \frac{|m|^2}{\Lambda^2} \\
&\quad - 2|m|^4 \log \frac{|m|^2}{\Lambda^2} - 2|m|^4 \log \frac{|m|^2}{\Lambda^2} \\
&= |m|^4 \left\{ \left(1 + 2\left|\frac{h\mu}{m}\right|^2\right)^2 \log \frac{|m|^2}{\Lambda^2} \left(1 + 2\left|\frac{h\mu}{m}\right|^2\right) \right. \\
&\quad \left. + \left(1 - 2\left|\frac{h\mu}{m}\right|^2\right)^2 \log \frac{|m|^2}{\Lambda^2} \left(1 - 2\left|\frac{h\mu}{m}\right|^2\right) - 2 \log \frac{|m|^2}{\Lambda^2} \right\} \\
&= |m|^4 \left\{ \left(1 + 2\left|\frac{h\mu}{m}\right|^2\right)^2 \log \left(1 + 2\left|\frac{h\mu}{m}\right|^2\right) \right. \\
&\quad \left. + \left(1 - 2\left|\frac{h\mu}{m}\right|^2\right)^2 \log \left(1 - 2\left|\frac{h\mu}{m}\right|^2\right) \right\} + 8|h\mu|^4 \log \frac{|m|^2}{\Lambda^2}.
\end{aligned}$$

The last (divergent) term accounts for the renormalization of the coupling of the quartic interaction in the potential, namely

$$|h|^2 \to |h|^2(1 + 8|h|^2 \log(|m|^2/\Lambda^2)).$$

Otherwise, the one-loop effective potential for $X = 0$ is clearly finite.

We note that $\mathcal{V}_{\text{eff}}^{(0)}(|\mu|^2) = \mathcal{V}_{\text{eff}}^{(0)}(-|\mu|^2)$ and, of course, $\mathcal{V}_{\text{eff}}^{(0)}(0) = 0$, so that

$$\mathcal{V}_{\text{eff}}^{(0)} \propto |\mu|^4.$$

In order to compute $\mathcal{V}_{\text{eff}}^{(1)}$, we need to expand the eigenvalues in $|X|^2$, and plug the result back into the expression for the Coleman-Weinberg effective potential. We will not attempt to do it here (but rather refer to K. A. Intriligator and N. Seiberg, "Lectures on Supersymmetry Breaking," Class. Quant. Grav. **24** (2007) S741 [arXiv:hep-ph/0702069]), and just state that the result of this procedure is that one finds a positive squared mass in front of the term $|X|^2$.

Hence the true quantum vacuum is at $X = 0$, and the complex plane of degenerate vacua is lifted as expected. Actually, we were willing to admit a circle of degenerate vacua, associated to a Goldstone boson. However it is easy to understand that $X = 0$ is the most symmetrical point on the complex plane, no symmetry is broken and then no bosonic mode should be massless. We will clarify in the next section of which global symmetry we are talking about.

7.4 SUSY breaking and R-symmetry

Let us concentrate now on global symmetries. Recall that for n chiral superfields and a generic superpotential $W(\Phi_i)$, the f-term conditions are also n generic holomorphic equations for the n complex variables ϕ_i:

$$\frac{\partial W}{\partial \phi_i} = 0. \tag{7.32}$$

These equations have generically a set of distinct solutions, which would imply the existence of isolated SUSY vacua. We thus learn that for the f-term equations not to have a solution, W must be non-generic. Now, a non-generic superpotential is acceptable if there is a symmetry which is responsible for the non-genericity. Otherwise it is not considered natural, but finely tuned.

If we consider an ordinary symmetry (usually referred to as axial), such that

$$\Phi_i \to e^{iq_i\alpha}\Phi_i,$$

we must require that $W(\Phi_i)$ is invariant. This means

$$\delta W = 0 \qquad \Leftrightarrow \qquad \sum_i q_i \phi_i \frac{\partial W}{\partial \phi_i} = 0, \tag{7.33}$$

for any value of ϕ_i (i.e. off-shell as well as on-shell). This condition is always satisfied by solutions of the f-terms (7.32). Thus, one might use a (complexified) symmetry transformation to rescale away one of the scalar fields, but the equation above tells us that simultaneously the n f-terms are not all independent, one is redundant. We are again in a situation with the same number of equations and variables. Hence an ordinary symmetry does not help.

Consider instead an R-symmetry. Here we must require that the superpotential $W(\Phi_i)$ be of charge 2

$$W \to e^{2i\alpha} W.$$

This means that

$$\delta W = 2i\alpha W \qquad \Leftrightarrow \qquad \sum_i q_i \phi_i \frac{\partial W}{\partial \phi_i} = 2W, \tag{7.34}$$

again for any value of ϕ_i. This tells us that the n f-terms are not all independent, but rather a combination of them sums up to the superpotential itself. However, from the algebraic point of view, this only tells us that the set of equations (7.32) sums to yet another non trivial equation, $W = 0$. As far as looking for F-flat solutions is concerned, this is generically not helpful, and hence we are left with n non-trivial equations to solve. On the other hand, we can still use a complexified R-symmetry transformation to set one field, say, to 1. We are thus left with $n - 1$ variables, and now generically the system of equations will not have a solution.

The argument above is quite formal, but is convincing of the fact that one needs an R-symmetry and a generic superpotential respecting it to have a chance of achieving spontaneous SUSY breaking at tree level. In practice, one writes W taking into account that it preserves the $U(1)_R$ symmetry and will find that $f_i = 0$ has no solutions.

Let us revisit in this light the O'Raifeartaigh model. Given the superpotential

$$W = hX(\Phi_1^2 - \mu^2) + m\Phi_1\Phi_2,$$

we see that it preserves an R-symmetry for which

$$R(X) = 2, \quad R(\Phi_1) = 0, \quad R(\Phi_2) = 2. \tag{7.35}$$

One can show that W is indeed of the most generic (renormalizable) form compatible with these charges, up to shifts and relabeling of the fields.

If we were to add R-symmetry breaking terms such as

$$\delta W = \tfrac{1}{2} m' X^2 \qquad \text{or} \qquad \delta' W = \tfrac{1}{2} m'' \Phi_2^2, \tag{7.36}$$

it is easy to see that we immediately reintroduce SUSY vacua. Nevertheless, there is still the possibility of keeping some *metastable* SUSY breaking vacua somewhere else in field space. For instance, this can be the case when the above corrections are small. However, the physics becomes a bit subtle since it generically involves playing one-loop terms (such as the ones coming from the Coleman-Weinberg effective potential) against tree-level ones.

As a last comment, note that because of the R-symmetry "theorem", when SUSY is broken R-symmetry is also most likely to be broken spontaneously. There will thus be a massless (real) Goldstone boson associated to it. However, it appears that, essentially because of holomorphy, at tree level the Goldstone boson is complexified. There is then automatically at least one pseudomodulus. It is to be noted that in the metastable vacua, the issue of the pseudomoduli is particularly subtle to discuss, since it could happen that the effective potential is such that the pseudomodulus could just roll down to the SUSY vacuum.

7.5 Explicit SUSY breaking: soft terms

Let us now discuss very briefly the other possibility for supersymmetry breaking, namely the introduction of explicit SUSY breaking terms in the Lagrangian.

As we have already argued, it is important that the SUSY breaking terms involve dimensionful, renormalizable couplings. Indeed, the virtue of SUSY resides in the cancellation of UV divergencies. The latter takes place because dimensionless Yukawa, gauge and quartic couplings are identified. If we were to spoil these identifications by introducing non-SUSY dimensionless couplings, we would destroy all effects of supersymmetry altogether. On the other hand, introducing couplings of positive mass dimension (like masses for scalars or fermions, or cubic scalar couplings) should not spoil SUSY cancellations for energies above the scale set by the new coupling itself. In other words, the SUSY breaking terms set the UV cut-off for cancellations of divergencies. Such SUSY breaking terms, that preserve the UV structure of SUSY theories, are called *soft*.

We will not here review all such terms and the phenomenology associated to them, but rather refer to the reviews by Martin and Terning cited in Chapter 1. We will just very briefly comment on the relation between explicit SUSY breaking by soft terms and spontaneous SUSY breaking.

An example of a soft term is, for instance, to add

$$\delta\mathcal{L}_{\text{susybr}} = m_s^2|\phi|^2 \tag{7.37}$$

to a massless Wess-Zumino model, or more generally just a correction to the scalar mass square that makes it different from the mass of the fermionic partner. Other bilinear soft terms are

$$\delta\mathcal{L}'_{\text{susybr}} = B_m\phi\tilde{\phi} + c.c. \tag{7.38}$$

or

$$\delta\mathcal{L}''_{\text{susybr}} = m_\lambda\lambda^2 + c.c., \tag{7.39}$$

where λ is a gaugino.

Terms such as (7.37) and (7.38) are for instance needed in the Higgs sector of the MSSM in order to correctly implement Electro-weak symmetry breaking. While of course terms such as (7.39) and (7.37) are also expected to give masses to the superpartners of the SM particles in order to explain why we have not yet seen them.

All of these terms, being mass terms, are clearly renormalizable. It is interesting to rewrite them in a supersymmetric fashion using an extra chiral superfield, called *spurion*, which we will denote by U. This background superfield has the property that it has a non vanishing f-term

$$U = \theta^2 F. \tag{7.40}$$

Using it, we can write the above soft terms as integrals over superspace. We have

$$\delta\mathcal{L}_{\rm susybr} = \int d^2\theta d^2\bar{\theta}\frac{U\bar{U}}{M^2}\Phi\bar{\Phi}, \tag{7.41}$$

$$\delta\mathcal{L}'_{\rm susybr} = \int d^2\theta \; U\Phi\tilde{\Phi} + c.c., \tag{7.42}$$

$$\delta\mathcal{L}''_{\rm susybr} = \int d^2\theta \frac{U}{M}\mathcal{W}^\alpha\mathcal{W}_\alpha + c.c. \tag{7.43}$$

In the above expressions, M is some UV scale. From them, we obtain soft masses like $|m_s|^2 = |F|^2/M^2$, $B_m = F$ and $m_\lambda = F/M$.

We now see that the gap from explicit to spontaneous SUSY breaking is small: all we have to do is to promote U to a dynamical superfield, which acquires the f-term spontaneously, and then generates the soft terms by its interactions with the rest of the superfields. From this point of view, it is then clear that all the virtues of a SUSY theory (i.e. the UV cancellations) are preserved in presence of the soft terms, as much as in a theory which breaks SUSY spontaneously.

We should here just add a remark concerning the terms (7.41) and (7.43). At face value, they look like non-renormalizable additions to a SUSY Lagrangian. This is evident from the inverse powers of M which, unless specified otherwise, must be generically a UV scale of the theory (such as the Planck mass M_p or the GUT scale). However from the component version of the same terms (7.37) and (7.39), it is obvious that they should not introduce spurious UV divergencies. What this really means is that, on one hand, it is only consistent, from the renormalization point of view, to introduce all such soft terms. On the other hand, in a fully renormalizable theory, where U is dynamical, these terms are to be thought of as effective terms coming from integrating out degrees of freedom at the scale M.

It is to be noted that on the other hand a soft term like (7.42) is fully renormalizable even from the SUSY point of view. Hence by adding it we do not expect to be forced to add (soft) counter-terms to the bare Lagrangian.

7.5.1 The mediation paradigm

This point of view on the soft terms and explicit SUSY breaking is even more physical by the following argument. Given that spontaneous SUSY breaking is not possible within the MSSM, we must assume that there is some *hidden* *sector* where SUSY is broken spontaneously. Then one has to assume that there exist fields that interact both with the hidden sector and the visible one (the MSSM), and which will then *mediate* SUSY breaking.

These fields can be chiral superfields, which couple to the visible sector through the superpotential. This is the simplest of the models, where mediation happens at tree level.

Another option is that the mediating fields are the gauge supermultiplets of the visible sector itself. In this case, typically, the SUSY breaking is induced in the gauge supermultiplet and then on the matter chiral superfields through radiative corrections, i.e. at the quantum level. Note that it is in this way that, for instance, the supertrace theorem is evaded.

If all other options are absent, and the hidden sector is completely decoupled from the visible one, there will always be (super)gravity that universally couples to both. Hence gravity will mediate SUSY breaking in this last scenario. Gravity mediation effects will take the form of effective terms like (7.41)–(7.43), where now $M \equiv M_p$.

It is clear that what all these models of mediation of SUSY breaking are all about, is to provide a theory of the soft terms, in other words a predictive pattern for these extra terms that one can (and must) add to the MSSM Lagrangian. There is a very large amount of research in this direction, which has produced several different schemes, each with its own successes and shortcomings. Obviously, the community is eagerly awaiting and hoping that from the LHC data a pattern of measured soft masses could start to emerge (if superpartners are out there at all!), so that one could finally confront models with experiment.

7.6 Dynamical SUSY breaking

Let us close this chapter on SUSY breaking by mentioning that the most interesting models of spontaneous SUSY breaking are not the ones where SUSY is broken at tree level, that is classically, but rather at the quantum level. Now, since we had demonstrated that there are no perturbative corrections to the vacuum energy, when we start from a classically SUSY vacuum, then the only option is that *non-perturbative* effects break SUSY. This is of course most likely to happen in a gauge theory.

This is interesting, because the scale of SUSY breaking will necessarily be related to the dynamically generated scale of the theory $\Lambda_{\rm dyn}$, which much as in QCD, is naturally hierarchically small with respect to any UV scale, for instance $\Lambda_{QCD} \ll M_{GUT}$. That would essentially explain the hierarchy between M_{EW}, which is related to the scale of SUSY breaking, and M_{GUT}.

However, dealing with strongly coupled gauge theories is not very simple, though supersymmetry helps a good deal. In some cases it is possible to reformulate the low-energy dynamics of a strongly coupled (confining) gauge theory through a theory involving chiral superfields only. Typically there will be a highly non-trivial (and non-calculable) kinetic term, but the superpotential can usually be guessed from symmetry arguments, and possibly instanton computations. If the superpotential happens to have the form of a generalized O'Raifeartaigh model, then we can deduce with some confidence that the model breaks SUSY dynamically.

There are many non-trivial checks one can do, and great advances have been made in the last years. This topic however would require a much deeper study of non-perturbative aspects of SUSY gauge theories. This is left for a more advaced course.