CONSTRUCTING STABLE GENERALIZED COMPLEX FOUR-MANIFOLDS

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Abstract. A generalized complex structure is said to be stable if its defining anticanonical section vanishes transversally, giving rise to a codimension-two type-change locus. These structures are those generalized complex structures that are closest to being symplectic, while exhibiting further interesting behavior. An alternate and fruitful viewpoint is to consider them as zero-residue symplectic structures in the elliptic tangent bundle, a Lie algebroid naturally associated to their type-change locus. We develop Gompf-Thurston methods for Lie algebroids to relate their existence to that of fibration-like maps. We define boundary Lefschetz fibrations and use them to construct stable structures out of log-symplectic structures. Moreover we discuss how to obtain such boundary Lefschetz fibrations on concrete four-manifolds, and in fact classify all such fibrations over the disk.

This note summarizes joint work with Stefan Behrens and Gil Cavalcanti.

Contents

1. Introduction

The goal of this note is to explain how to obtain effective existence results for stable generalized complex structures and make them compatible with specific fibration-like structures. Indeed, we relate their existence on a manifold to specific types of maps the manifold admits.

1.1. **Results.** We extend techniques by Gompf to construct stable GCSs on four-manifolds.

Theorem A (Cavalcanti–K. [3]). *Let $f\colon (X^4, D^2) \to (\Sigma^2, Z^1)$ be a boundary Lefschetz fibration such that (Σ, Z) carries a log-symplectic structure (i.e. such that $[Z] = 0 \in H^1(\Sigma; \mathbb{Z}_2)$). Assume that D is coorientable and that the generic fiber F satisfies $[F] \neq 0 \in H_2(X \setminus D; \mathbb{R})$. Then (X, D) admits a stable generalized complex structure.*

To make effective use of this result, one needs an understanding of boundary Lefschetz fibrations. We have classified total spaces of boundary Lefschetz fibrations over $(\mathbb{D}^2, \partial\mathbb{D}^2)$.

Theorem B (Behrens–Cavalcanti–K. [2]). *Let $f\colon (X^4, D) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ be a relatively minimal boundary Lefschetz fibration. Then X is diffeomorphic to one of the following:*

(1) $S^1 \times S^3$;

Partly based on joint work with Gil Cavalcanti, and some with Stefan Behrens and Gil Cavalcanti. This draws from arXiv:1703.03798, [3], and 1706.09207, [2], as well as my Utrecht University Ph.D. thesis [7].

(2) $\#m(S^2 \times S^2)$, *including* S^4 *for* $m = 0$;
(3) $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ *with* $m > n \geq 0$.

In all cases the generic fiber is nontrivial in $H_2(X \setminus D; \mathbb{R})$. *In case* (1), D *is coorientable, while in cases* (2) *and* (3), D *is coorientable if and only if* m *is odd.*

This result can be combined with Theorem A to equip the listed four-manifolds X with coorientable D (and all of their blow-ups) with stable generalized complex structures.

2. Generalized complex geometry

We start by briefly defining the geometric structures of interest to us.

2.1. $\mathbb{T}X$ **as a Courant algebroid.** Let (X, H) be a $2n$-dimensional manifold equipped with a closed three-form $H \in \Omega^3_{\rm cl}(X)$. Recall that the *double tangent bundle* $\mathbb{T}X := TX \oplus T^*X$ is a Courant algebroid with anchor the projection $p\colon \mathbb{T}X \to TX$. It carries a natural pairing of split signature, and an H-twisted *Courant bracket*, viz. for $V, W \in \Gamma(TX)$ and $\xi, \eta \in \Gamma(T^*X)$:

$$\langle V + \xi, W + \eta\rangle = \frac{1}{2}(\eta(V) + \xi(W)), \qquad [[V + \xi, W + \eta]]_H = [V, W] + \mathcal{L}_V\eta - \iota_W d\xi + \iota_V \iota_W H$$

Two-forms $B \in \Omega^2(X)$ act via *B-field* (or *gauge*) *transformations* $e^B\colon \mathbb{T}X \to \mathbb{T}X$ given by $e^B\colon V + \xi \mapsto V + \xi + \iota_V B$. This takes $(\mathbb{T}X, H)$ to $(\mathbb{T}X, H + dB)$, leading to the *Ševera class* $[H] \in H^3(X; \mathbb{R})$ determining $\mathbb{T}X$ up to Courant isomorphism. The Courant automorphisms of $\mathbb{T}X$ are generated by the diffeomorphisms and closed B-field transformations.

2.2. **Generalized complex structures.**

Definition 2.1. A *generalized complex structure* on (X, H) is a complex structure $\mathcal{J}$ on $\mathbb{T}X$ that is orthogonal with respect to $\langle\cdot,\cdot\rangle$, and whose $+i$-eigenbundle is involutive under $[[\cdot,\cdot]]_H$.

There is an alternative definition of a generalized complex structure using spinors. To state it, recall that sections $v = V + \xi \in \Gamma(\mathbb{T}X)$ of the double tangent bundle act on differential forms via Clifford multiplication, given by $v \cdot \rho = \iota_V\rho + \xi \wedge \rho$ for $\rho \in \Omega^\bullet(X)$.

Definition 2.2. A *GCS* on (X, H) is given by a pure spinor line $K_{\mathcal{J}} \subset \wedge^\bullet T^*_{\mathbb{C}}X$ such that

- Pointwise $K_{\mathcal{J}} = \langle\rho\rangle$ for a pure spinor $\rho = e^{B+i\omega} \wedge \Omega$ with Ω a decomposable k-form;
- This pure spinor satisfies $(\rho, \overline{\rho})_{\rm Ch} \sim \Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$;
- For any local generator $\rho \in \Gamma(K_{\mathcal{J}})$ there exists $v \in \Gamma(\mathbb{T}X)$ such that $d\rho + H \wedge \rho = v \cdot \rho$.

Both definitions are related using that $K_{\mathcal{J}} = {\rm Ann}(E_{\mathcal{J}})$ is the annihilator under the Clifford action of $E_{\mathcal{J}}$, the $+i$-eigenbundle of $\mathcal{J}$. The bundle $K_{\mathcal{J}}$ is called the *canonical bundle* of $\mathcal{J}$. There is, if you want, a third definition, namely a complex Dirac structure $E_{\mathcal{J}} \subsetneq \mathbb{T}_{\mathbb{C}}X$, i.e. $E_{\mathcal{J}}$ is a maximal isotropic and involutive subspace, which further satisfies $E_{\mathcal{J}} \cap \overline{E}_{\mathcal{J}} = \emptyset$,

2.3. **Type change.** We next introduce the type of a generalized complex structure $\mathcal{J}$, which colloquially provides a measure for how many complex directions there are. The type is an integer-valued upper semicontinuous function on X whose parity is locally constant.

Definition 2.3. Let $\mathcal{J}$ be a generalized complex structure on X. The *type* of $\mathcal{J}$ is a map ${\rm type}(\mathcal{J})\colon X \to \mathbb{Z}$ whose value at a point $x \in X$ is the integer k above, the degree of Ω. The *type change locus* $D_{\mathcal{J}}$ of $\mathcal{J}$ is the subset of X where ${\rm type}(\mathcal{J})$ is not locally constant.

The complement of $D_{\mathcal{J}}$ is an open dense set where the type is minimal. Using the type, generalized complex structures are seen to interpolate between symplectic and complex structures. At points where $\text{type}(\mathcal{J}) = 0$, the generalized complex structure is equivalent to a symplectic structure, in that it is equivalent to the generalized complex structure $\mathcal{J}_\omega$ of a symplectic structure under a B-field transformation.

Any generalized complex structure $\mathcal{J}$ determines a Poisson structure $\pi_{\mathcal{J}}$ as the composition $\pi^\sharp_{\mathcal{J}} := p \circ \mathcal{J}|_{T^*X}$. The type of $\mathcal{J}$ is related to the rank of $\pi_{\mathcal{J}}$ through the formula $\text{rank}(\pi_{\mathcal{J}}) = 2n - 2\,\text{type}(\mathcal{J})$. Using $\pi_{\mathcal{J}}$ one can view a generalized complex structure $\mathcal{J}$ as a foliation on X with symplectic leaves, and a suitably compatible complex structure transverse to the leaves.

2.4. **Examples.**

Example 2.4. The following provide examples of generalized complex structures on $(X, 0)$.

- Let $\omega \in \text{Symp}(X)$ be symplectic. Then $K_{\mathcal{J}_\omega} := \langle e^{i\omega} \rangle$ defines a GCS $\mathcal{J}_\omega$.
- Let $I \in \text{CS}(X)$ be complex with $K_I = \wedge^{n,0}T^*X$. Then $K_{\mathcal{J}_I} := K_I$ defines a GCS $\mathcal{J}_I$.
- Let $P \in \Gamma(\wedge^{2,0}TX)$ be holomorphic Poisson structure with respect to a complex structure I. Then $K_{\mathcal{J}_{P,I}} := e^P K_I$ defines a GCS $\mathcal{J}_{P,I}$.

The automorphisms $\mathcal{J}_\omega$, $\mathcal{J}_I$ and $\mathcal{J}_{P,I}$ are given by, with $\pi = \text{Im}(P)$:

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad \mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \mathcal{J}_{P,I} = \begin{pmatrix} -I & \pi \\ 0 & I^* \end{pmatrix}.$$

We wish to point out in particular that $\mathcal{J}_\omega$ has type constant equal to zero.

2.5. **Stable generalized complex structures.** Generalized complex structures which are stable were introduced in [4, 6], and are not far from being symplectic.

Let $\mathcal{J}$ be a generalized complex structure on (X, H). The anticanonical bundle $K^*_{\mathcal{J}}$ has a section $s \in \Gamma(K^*_{\mathcal{J}})$, given by $s(\rho) := \rho_0$ for $\rho \in \Gamma(K_{\mathcal{J}})$, with ρ_0 the degree-zero part of ρ. Note the zero set of s cuts out the degeneracy locus $D_{\mathcal{J}}$ of $\mathcal{J}$, if $\mathcal{J}$ is generically of type zero.

Definition 2.5. A generalized complex structure $\mathcal{J}$ on (X, H) is *stable* if s is transverse to the zero section in $K^*_{\mathcal{J}}$. The set $D_{\mathcal{J}} := s^{-1}(0)$ is a codimension-two smooth submanifold of X called the *anticanonical divisor* of $\mathcal{J}$.

Outside of $D_{\mathcal{J}}$, the section s is nonvanishing hence the type of $\mathcal{J}$ is equal to zero, while over $D_{\mathcal{J}}$ it is equal to two. Consequently, stable generalized complex structures can be seen as generalized complex structures which are close to being symplectic.

Example 2.6. Consider $(\mathbb{C}^2, 0)$ with holomorphic Poisson structure $\pi = z\partial_z \wedge \partial_w$. This gives a stable generalized complex structure with $K_{\mathcal{J}} = \langle z + dz \wedge dw \rangle$ and $D_{\mathcal{J}} = \{z = 0\}$.

Any stable generalized complex structure $\mathcal{J}$ is locally equivalent around points in $D_{\mathcal{J}}$ to $\langle e^{i\omega_0}(z + dz \wedge dw) \rangle$ on $\mathbb{C}^2 \times \mathbb{R}^{2n-4}$, with ω_0 the standard symplectic form on $\mathbb{R}^{2n-4}$, and $\langle z + dz \wedge dw \rangle$ the structure on $\mathbb{C}^2$ of Example 2.6 with $D_{\mathcal{J}} = \{z = 0\}$, see [4, Section 3.4]. On $D_{\mathcal{J}}$ there is an induced type-1 generalized Calabi–Yau structure [4, Theorem 2.13].

Any compact type-1 generalized Calabi–Yau manifold, such as $D_{\mathcal{J}}$, fibers over the torus T^2 [1]. Moreover, the semilocal form of a stable generalized complex structure around its type change locus is given by its *linearization* along $D_{\mathcal{J}}$, which is the stable generalized complex structure naturally present on the normal bundle to this type-1 generalized Calabi–Yau manifold. We will not elaborate on this further here and instead refer to [1, 4].

3. Divisors and Lie algebroids

3.1. **Divisors on smooth manifolds.** We introduce the divisor (ideals) to keep track of degeneracies. Let X be a smooth n-dimensional manifold (*without* any complex structure).

Definition 3.1. A *divisor* on X is a pair (U, σ) where $U \to X$ is a real/complex line bundle and $\sigma \in \Gamma(U)$ is a section whose zero set is nowhere dense. Its *divisor ideal* is given by $I_\sigma := \sigma(\Gamma(U^*)) \subseteq C^\infty(X)$, and is obtained by evaluation using the map $\sigma\colon \Gamma(U^*) \to C^\infty(X)$.

A morphism of divisors is a map $f\colon (X, I) \to (Y, I')$ such that $f^*I' = I$.

Example 3.2. The following are examples of divisors that are important to us.

- A *complex log divisor*: (U, σ), $\sigma \pitchfork 0$. Have $D^{n-2} = \sigma^{-1}(0)$ and $I_D = \langle w \rangle \subseteq C^\infty(X; \mathbb{C})$;
- A *(real) log divisor*: (L, s), $s \pitchfork 0$. Have $Z^{n-1} = s^{-1}(0)$ and $I_Z = \langle x \rangle \subseteq C^\infty(X)$;
- An *elliptic divisor*: (R, q), $\mathrm{Hess}(q) \in \Gamma(D; \mathrm{Sym}^2 N^*D \otimes R)$ a definite bilinear form. Have $D^{n-2} = q^{-1}(0)$ and $I_{|D|} = \langle r^2 \rangle \subseteq C^\infty(X)$. Note that we have $\langle w\overline{w} \rangle = \langle r^2 \rangle$.

For a stable generalized complex structure $\mathcal{J}$, the pair $(K^*_{\mathcal{J}}, s)$ is a complex log divisor.

3.2. **Lie algebroids coming from divisors.** As a tool to study geometric structures we introduce Lie algebroids which adhere to the degeneracies specified by a divisor ideal.

Definition 3.3. Let $\mathcal{A} \to X$ be a Lie algebroid. The *isomorphism locus* of a Lie algebroid $\mathcal{A}$ is the open set $X_\mathcal{A} \subseteq X$ where $\rho_\mathcal{A}$ is an isomorphism. The associated divisor is $\mathrm{div}(\mathcal{A}) := (\det(\mathcal{A}^*) \otimes \det(TX), \det(\rho_\mathcal{A}))$, viewing $\rho_\mathcal{A}$ as a section of $\mathcal{A}^* \otimes TX$, and divisor ideal $I_\mathcal{A}$.

Next let I be a divisor ideal and consider the module of vector fields preserving I:

$$\mathcal{V}(I) := \{v \in \Gamma(TX) \,|\, \mathcal{L}_v I \subset I\}.$$

This defines an involutive submodule of $\Gamma(TX)$. If it is in addition locally finitely generated projective, it gives rise to a Lie algebroid $\mathcal{A}_I$ with $\Gamma(\mathcal{A}_I) = \mathcal{V}(I)$ by the Serre–Swan theorem.

Example 3.4. The above construction is fruitful for each of the three divisor-types.

- The *complex log tangent bundle*: $TX(-\log D) \to T_\mathbb{C}X$, locally $\langle w\partial_w, \partial_{\overline{w}}, \partial_{x_i} \rangle$;
- The *(real) log tangent bundle*: $TX(-\log Z) \to TX$, locally $\langle x\partial_x, \partial_{x_i} \rangle$;
- The *elliptic tangent bundle*: $TX(-\log |D|) \to TX$, locally $\langle r\partial_r, \partial_\theta, \partial_{x_i} \rangle$.

Each of these Lie algebroids satisfy the condition that $I_{\mathcal{A}_I} = I$. There are natural morphisms

$$TX(-\log |D|) \otimes \mathbb{C} \xrightarrow{\iota} TX(-\log D) \xrightarrow{\rho_{\mathcal{A}_D}} T_\mathbb{C}X.$$

We discuss very briefly the Lie algebroid cohomology of these Lie algebroids.

- For $TX(-\log D)$ we have $H^k(\log D) \cong H^k(X \setminus D; \mathbb{C})$;
- For $TX(-\log Z)$ we have $H^k(\log Z) \cong H^k(X) \oplus H^{k-1}(Z)$;
- For $TX(-\log |D|)$ we have
  - $H^k(\log |D|) \cong H^k(X \setminus D) \oplus H^{k-1}(S^1 ND)$;
  - $H^k_0(\log |D|) \cong H^k(X \setminus D) \oplus H^{k-1}(D)$.

Here we use a residue map $\mathrm{Res}_q\colon \Omega^k(\log |D|) \to \Omega^{k-2}(D, \mathfrak{k}^*)$ called the *elliptic residue*, where $\mathfrak{k} \to D$ is a line bundle which is trivialized by a choice of coorientation for D.[1] The kernel

[1] This map arises from the short exact sequence given by the anchor

$$0 \to \underline{\mathbb{R}} \oplus \mathfrak{k} \to TX(-\log |D|)|_D \to TD \to 0, \tag{3.1}$$

with $\underline{\mathbb{R}}$ generated by the Euler vector field of ND, and $\mathfrak{k} \cong \wedge^2 N^*D \otimes R$.

$\Omega_0^\bullet(\log|D|)$ of Res_q is a subcomplex of $\Omega^\bullet(\log D)$. There are two further residues which we will use in passing, namely the *radial* and *θ-residues*, which are given on $\Omega_0^\bullet(\log|D|)$ by

$$(\mathrm{Res}_r, \mathrm{Res}_\theta)\colon \Omega_0^k(\log|D|) \to \Omega^{k-1}(D), \qquad \alpha d\log r + \beta d\theta + \gamma \mapsto (i_D^*\alpha, i_D^*\beta).$$

3.3. **Lie algebroid symplectic structures.** Given these Lie algebroids, we can define $\mathcal{A}$-*symplectic structures* as nondegenerate closed two-forms $\omega_{\mathcal{A}} \in \Omega^2_{\mathrm{cl}}(\mathcal{A})$. The reason for introducing these Lie algebroids is the following.

Theorem 3.5 (Cavalcanti–Gualtieri [4])**.** *There are one-to-one correspondence between*

- *stable GCSs $\mathcal{J}$ on (X,H) $\longleftrightarrow$ complex log-symplectic structures: $\omega_{\mathcal{J}} \in \Omega^2(X;\log D)$ such that $d\omega_{\mathcal{J}} = \rho^*_{\mathcal{A}_D} H$ and $\iota^*\omega_{\mathcal{J}} = b + i\omega$ has nondegenerate imaginary part;*
- *gauge equivalence classes of stable GCSs and three-forms $(\mathcal{J}, H)$ on X $\longleftrightarrow$ elliptic symplectic structures and D-coorientations $(\pi_{\mathcal{J}}^{-1}, \mathfrak{o}_D)$ where $\mathrm{Res}_q(\pi_{\mathcal{J}}^{-1}) = 0$.*

Here $[H] = \mathrm{Res}_r([\pi_{\mathcal{J}}^{-1}]) \wedge \mathrm{PD}[D]$, using the coorientation for D to define the Poincaré dual.

We further use *log-symplectic structures* $\omega \in \mathrm{Symp}(TX(-\log Z))$, which are dual to *log-Poisson bivectors* π for which $(\wedge^{2n}TX, \wedge^n\pi)$ is a log divisor, with $Z = (\wedge^n\pi)^{-1}(0)$. These are locally given by $\pi = x\partial_x \wedge \partial_y + \omega_0^{-1}$, where $x \in I_Z$ is a generator. Moreover, they induce on Z an (equivalence class) of *cosymplectic structures* (α,β) with $\alpha \wedge \beta^{n-1} \neq 0$. From these one can build the semi-global model $\pi^{-1} = d\log|x| \wedge p^*(\alpha) + p^*(\beta)$ for $p\colon NZ \to Z$. After perturbation one can arrange they are *proper*, so that we have $p_Z\colon Z \to S^1$ using α.

Due to this, from now we will work purely with Lie algebroid objects.

4. Boundary maps and boundary Lefshetz fibrations

As was stated in the introduction, we aim to relate the existence of stable generalized complex structures to the existence of certain types of maps. These we will now introduce.

We will use the following language. A *pair* (X,D) consists of a manifold X and a submanifold $\mathcal{D} \subseteq X$. A *map of pairs* $f\colon (X,D) \to (\Sigma, Z)$ is a map $f\colon X \to \Sigma$ for which $f(D) \subseteq Z$. A *strong map of pairs* is a map of pairs $f\colon (X,D) \to (\Sigma,Z)$ for which $f^{-1}(Z) = D$.

Definition 4.1. Let $f\colon (X^{2n}, D^{2n-2}) \to (\Sigma^2, Z^1)$ be a strong map of pairs which is proper and for which D and Z are compact.

- The map f is a *boundary map* if the normal Hessian of f along D is definite;
- The map f is a *boundary fibration* if it is a boundary map and the following two maps are submersions:
  a) $f|_{X\setminus D}\colon X \setminus D \to \Sigma \setminus Z$, and
  b) $f|_D\colon D \to Z$.

  The condition that f is a boundary fibration (in a neighbourhood of D) is equivalent to the condition that for every $x \in D$, there are coordinates $(x_1,\dots,x_{2n})$ centred at x and (y_1,y_2) centred at $f(x)$ such that f takes the form
  $$f(x_1,\dots,x_{2n}) = (x_1^2 + x_2^2, x_3),$$
  where D corresponds to the locus $\{x_1 = x_2 = 0\}$ and Z to the locus $\{y_1 = 0\}$;
- The map f is a *boundary Lefschetz fibration* if X and Σ are oriented, f is a boundary fibration from a neighbourhood of D to a neighbouhood of Z and $f|_{X^{2n}\setminus D}\colon X \setminus D \to \Sigma \setminus Z$ is a proper *Lefschetz fibration*, that is, for each critical point $x \in X \setminus D$ and

corresponding singular value $y \in \Sigma \setminus Z$, there are complex coordinates centred at x and y compatible with the orientations for which f acquires the form

$$f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2.$$

These give us morphisms of divisors between the induced divisor structures.

Proposition 4.2. *Let $f\colon (X, D^{-2}) \to (Y, Z^{-1})$ be a map of pairs. Then:*

- *f is a boundary map if and only if $I_{|D|} := f^* I_Z$ is an elliptic ideal and f a morphism;*
- *Any such f induces $(\varphi, f)\colon TX(-\log |D|) \to TY(-\log Z)$ such that $\varphi \equiv df$ on $\Gamma(\cdot)$.*

Boundary fibrations can be put in normal form around the degeneracy loci.

Theorem 4.3. *Let $f\colon (X^{2n}, D^{2n-2}) \to (\Sigma^2, Z^1)$ be a boundary map which is a boundary fibration on neighbourhoods of D and Z and for which Z is coorientable. Then there are*

- *neighbourhoods U of D and V of Z and diffeomorphisms between these sets and neighbourhoods of the zero sections of the corresponding normal bundles*

$$\Phi_D\colon U \to ND \qquad \text{and} \qquad \Phi_Z\colon V \to \mathbb{R} \times Z, \text{ and}$$

- *a bundle metric g on ND,*

such that the following diagram commutes, where $\mathrm{pr}_D\colon ND \to D$ is the bundle projection:

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ \big\downarrow{\scriptstyle \Phi_D} & & \big\downarrow{\scriptstyle \Phi_Z} \\ ND & \xrightarrow{(\|\cdot\|_g^2,\, f|_D \circ \mathrm{pr}_D)} & \mathbb{R} \times Z \end{array}$$

5. Sketch of proof of Theorem A

We will use the following result, whose proof would take too much time.

Proposition 5.1. *Let $f\colon (X^4, D^2) \to (\Sigma^2, Z^1)$ be a boundary map with $f|_D$ submersive, $[F] \neq 0 \in H_2(X \setminus D)$ near D, and $f(D)$ coorientable. Then there exists a closed elliptic two-form $\eta \in \Omega^2(\log |D|)$ with $\mathrm{Res}_q(\eta) = \mathrm{Res}_r(\eta) = 0$ and $\eta|_{\ker \varphi}$ nondegenerate near D.*

The proof of this result relies crucially on the Lie algebroid cohomology isomorphism $H_0^k(\log |D|) \cong H^k(X \setminus D) \oplus H^{k-1}(D)$. We can now prove our first main result.

Proof of Theorem A. Equip D and Z with the appropriate divisor structures making f induce a boundary Lefschetz fibration. Ensure that f has connected fibers after replacing (Y, Z) by $(Y', \partial Y')$. Then construct the Lie algebroid two-form η using Proposition 5.1. This form can be adjusted using Gompf's proof [5] to a closed elliptic two-form η' which is nondegenerate on $\ker \varphi$ over the entirety of X. Next, let $\omega_\Sigma \in \mathrm{Symp}(T\Sigma(-\log Z))$, and define for $t > 0$:

$$\omega_t := \varphi^* \omega_\Sigma + t\eta'.$$

Due to compactness of X, for $t > 0$ small this is nondegenerate, hence symplectic. Finally:

- $\mathrm{Res}_q(\omega_t) = \mathrm{Res}_q(\varphi^* \omega_Z) + t\,\mathrm{Res}_q(\eta') = 0$, as desired; and
- $\mathrm{Res}_r(\omega_t) = \mathrm{Res}_r(\varphi^* \omega_Z) + t\,\mathrm{Res}_r(\eta') = f^* \mathrm{Res}_Z(\omega_\Sigma) = f^* \alpha \in \Omega^1_{\mathrm{cl}}(X)$. □

Corollary 5.2. *Under the assumptions of Theorem A, given a log-symplectic structure ω_Y on (Y, Z), the elliptic two-form ω_X can be chosen such that $p \circ p_D = p_Z \circ f|_D$.*

In other words, we have a full commutative diagram around D and Z as follows:

$$\begin{array}{ccccc} ND & \xrightarrow{\mathrm{pr}_D} & D & \xrightarrow{p_D} & T^2 \\ {\scriptstyle f}\downarrow & & {\scriptstyle f|_D}\downarrow & & {\scriptstyle p}\downarrow \\ NZ & \xrightarrow{\mathrm{pr}_Z} & Z & \xrightarrow{p_Z} & S^1 \end{array}$$

6. Classifying boundary Lefschetz fibrations (Theorem B)

In this final part of the talk we aim to discuss boundary Lefschetz fibrations on four-manifolds in some more detail, and in particular sketch the proof of Theorem B.

6.1. **Monodromy description of boundary Lefschetz fibrations.**

It is very standard that Lefschetz fibrations admit a combinatorial description in terms of monodromy representations and vanishing cycles. This can be extended to boundary Lefschetz fibrations. A very rough description of this is the following.

For simplicity, we focus on fibrations over $(\mathbb{D}^2, \partial\mathbb{D}^2)$ and assume they are injective on their Lefschetz singularities. The latter condition can always be achieved by a small perturbation and the generalization to general base surfaces is similar to the Lefschetz case.

Definition 6.1 (Hurwitz systems)**.** Let $f\colon (X^4, D^2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ be a boundary Lefschetz fibration with ℓ Lefschetz singularities, and let $y \in D^2$ be a regular value. A *Hurwitz system* for f based at y is a collection of embedded arcs $\eta_0, \eta_1, \ldots, \eta_\ell \subset D^2$ such that

(1) η_0 connects y to $\partial\mathbb{D}^2$ and is transverse to $\partial\mathbb{D}^2$,
(2) η_i connects y to a critical value y_i,
(3) the arcs intersect pairwise transversely in y and are otherwise disjoint, and
(4) the order of the arcs is counterclockwise around y.

Such a Hurwitz system admits the notion of *Hurwitz equivalence* (see Figure 1).

6.2. **Extension to boundary Lefschetz fibrations.**

We now turn to boundary Lefschetz fibrations. We have not explicitly mentioned the following.

Proposition 6.2. *Let $f\colon (X^4, D^2) \to (\Sigma^2, Z^1)$ be a boundary fibration with connected fibers, where Z is coorientable and X is connected and orientable. Then its generic fibers are tori.*

This follows from the normal form result, Theorem 4.3. Another consequence of this is:

Proposition 6.3. *Let $f\colon (X^4, D^2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ be a boundary Lefschetz fibration.*
If $(a; b_1, \ldots, b_\ell)$ is any cycle system for f, then

$$B_\ell \circ \cdots \circ B_1 = \pm A^k \in \mathcal{M}(T^2) \tag{6.1}$$

for some $k \in \mathbb{Z}$, where the sign is positive if and only if D is coorientable. The integer k agrees with the Euler number of the normal bundle of D in X.

Thus the total monodromy around the boundary of $\mathbb{D}^2$ is a (signed) power of a Dehn twist.

Definition 6.4 (Abstract cycle systems)**.** An ordered collection of curves $(a; b_1, \ldots, b_\ell)$ in T^2 is called an *abstract cycle system* if it satisfies the condition in Equation (6.1). The notion of their *Hurwitz equivalence* is defined exactly as for Hurwitz systems, Definition 6.1.

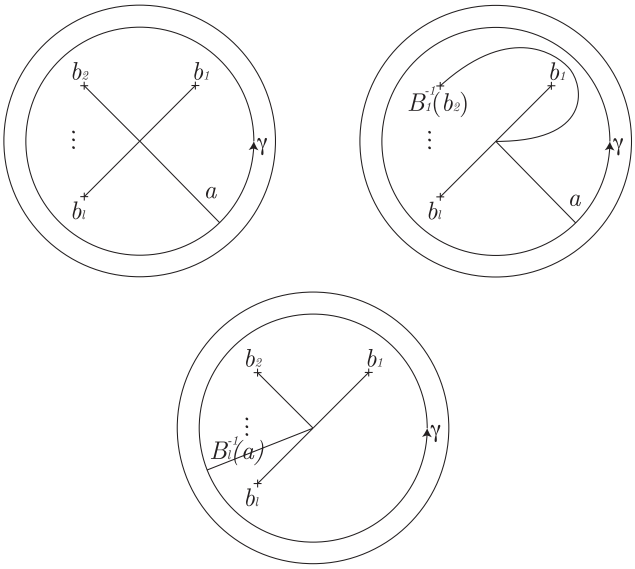

FIGURE 1. The origin of Hurwitz equivalence: here we illustrate the equivalences $\big(a; b_1, \dots, b_\ell\big) \sim \big(a; B_1^{-1}(b_2), b_1, b_3, \dots, b_\ell\big) \sim \big(B_\ell^{-1}(a); b_\ell, b_1, \dots, b_{\ell-1}\big)$.

Further, there is the notion of *relative minimality* of a cycle system. This is the condition that there is no b_i which is either null-homotopic or isotopic to a. If a cycle system of a ∂LF is not relatively minimal, one can blow-down a copy of $\overline{\mathbb{C}P}^2$ and find a new ∂LF.

Using cycle systems and Kirby calculus, in case of few Lefschetz singularities one can obtain:

Proposition 6.5. *Let $f\colon (X^4, D^2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ be a boundary Lefschetz fibration. If f has n Lefschetz singularities, then:*

- *If $n = 0$, then $X \cong S^1 \times S^3$ and D is coorientable;*
- *If $n = 1$, then $X \cong (S^1 \times S^3)\#\overline{\mathbb{C}P}^2$ and D is coorientable, with f not rel. minimal;*
- *If $n = 2$ and f is rel. minimal, then $X \cong S^4$ and D is not coorientable.*

To proceed and prove Theorem B, we plan to use an inductive procedure by splitting off certain parts of a cycle system, once it is in a desired normal form. For this we make use of a result by Hayano in the closely-related context of genus-one broken Lefschetz fibrations.

Theorem 6.6 (Hayano Factorisation Theorem)**.** *Any abstract cycle system $(a; b_1, \dots, b_\ell)$ in the sense of Definition 6.4 is Hurwitz equivalent to one of the form*

$$\big(a;\, a, \dots, a,\, b + k_1 a, \dots, b + k_r a\big).$$

Moreover, for some $1 \leq i < r$ we must have $k_i - k_{i+1} \in \{1, 2, 3\}$.

By blowing-down and using Hurwitz moves to bring desired terms to the front, we have:

Corollary 6.7. *Let $f\colon (X^4, D^2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ be a relatively minimal boundary Lefschetz fibration. Then f has a cycle system of the following form, with $k \in \{1, 2, 3\}$ and $n \in \mathbb{Z}$:*

$$\big(a; b + ka, b, b + na, \dots\big).$$

6.3. **Proof of Theorem B.** To complete the proof of Theorem B, up to omitted details:

Proposition 6.8. *Let $f\colon (X^4, D^2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ be a boundary Lefschetz fibration with cycle system of the form $\big(a; b + ka, b, b + na, b_4, \dots, b_\ell\big)$ with $k \in \{1, 2, 3\}$ and $n \in \mathbb{Z}$.*

- *If $k = 1$, then f is not relatively minimal, i.e. $X = X'\#\overline{\mathbb{C}P}^2$ where X' carries a boundary Lefschetz fibration whose divisor has the same coorientability as D;*
- *If $k = 3$, then there is a ∂LF $f'\colon (X', D') \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ with one fewer Lefschetz singularity. We have $X = X'\#\mathbb{C}P^2$ and the coorientability of D' is opposite to that of D. A cycle system for f' is given by $(a; b - a, b_3, \dots, b_\ell)$.*
- *If $k = 2$, the behavior depends on the parity of n:*
  - *If n is even, there is a ∂LF $f'\colon (X', D') \to (\mathbb{D}^2, \partial\mathbb{D}^2)$ with two fewer Lefschetz singularities. We have $X = X'\#S^2 \times S^2$ and the coorientability of D' is opposite to that of D. A cycle system for f' is given by $(a; b + na, b_4, \dots, b_\ell)$;*
  - *If n is odd, the cycle system is equivalent to one with $k = 1$ or $k = 3$.*

Theorem B follows by inductively applying the above result. Finally, we remark that the steps in the previous proposition are reversible.

Corollary 6.9. *Let $(a; b_1, \dots, b_\ell)$ be a cycle system for an elliptic pair $(X, |D|)$. Passing to*

(1) *$(a; a, b_1, \dots, b_\ell)$ realizes $X\#\overline{\mathbb{C}P}^2$. The coorientability $\mathfrak{o}_D$ of the divisor is preserved;*
(2) *$(a; b_1 + 4a, b_1 + a, \dots, b_\ell)$ if $\langle a, b_1\rangle = 1$ realizes $X\#\mathbb{C}P^2$ and preserves $\mathfrak{o}_D$;*
(3) *$(a; b_1, b_1 - 2a, b_1 \dots, b_\ell)$ if $\langle a, b_1\rangle = 1$ realizes $X\#(S^2 \times S^2)$ and reverses $\mathfrak{o}_D$.*

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