ON CROSS PRODUCT HOPF ALGEBRAS

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ABSTRACT. Let $A$ and $B$ be algebras and coalgebras in a braided monoidal category $C$, and suppose that we have a cross product algebra and a cross product coalgebra structure on $A \otimes B$. We present necessary and sufficient conditions for $A \otimes B$ to be a bialgebra, and sufficient conditions for $A \otimes B$ to be a Hopf algebra. We discuss when such a cross product Hopf algebra is a double cross (co)product, a biproduct, or, more generally, a smash (co)product Hopf algebra. In each of these cases, we provide an explicit description of the associated Hopf algebra projection.

INTRODUCTION

Given algebras $A$ and $B$ in a monoidal category, and a local braiding between them, this is a morphism $\psi : B \otimes A \rightarrow A \otimes B$ satisfying four properties, we can construct a new algebra $A \#_\psi B$ with underlying object $A \otimes B$, called cross product algebra. If $C$ is braided, then the tensor product algebra and the smash product algebra are special cases. A dual construction is possible: given two coalgebras $A$ and $B$, and a morphism $\phi : A \otimes B \rightarrow B \otimes A$ satisfying appropriate conditions, we can form the cross product coalgebra $A \#^\phi B$.

Cross product bialgebras were introduced independently in [6] (in the category of vector spaces) and in [3] (in a general braided monoidal category). The construction generalizes biproduct bialgebras [13] and double cross (co)product bialgebras [9, 11]. It can be summarized easily: given algebras and coalgebras $A$ and $B$, and local braidings $\psi$ and $\phi$, we can consider $A \#_\psi B$, with underlying algebra $A \#_\psi B$ and underlying coalgebra $A \#^\phi B$. If this is a bialgebra then we call $A \#_\psi B$ a cross product bialgebra, denote it by $A \times_B^\psi B$, and say that $(A, B, \psi, \phi)$ is a bialgebra admissible tuple. Cross product bialgebras can be characterized using injections and projections, see [3, Prop. 2.2], [6, Theorem 4.3] or Proposition 7.1.

If $A \#_\psi B$ is a cross product algebra, and $A$ and $B$ are augmented, then $A$ is a left $B$-module, and $B$ is a right $A$-module. Similarly, if $A \#^\phi B$ is a cross product algebra, and $A$ and $B$ are coaugmented, then $A$ is a left $B$-comodule, and $B$ is a right $A$-comodule, we will recall these constructions in Lemmas 2.2 and 2.4. In [3], an attempt was made to characterize cross product bialgebras in terms of these actions and coactions. A Hopf datum consists of a pair of algebras and coalgebras $A$ and $B$ that act and coact on each other as above, satisfying a list of compatibility conditions, that we will refer to as the Bupslov-Drabant list [3, Def. 2.5]. If $A \#^\phi B$ is a cross product algebra, then $A$ and $B$ together with the actions and coactions from Lemmas 2.2 and 2.4 form a Hopf datum, [3, Prop. 2.7]. Conversely, if we have a Hopf datum, then we can find $\psi$ and $\phi$ such that $A \#^\phi B$ is a cross product algebra and coalgebra, but we are not able to show that it is a bialgebra, see [3, Prop. 2.6]. Roughly stated, the Bespalov-Drabant list is a list of necessary conditions but we do not know whether it is also sufficient.

The main motivation of this paper was to fill in this gap: in Sections 4 and 5, we will present some alternatives to the Bespalov-Drabant list, consisting of necessary and sufficient conditions. Our first main result is Theorem 4.6, in which we provide a set of lists of necessary and sufficient conditions,

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in terms of the local braidings $\phi$ and $\psi$. Another set, now in terms of the actions and coactions, will be given in Theorem 5.4.

As we have already mentioned, smash product algebras are special cases of cross product algebras, and they can be characterized, see Section 3. In Section 6, we first show that a cross product bialgebra is a smash cross product bialgebra if and only if $\psi$ satisfies a (left) normality condition, see Definition 6.1. In this situation, the necessary and sufficient conditions from Theorems 4.6 and 5.4 take a more elegant form. As a consequence, we obtain that any cross product bialgebra with $\psi$ left or right conormal is defined by a (trivalent) Hopf datum; in other words, in this particular situation the set of bialgebra admissible tuples is in a bijective correspondence with the set of Bespalov-Drabant Hopf data, see Theorem 6.4. We have a dual version, characterizing smash cross coproduct bialgebras (with cross product coalgebra as underlying coalgebra), from where we obtain that any trivalent cross product bialgebra is either a smash cross product or coproduct bialgebra.

Furthermore, a combination of the two versions yields a characterization of Radford's biproducts, see Corollary 6.3: a cross product bialgebra is a left/right Radford biproduct if and only if $\psi$ is left/right conormal and $\phi$ is left/right normal. In Theorem 6.4, we also present sufficient conditions for a smash cross product bialgebra to be a Hopf algebra. All these results have a left and right version; combining the left and right version, we have the following interesting application, see Corollary 6.7: a cross product bialgebra is a double cross product in the sense of Majid [9] if and only if $\phi$ is left and right normal. In this situation, $\phi$ coincides with the braiding of $A$ and $B$.

Otherwise stated: Majid's double cross product bialgebras are precisely the cross product bialgebras for which the underlying coalgebra is the cotensor coalgebra. Consequently, in the category of sets any cross product Hopf algebra is a bicrossed product of groups in the sense of [18], see Corollary 6.8. We have already mentioned that cross product bialgebras can be characterized using injections and projections. The aim of Section 7 is to study this characterization in the case of smash cross product algebras. The structure of Hopf algebras with a projection was described completely by Radford in [13]: if $H$ and $B$ are Hopf algebras, and there exist Hopf algebra maps $i : B \to H$ and $\pi : H \to B$ such that $\pi i = \text{Id}_B$, then $H$ is isomorphic to a biproduct Hopf algebra. Several generalizations of this result have appeared in the literature. In [15], the condition on $\pi$ is relaxed: if $\pi$ is a left $B$-linear coalgebra map then $H$ is isomorphic to a smash product coalgebra, with an algebra structure given by a complicated formula that does not imply in general that $H$ is isomorphic to a cross product bialgebra. The situation where $\pi$ is a right $B$-linear coalgebra morphism was studied using different methods in [2]. The situation where $\pi$ is a Hopf algebra morphism and $i$ is a coalgebra morphism is studied in [4], and the case where $\pi$ is a morphism of bialgebras and $i$ is a $B$-bicolinear algebra map is studied in [1].

With these examples in mind, we have been looking for the appropriate projection context on a Hopf algebra, that ensures that the Hopf algebra is isomorphic to a smash cross product Hopf algebra. Here the idea is the following. If $H = A \#^\psi B$ is a cross product coalgebra, then we have algebra morphisms $i, j$ and coalgebra morphisms $p, \pi$, as in [3, Prop. 2.2] and Proposition 7.1. If $H$ is a smash cross product bialgebra, then $\pi$ is a bialgebra morphism, and $(A, p, j)$ can be reconstructed from $(B, \pi, i)$: $(A, j)$ is the equalizer of a certain pair of morphisms, see Lemma 7.2. Conversely, if we have a bialgebra $B$, and a bialgebra map $\pi : H \to B$ and an algebra map $i : B \to H$ such that $\pi$ is left inverse of $i$, then we can construct $(A, j)$ as an equalizer, and show that $A$ is an algebra and a coalgebra, and $j$ has a left inverse $p$, see Lemma 7.4. The definition of the coalgebra structure on $A$ requires the fact that $B$ is a Hopf algebra. At this point, we can explain why we have to restrict attention to smash cross product Hopf algebras, that is, the case where $B$ is a bialgebra. In the general case where $H$ is a cross product Hopf algebra, and $B$ is only an algebra and a coalgebra, one could simply require the existence of a convolution inverse of $\text{Id}_B$. But this does not work, as we need in the construction that the antipode is an anti-algebra and an anti-coalgebra map. We also show that $(A, p)$ is a coequalizer. The main result is Theorem 7.5, characterizing smash cross product Hopf algebras in terms of projections. As a special case, we recover Radford’s result that $H$ can be written as a biproduct Hopf algebra if and only if we have a split Hopf algebra map $\pi : H \to B$, see Corollary 7.6. As another application, we characterize double cross coproduct
Hopf algebras in terms of projections, see Corollary 7.11. This improves [2, Theorem 2.15]. We end Section 7 with a sketch of the dual theory, characterizing smash cross coproduct Hopf algebras. In Section 8 we present some examples. We will show that the Sweedler’s Hopf algebra \( H_4 \) can be described both as a smash product and coproduct Hopf algebra. Also, the eight dimensional Hopf algebra \( A_{\ell} \) considered in [17] can be characterized as a smash cross coproduct Hopf algebra. Since \( A_{\ell} \) is not a selfdual Hopf algebra it follows that its dual yields another example of a smash cross product Hopf algebra.

1. Preliminary results

We assume that the reader is familiar with the basic theory of braided monoidal categories, and refer to [5, 7, 11] for more details. Throughout this paper, \( \mathcal{C} \) will be a braided monoidal category with tensor product \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \), unit object \( 1 \) and braiding \( c : \otimes \to \otimes \circ \tau \). Here \( \tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C} \) is the twist functor. We will assume implicitly that the monoidal category \( \mathcal{C} \) is strict, that is, the associativity and unit constraints are all identity morphisms in \( \mathcal{C} \). Our results will remain valid in arbitrary monoidal categories, since every monoidal category is monoidal equivalent to a strict one, see for example [5, 7].

For \( X, Y \in \mathcal{C} \), we write \( c_{X,Y} = \frac{X}{Y} \frac{Y}{X} \) and \( c_{X,Y}^{-1} = \frac{X}{Y} \frac{Y}{X} \). Recall that \( c \) satisfies

\[
(1.1) \quad c_{X,Y \otimes Z} = \frac{X}{Y} \frac{Y}{Z} \frac{Z}{X} \quad \text{and} \quad c_{X \otimes Y, Z} = \frac{X}{Y} \frac{Y}{Z} \frac{Z}{X},
\]

for all \( X, Y, Z \in \mathcal{C} \). The naturality of \( c \) means that \((g \otimes f)c_{M,N} = c_{U,V}(f \otimes g)\), for any \( f : M \to U \), \( g : N \to V \) in \( \mathcal{C} \). In particular, for \( T \in \mathcal{C} \) and \( \frac{X}{Y} \frac{Y}{Z} : X \otimes Y \to Z \) in \( \mathcal{C} \), we have

\[
(1.2) \quad \frac{T}{Z} \frac{X}{Y} \frac{Y}{Z} = \frac{T}{Z} \frac{X}{Y} \frac{Y}{Z} \quad \text{and} \quad \frac{X}{Y} \frac{Y}{Z} \frac{Z}{T} = \frac{X}{Y} \frac{Y}{Z} \frac{Z}{T}.
\]

In a similar way, for a morphism \( \frac{X}{Y} \frac{Y}{Z} \) between \( X \) and \( Y \otimes Z \), we have that

\[
(1.3) \quad \frac{X}{T} \frac{T}{Y} = \frac{X}{T} \frac{T}{Y} \quad \text{and} \quad \frac{T}{X} \frac{X}{Y} \frac{Y}{Z} = \frac{T}{X} \frac{X}{Y} \frac{Y}{Z}.
\]

\( c_{1,X} : 1 \otimes X = X \to X \otimes 1 = X \) and \( c_{X,1} : X \otimes 1 = X \to 1 \otimes X = X \) are equal to the identity morphism of \( \text{Id}_X = \frac{X}{X} \), see [7, Prop. XIII.1.2].

Let us now recall the notions of algebra and coalgebra in a monoidal category \( \mathcal{C} \), and of bialgebra and Hopf algebra in a braided monoidal category \( \mathcal{C} \). An algebra in \( \mathcal{C} \) is a triple \((A, m_A, \eta_A)\), where \( A \in \mathcal{C} \), and

\[
m_A = \frac{A}{A} : A \otimes A \to A \quad \text{and} \quad \eta_A = \frac{1}{A} : 1 \to A \quad \text{are morphisms in} \ \mathcal{C} \quad \text{satisfying the associativity and unit conditions} \quad m_A \circ (m_A \otimes \text{Id}_A) = m_A \circ (\text{Id}_A \otimes m_A) \quad \text{and} \quad m_A \circ (\eta_A \otimes \text{Id}_A) = m_A \circ (\text{Id}_A \otimes \eta_A) = \text{Id}_A.
\]
A coalgebra in $C$ is a triple $(B, \Delta_B, \varepsilon_B)$, where $B \in C$, and $\Delta_B = \frac{B}{B \otimes B} : B \to B \otimes B$ and $\varepsilon_B = \frac{1}{B} : B \to 1$, satisfying appropriate coassociativity and counit conditions.

A bialgebra in $C$ is a fivetuple $(B, m_B, \eta_B, \Delta_B, \varepsilon_B)$, such that $(B, m_B, \eta_B)$ is an algebra and $(B, \Delta_B, \varepsilon_B)$ is a coalgebra such that $\Delta_B : B \to B \otimes B$ and $\varepsilon_B : B \to 1$ are algebra morphisms. $B \otimes B$ has the tensor product algebra structure (using the braiding on $C$), and $1$ is an algebra, with both the multiplication and unit map equal to the identity on $1$. For later reference, we give explicit formulas for the axioms of a bialgebra $B$:

\[
\begin{align*}
\id = B_B & \quad \text{and} \quad 1 \otimes 1 = 1,
\end{align*}
\]

(1.4)

For a bialgebra $B$, we can introduce the category of left $B$-modules $B_C$ and the category of left $B$-comodules $B_C$. The left $B$-action on $X \in B_C$ is denoted by $B_X$, and the left $B$-coaction on $X \in B_C$ by $X_B$. $B_C$ and $C_B$ are monoidal categories; for $X, Y \in B_C$ (resp. $B_C$), then $X \otimes Y$ is a left $B$-module (resp. left $B$-comodule) via the action (resp. coaction)

\[
\begin{align*}
\quad & \quad 
\end{align*}
\]

A Hopf algebra in a braided monoidal category $C$ is a bialgebra $B$ in $C$ together with a morphism $S : B \to B$ in $C$ (the antipode) satisfying the axioms

\[
\begin{align*}
\id = B_B & \quad \text{and} \quad 1 \otimes 1 = 1,
\end{align*}
\]

(1.5)

It is well-known, see [10, Lemma 2.3], that the antipode $S$ of a Hopf algebra $B$ in a braided monoidal category $C$ is an anti-algebra and anti-coalgebra morphism, in the sense that

\[
\begin{align*}
\quad & \quad 
\end{align*}
\]

(1.6)
2. Cross product algebras and coalgebras

Let $A$ and $B$ be algebras and coalgebras in $C$, but not necessarily bialgebras. Consider morphisms $\psi$ and $\phi$ such that $A\#\psi B$ is a cross product algebra of $A$ and $B$ with the multiplication induced by $\psi$ and the comultiplication induced by $\phi$, with unit map $\eta_{A} \otimes \eta_{B}$; $A\#\phi B$ is a cross product coalgebra of $A$ and $B$ with the multiplication induced by $\phi$, and with counit map $\varepsilon_{A} \otimes \varepsilon_{B}$. If $A\#\psi B$ is an algebra in $C$, then we say that $A\#\psi B$ is a cross product algebra of $A$ and $B$; if $A\#\phi B$ is a coalgebra in $C$, then we say that $A\#\phi B$ is a cross product coalgebra of $A$ and $B$. $A \otimes B$ together with the multiplication induced by $\psi$, the comultiplication induced by $\phi$, unit $\eta_{A} \otimes \eta_{B}$ and counit $\varepsilon_{A} \otimes \varepsilon_{B}$ will be denoted by $A\times^{\phi}_{\psi}B$. If $A\times^{\phi}_{\psi}B$ is a bialgebra in $C$, then we call it a cross product bialgebra, and we denote it by $A \times B$.

A morphism in $\Sigma_{A}$ between $(X, \psi_{X,A})$ and $(Y, \psi_{Y,A})$ is a morphism $\mu : X \to Y$ in $C$ such that $(\text{Id}_{A} \otimes \mu) \circ \psi_{X,A} = \psi_{Y,A} \circ (\mu \otimes \text{Id}_{A})$. $\Sigma_{A}$ is a strict monoidal category, with unit object $(1, \text{Id}_{A})$ and tensor product

$$(X, \psi_{X,A}) \otimes (Y, \psi_{Y,A}) = (X \otimes Y, \psi_{X \otimes Y,A}),$$

with $\psi_{X \otimes Y,A} := (\psi_{X,A} \otimes \text{Id}_{Y})(\text{Id}_{X} \otimes \psi_{Y,A})$.

The category $\Sigma_{A}$ of left transfer morphisms through $A$ is defined in a similar way, and is also a strict monoidal category. Then we have the following result, going back to [19], see also [16, Sec. 4].

**Proposition 2.1.** Let $A$ and $B$ be algebras in a strict monoidal category $C$, and $\psi : B \otimes A \to A \otimes B$ a morphism in $C$. Then the following assertions are equivalent:

(i) $A\#\psi B$ is a cross product algebra;
(ii) $(B, \psi)$ is an algebra in $\Sigma_{A}$;
(iii) $(A, \psi)$ is an algebra in $B\Sigma$.

**Proof.** Observe $(B, \psi) \in \Sigma_{A}$ is equivalent to (2.1,b-c); if these hold, then (2.1,a) and (2.1,d) mean precisely that $(B, \psi)$ is an algebra in $\Sigma_{A}$. This proves the equivalence of (i) and (ii). The equivalence of (i) and (iii) can be proved in a similar way: (2.1,a) and (2.1,d) are equivalent to $(A, \psi) \in B\Sigma$, and then the two other conditions mean that $(A, \psi)$ is an algebra in $B\Sigma$. \[\square\]
Recall that an augmented algebra is a pair \((B, \varepsilon_B)\), where \(B\) is an algebra, and \(\varepsilon_B : B \rightarrow 1\) is an algebra morphism.

**Lemma 2.2.** Let \(A \#_\psi B\) be a cross product algebra. If \((B, \varepsilon_B)\) is an augmented algebra then \(A \in B\mathcal{C}\) via (2.2.a). If \((A, \varepsilon_A)\) is an augmented algebra then \(B \in \mathcal{C}A\) via (2.2.b).

\[
(2.2) \quad \begin{align*}
(a) \quad & \begin{array}{c}
\xymatrix{BA \ar[rr] & & BA \ar[rr] & & BA \\
A & & A & & A
}
\end{array} \quad ; \\
(b) \quad & \begin{array}{c}
\xymatrix{BA \ar[rr] & & BA \ar[rr] & & BA \\
B & & B & & B
}
\end{array}
\end{align*}
\]

**Proof.** Composing (2.1.a) and (2.1.d) to the left with \(\text{Id}_A \otimes \varepsilon_B\), we find that \(A\) is a left \(B\)-module. \(\square\)

For further reference, we record the dual results. We leave it to the reader to introduce the monoidal categories \(A\mathcal{T}\) and \(\mathcal{T}A\) of left and right transfer morphisms through the coalgebra \(A\).

**Proposition 2.3.** Let \(A\) and \(B\) be coalgebras, and let \(\phi : A \otimes B \rightarrow B \otimes A\) be a morphism in \(\mathcal{C}\).

The following statements are equivalent:

1) \(A \#_\phi B\) is a cross product coalgebra;
2) the following relations hold:

\[
(2.3) \quad \begin{align*}
(a) \quad & \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \quad = \quad \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \\
(b) \quad & \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \quad = \quad \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \\
(c) \quad & \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \quad = \quad \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \\
(d) \quad & \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \quad = \quad \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array}
\end{align*}
\]

3) \((B, \phi)\) is a coalgebra in \(A\mathcal{T}\);
4) \((A, \phi)\) is a coalgebra in \(\mathcal{T}B\).

A coaugmented coalgebra is a pair \((B, \eta_B)\), where \(B\) is a coalgebra, and \(\eta_B : 1 \rightarrow B\) is a coalgebra morphism.

**Lemma 2.4.** Assume that \(A \#_\psi B\) is a cross product coalgebra. If \((B, \eta_B)\) is a coaugmented coalgebra, then \(A \in B\mathcal{C}\) via (2.4.a). If \((A, \eta_A)\) is a coaugmented coalgebra, then \(B \in \mathcal{C}A\) via (2.4.b).

\[
(2.4) \quad \begin{align*}
(a) \quad & \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \quad = \quad \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \\
(b) \quad & \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array} \quad = \quad \begin{array}{c}
\xymatrix{AB \ar[rr] & & AB \ar[rr] & & AB \\
BA & & BA & & BA
}
\end{array}
\end{align*}
\]

### 3. Smash product algebras and coalgebras

These are particular examples of cross product algebras and coalgebras. Assume that \(B\) is a bialgebra, so that \(B\mathcal{C}\), the category of left \(B\)-modules, and \(\mathcal{B}C\), the category of left \(B\)-comodules, are monoidal categories. For an algebra \(A\) in \(B\mathcal{C}\), we have a cross product algebra \(A \#_\psi B\), with

\[
\psi = \begin{array}{c}
\xymatrix{BA \ar[rr] & & BA \ar[rr] & & BA \\
A & & A & & A
}
\end{array},
\]

where the left \(B\)-action on \(A\) is \(\begin{array}{c}
\xymatrix{BA \ar[rr] & & BA \ar[rr] & & BA \\
A & & A & & A
}
\end{array}\). This algebra is called the left smash product algebra of \(A\) and \(B\). In a similar way, for a coalgebra \(A\) in \(B\mathcal{C}\), we have a cross product coalgebra \(A \#_\phi B\), with

\[
\phi = \begin{array}{c}
\xymatrix{BA \ar[rr] & & BA \ar[rr] & & BA \\
A & & A & & A
}
\end{array},
\]

where the left \(B\)-coaction on \(A\) is \(\begin{array}{c}
\xymatrix{BA \ar[rr] & & BA \ar[rr] & & BA \\
A & & A & & A
}
\end{array}\). This coalgebra is called the left smash product coalgebra. We remark that right smash product algebras and coalgebras can be considered as well. Assume that \(B\) is a bialgebra, and that \(A \#_\psi B\) is a cross product algebra. In Proposition 3.1, we discuss when \(A \#_\psi B\) is a smash product algebra.
**Proposition 3.1.** Let $B$ be a bialgebra, and let $A$ be an algebra, and consider $\psi : B \otimes A \to A \otimes B$ such that $A \#_\psi B$ is a cross product algebra. $A \#_\psi B$ is a left smash product algebra if and only if

$$\psi = \begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array}$$

Moreover, the full subcategory $B^\Sigma$ of $B\Sigma$, with objects of the form $(X, \psi)$, where $\psi$ satisfies (3.1), with $A$ replaced by $X$, is a monoidal subcategory of $B\Sigma$ that is monoidal isomorphic to $B\mathcal{C}$.

**Proof.** Assume first that $A \#_\psi B$ is a smash product algebra. Then $A$ is an algebra in $B\mathcal{C}$ and

$$\psi = \begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array}$$

Composing (3.2) to the right with $\eta_A \otimes \text{Id}_B \otimes \eta_B$, we obtain that

$$\begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array} = \begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array}$$

and this implies (3.1).

Conversely, assume that $A \#_\psi B$ is a cross product algebra and that $\psi$ satisfies (3.1). We know that $A \in B\mathcal{C}$, with left $B$-action (2.2.a). $A$ is an algebra in $B\mathcal{C}$ since

$$\begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array} = \begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array}$$

The multiplication on the smash product algebra is

$$\begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array} = \begin{array}{c} \begin{array}{c} B \ A \\ \ \ A \ B \ \end{array} \end{array}$$

and coincides with the multiplication on the cross product algebra $A \#_\psi B$. This finishes the proof of the first statement.

We next show that $B^\Sigma$ is closed under the tensor product: if $(X, \psi_{B,X})$, $(Y, \psi_{B,Y}) \in B^\Sigma$, then
Finally, we will construct a monoidal isomorphism $F: B^\mathcal{T}' \to B^\mathcal{C}$. Take $(X, \psi) \in B^\mathcal{T}'$. In the first part of the proof, we have seen that $X \in B^\mathcal{C}$ via the $B$-action $B X \cong B Y$, and this defines $F$ at the level of objects. At the level of morphisms, $F$ acts as the identity. Now we define a functor $G: B^\mathcal{C} \to B^\mathcal{T}'$. Take a left $B$-module $X$, and let $\psi = B X \cong X B$, and therefore $\psi$ satisfies (3.1). $(X, \psi_{B,X})$ is an object of $B^\mathcal{C}$ since

We conclude that $(X, \psi_{B,X}) \in B^\mathcal{T}'$, and we define $G(X) = (X, \psi_{B,X})$. At the level of morphisms, $G$ acts as the identity. Using (3.1), we can show that $F$ and $G$ are inverses. Finally, using the coassociativity of the comultiplication on $B$ and (1.3), we can prove that

and this implies that $F$ is a strictly monoidal functor. □
We end this Section with the dual version of Proposition 3.1. Verification of the details is left to the reader.

**Proposition 3.2.** Let $B$ be a bialgebra, and let $A$ be a coalgebra. Assume that $\phi : A \otimes B \to B \otimes A$ is such that $A \#^\phi B$ is a cross product coalgebra. $A \#^\phi B$ is a smash product coalgebra if and only if

$$\phi = \begin{array}{c}
A \\
\downarrow \\
B \\
\downarrow \\
A
\end{array}$$

The full subcategory of $\Sigma^B$ consisting of objects $(X, \phi)$ satisfying (3.3), with $A$ replaced by $X$, is strictly monoidal and can be identified with $\mathcal{C}^B$ as a monoidal category.

### 4. Cross product bialgebras

Suppose that $A$ and $B$ are algebras and coalgebras, and that we have morphisms $\psi : B \otimes A \to A \otimes B$ and $\phi : A \otimes B \to B \otimes A$ such that $A \#^\psi B$ is a cross product algebra and $A \#^\phi B$ is a cross product coalgebra. Then we will call $(A, B, \psi, \phi)$ a cross product algebra-coalgebra datum. In [3, Sec. 2], $(A, B, \psi, \phi)$ is called a bialgebra admissible tuple, or a BAT, if $A \#^\psi_\phi B$ is a cross product bialgebra. Take a cross product algebra-coalgebra datum $(A, B, \psi, \phi)$. We will produce a list of properties that are satisfied if $(A, B, \psi, \phi)$ is an admissible tuple; otherwise stated, we will make a list of necessary conditions for $A \#^\psi_\phi B$ being a cross product bialgebra. The results will be summarized in Theorem 4.6.

$A \#^\psi_\phi B$ is a cross product bialgebra if and only if the comultiplication and counit are algebra maps; these conditions come down to the following equalities:

$$\begin{align*}
(a) & \quad A B A B = A B A B \\
(b) & \quad 1 r \circ 1 r = 1 r
\end{align*}$$

$$\begin{align*}
(c) & \quad 1 = 1 r \\
(d) & \quad 1 r \circ 1 r
\end{align*}$$

Note that the first composition at the left hand side of (4.1.d) is the composition of the counit and the unit of $A$, and the second one is the composition of the counit and the unit of $B$. Using (4.1.d)
and the counit conditions (2.3.c-d), we can see that (4.1.b) is equivalent to

\[
\begin{align*}
(4.2) \quad (a) & \quad \frac{1}{A A} = \frac{1}{A A}, \\
(b) & \quad \frac{1}{B B} = \frac{1}{B B}, \\
(c) & \quad \frac{1}{B A} = \frac{1}{B A}.
\end{align*}
\]

In a similar way, (4.1.c) is equivalent to

\[
\begin{align*}
(4.3) \quad (a) & \quad \frac{1}{A A} = \frac{1}{A A}, \\
(b) & \quad \frac{1}{B B} = \frac{1}{B B}, \\
(c) & \quad \frac{1}{B A} = \frac{1}{B A},
\end{align*}
\]

(4.1.d) is equivalent to \( \varepsilon_A \circ \eta_A = \text{Id}_1 = \varepsilon_B \circ \eta_B; \) this follows from the observation that both compositions are invertible idempotents of \( \text{End}_C(1) \).

We can now formulate a first list of properties of bialgebra admissible tuples.

**Proposition 4.1.** A bialgebra admissible tuple \((A, B, \psi, \phi)\) satisfies the following properties.

\[
\begin{align*}
(4.4) \quad (d) & \quad \frac{1}{A A} = \frac{1}{A A}, \\
(e) & \quad \frac{1}{B B} = \frac{1}{B B}, \\
(f) & \quad \frac{1}{B A} = \frac{1}{B A},
\end{align*}
\]

\[
\begin{align*}
(4.5) \quad (g) & \quad \frac{1}{B A} = \frac{1}{B A}.
\end{align*}
\]

**Proof.** Compose (4.1.a) to the right with \( \text{Id}_A \otimes \eta_B \otimes \text{Id}_{A \otimes B} \) and to the left with \( \text{Id}_{A \otimes B \otimes A} \otimes \varepsilon_B \). Applying (2.1.d), (4.2.b) and (4.3.b), and the fact that \( \varepsilon_B \eta_B = \text{Id}_1 \) we obtain

\[
\begin{align*}
(4.5) \quad (g) & \quad \frac{1}{B A} = \frac{1}{B A}.
\end{align*}
\]

We find (4.4.a) after we compose (4.5) to the right with \( \text{Id}_{A \otimes A} \otimes \eta_B \). Composing (4.5) to the right with \( \text{Id}_A \otimes \eta_A \otimes \text{Id}_B \) and to the left with \( \varepsilon_A \otimes \text{Id}_{B \otimes A} \), and with the help of (4.2.a) and (2.1.c), we
deduce (4.4.c). Now compose (4.1.a) to the right with $\text{Id}_{A B} \otimes \eta_A \otimes \text{Id}_B$ and to the left with $\xi_A \otimes \text{Id}_{B \otimes A \otimes B}$.

Combining the resulting equation with (2.1.c), we obtain that

\[(4.6) \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram1.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram2.png}
\end{array}.
\]

After we compose (4.6) to the right with $\eta_A \otimes \text{Id}_{B \otimes A}$, we obtain (4.4.b).

Compose (4.1.a) to the right with $\text{Id}_{A \otimes B \otimes A} \otimes \eta_B$ and to the left with $\text{Id}_A \otimes \xi_B \otimes \text{Id}_{A \otimes B}$. Then by (2.3.d), we obtain that

\[(4.7) \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram3.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram4.png}
\end{array}.
\]

Composing (4.7) to the right with $\eta_A \otimes \text{Id}_{B \otimes A}$ and to the left with $\text{Id}_{A} \otimes \xi_A \otimes \text{Id}_{B}$, and using (2.3.c) and (4.3.a), we find (4.4.g). Composing (4.7) to the left with $\text{Id}_{A \otimes B} \otimes \xi_A \otimes \text{Id}_{B}$, we find (4.4.e).

Now compose (4.1.a) to the left with $\text{Id}_{A \otimes B} \otimes \xi_A \otimes \text{Id}_{B}$ and to the right with $\eta_A \otimes \text{Id}_{B \otimes A \otimes B}$. This gives

\[(4.8) \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram5.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram6.png}
\end{array}.
\]

Composing (4.8) to the left with $\xi_A \otimes \text{Id}_{B \otimes A}$, we obtain (4.4.f). (4.4.d) follows after we compose (4.1.a) to the right with $\eta_A \otimes \text{Id}_{B \otimes A} \otimes \eta_B$, and then use (4.2.a-b). Finally, (4.4.h) follows after we compose (4.1.a) to the left with $\xi_A \otimes \text{Id}_{B \otimes A \otimes B} \otimes \xi_B$, and then use (4.3.a-b).

Observe that we could have skipped half of the proof: (4.4.e-h) follow from (4.4.a-d) using duality arguments. Applying Proposition 4.1, we find some more properties of bialgebra admissible tuples. They deserve a separate formulation for two reasons: they appear also in the Bespalov-Drabant list, and they play a key role in the formulation of Theorem 4.6.

**Corollary 4.2.** If $A \# \otimes B$ is a cross product bialgebra then the following equalities hold:

\begin{align*}
\text{algebra-coalgebra compatibility} : & \quad (a) \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram7.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram8.png}
\end{array}, \quad (b) \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram9.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram10.png}
\end{array}.
\end{align*}
(4.9) comodule-algebra compatibility : (c) 

\[
\begin{align*}
A A & = \quad B A \\
B A & = \quad A A
\end{align*}
\]

, (d) 

\[
\begin{align*}
B B & = \quad A A \\
A A & = \quad B B
\end{align*}
\]

\[
(4.9)
\]

module-coalgebra compatibility : (e) 

\[
\begin{align*}
B A & = \quad B A \\
A A & = \quad A A
\end{align*}
\]

and (f) 

\[
\begin{align*}
B A & = \quad B A \\
B B & = \quad B B
\end{align*}
\]

Proof. (4.9.a) follows after we compose (4.4.a) to the left with \(\text{Id}_A \otimes \varepsilon_B \otimes \text{Id}_A\), and (4.9.b) follows after we compose (4.4.b) to the left with \(\varepsilon_B \otimes \text{Id}_A \otimes \text{Id}_B\). In a similar way, (4.9.c) follows after we compose (4.4.a) to the left with \(\varepsilon_A \otimes \text{Id}_B \otimes \varepsilon_A\), and (4.9.d) follows after we compose (4.4.b) to the left with \(\text{Id}_B \otimes \varepsilon_A \otimes \varepsilon_B\). Finally, (4.9.e) follows after we compose (4.4.d) to the left with \(\varepsilon_A \otimes \varepsilon_B \otimes \varepsilon_A \otimes \varepsilon_B\), and (4.9.f) follows after we compose (4.4.d) to the left with \(\varepsilon_A \otimes \varepsilon_B \otimes \varepsilon_A \otimes \varepsilon_B\). Note that in all these computations, we have to use freely the relations (4.2) and (4.3), and the fact that \(\varepsilon_A \eta_A = \text{Id}_1 = \varepsilon_B \eta_B\). \(\square\)

Proposition 4.3. Let \((A, B, \psi, \phi)\) be a crossed product algebra-coalgebra datum. If (4.4.a-d) or (4.4.e-h) holds, then (4.1.a) holds, that is, the comultiplication on \(A \#_B B\) is multiplicative.

Proof. The first assertion follows from the following computation
as desired. At some steps, we used the associativity of $m_A$ and $m_B$, and the naturality of the braiding. The proof of the second one is similar, and can also be obtained by duality arguments. □

Corollary 4.4. For a cross product algebra-coalgebra datum $(A, B, \psi, \phi)$, the following assertions are equivalent:

(i) $(A, B, \psi, \phi)$ is a bialgebra admissible tuple, that is, $A \# B$ is a cross product bialgebra;
(ii) $\varepsilon_X = \eta_X = \text{Id}_X$ for $X \in \{A, B\}$, and (4.2), (4.3) and (4.4.a-d) hold;
(iii) $\varepsilon_X = \eta_X = \text{Id}_X$ for $X \in \{A, B\}$, and (4.2), (4.3) and (4.4.e-h) hold.

Corollary 4.4 is a first list of sets of necessary and sufficient conditions for a cross product algebra-coalgebra datum being a bialgebra admissible tuple. Before we can extend this list, we need another Lemma.

Lemma 4.5. Let $(A, B, \psi, \phi)$ be a cross product algebra-coalgebra datum and assume that $\varepsilon_X \eta_X = \text{Id}_X$, for $X \in \{A, B\}$, and that (4.2-4.3) hold.
(i) If (4.4.g) holds then (4.4.a) is equivalent to (4.9.a,c), and (4.4.b) is equivalent to (4.9.b,d).
(ii) If (4.4.c) holds then (4.4.e) is equivalent to (4.9.a,e), and (4.4.f) is equivalent to (4.9.b,f).

Proof. We will prove the first statement of (i), the proof of all the other assertions is similar. If (4.4.a) holds, then (4.9.a), resp. (4.9.c), follows after we compose (4.4.a) to the left with \( \text{Id}_A \otimes \varepsilon_B \otimes \text{Id}_A \), resp. \( \varepsilon_A \otimes \text{Id}_B \otimes \text{Id}_A \), see the proof of Corollary 4.2. The proof of the converse implication follows from (4.9.a), (4.9.c), (1.3), (2.3), again (1.3), and (4.4.g), we leave the verification of the details to the reader. \( \square \)

Theorem 4.6 is the main result of this Section. Two new conditions will appear, namely

\[
\text{(4.10) (a) } \begin{array}{c}
A & B & A \\
B & A & B
\end{array}
\begin{array}{c}
\oplus \\
\otimes
\end{array}
\begin{array}{c}
A & B & A \\
B & A & B
\end{array}
\text{ and (b) } \begin{array}{c}
A & B & A \\
B & A & B
\end{array}
\begin{array}{c}
\oplus \\
\otimes
\end{array}
\begin{array}{c}
B & A & B \\
B & A & B
\end{array}
\]

Theorem 4.6. Let \((A,B,\psi,\phi)\) be a cross product algebra-coalgebra datum. Then the following assertions are equivalent,
(i) \( A \#^\psi \phi B \) is a cross product bialgebra;
(ii) \( \varepsilon_X \eta_X = \text{Id}_1 \) for \( X \in \{A, B\} \), and (4.2), (4.3) and (4.4.a-d) hold;
(iii) \( \varepsilon_X \eta_X = \text{Id}_1 \) for \( X \in \{A, B\} \), and (4.2), (4.3) and (4.4.c,d,g) hold;
(iv) \( \varepsilon_X \eta_X = \text{Id}_1 \) for \( X \in \{A, B\} \), and (4.2), (4.3), (4.4.c,d,g) and (4.9.a-d) hold;
(v) \( \varepsilon_X \eta_X = \text{Id}_1 \) for \( X \in \{A, B\} \), and (4.2), (4.3), (4.4.c,g,h) and (4.9.a,b,c,f) hold;
(vi) \( \varepsilon_X \eta_X = \text{Id}_1 \) for \( X \in \{A, B\} \), and (4.2), (4.3), (4.4.c,g,h) and (4.9.a,b,c,e,f) hold;
(vii) \( \varepsilon_X \eta_X = \text{Id}_1 \) for \( X \in \{A, B\} \), and (4.2), (4.3), (4.4.c,g,h) and (4.9.a,b,c,e,f) hold.

Proof. We have already seen in Corollary 4.4 that (i), (ii) and (iii) are equivalent. The equivalences (ii) \( \Leftrightarrow \) (iv) and (iii) \( \Leftrightarrow \) (v) follow from Lemma 4.5.
(ii) \( \Rightarrow \) (vi). We start by showing that (4.10.a) holds. To this end, we compute that
as needed (note that in the last but one equality we also applied the naturality of the braiding to the morphism \( \psi \)). Furthermore, we have seen in the proof of Corollary 4.2 that (4.4.a) implies (4.9.a,c), and that (4.4.b) implies (4.9.b,d). (4.4.g) follows after we compose (4.4.d) to the left with \( \varepsilon_A \otimes \varepsilon_B \otimes \text{Id}_A \otimes B \), so we are done (vi) \( \Rightarrow \) (ii). (4.4.d) can be proved as follows:

\[
\begin{align*}
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(4.4.g) \\
(4.9.e) \\
(4.10.a) \\
(1.3) \\
\end{array} \\
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(1.3) \\
\end{array} \\
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(2.3) \\
\end{array} \\
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(4.4.g) \\
\end{array} \\
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(1.3) \\
\end{array} \\
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(2.3) \\
\end{array} \\
&\begin{array}{c}
B \ A \\
A & B \\
B & A \\
A & B \\
\end{array} \\
&\begin{array}{c}
(4.4.g) \\
\end{array} \\
\end{align*}
\]

We know from Lemma 4.5 that (4.9.a,c) imply (4.4.a), and that (4.9.b,d) imply (4.4.b). Also, it is easy to see that (4.10.a) implies (4.9.c,f).

Finally, the proof of (iii) \( \Leftrightarrow \) (vii) is similar to the proof of (ii) \( \Leftrightarrow \) (vi).

If \( C \) is a coalgebra and \( A \) is an algebra, then \( \text{Hom}_C(C,A) \) is a monoid, with the convolution \( f * g = m_A(f \otimes g)\Delta_C \) as multiplication, and unit \( \eta_C \). We will now discuss some sufficient conditions for a cross product bialgebra to be a Hopf algebra.

**Proposition 4.7.** Let \( A \times_\psi B \) be a cross product bialgebra, and assume that \( \text{Id}_A \) and \( \text{Id}_B \) are convolution invertible. Then \( A \times_\psi B \) is a Hopf algebra.

**Proof.** Let \( S \) be the convolution inverse of \( \text{Id}_A \), and \( \hat{S} \) the convolution inverse of \( \text{Id}_B \). We then claim that \( \psi(\hat{S} \otimes S)\phi \) is the antipode for \( A \times_\psi B \). Indeed, it is a left inverse of \( \text{Id}_{A \times_\psi B} \) since

\[
\begin{align*}
&\begin{array}{c}
A \ B \\
\end{array} \\
&\begin{array}{c}
(2.1) \\
(2.3) \\
\end{array} \\
&\begin{array}{c}
A \ B \\
\end{array} \\
&\begin{array}{c}
(1.5) \\
(1.3) \\
\end{array} \\
&\begin{array}{c}
A \ B \\
\end{array} \\
&\begin{array}{c}
(2.1) \\
(2.3) \\
\end{array} \\
&\begin{array}{c}
A \ B \\
\end{array} \\
&\begin{array}{c}
(1.5) \\
(1.3) \\
\end{array} \\
&\begin{array}{c}
A \ B \\
\end{array} \\
&\end{align*}
\]

The proof of the fact that it is also a right inverse of \( \text{Id}_{A \times_\psi B} \) is similar. \( \Box \)
Remark 4.8. Some of the sufficient conditions in Proposition 4.7 are also necessary. More precisely, if $A \times B$ admits $\delta$ as antipode, then (1.5) specializes to

\[
\begin{align*}
A \times B & = A \times B \\
A \times B & = A \times B
\end{align*}
\]

(4.11)

If we compose the first equality to the left with $\xi_A \otimes \text{Id}_B$ and to the right with $\eta_A \otimes \text{Id}_B$, and the second equality to the left with $\text{Id}_A \otimes \xi_B$ and to the right with $\text{Id}_A \otimes \eta_B$, we deduce that

\[
\begin{align*}
B & = \quad B \\
A & = \quad A
\end{align*}
\]

(4.12)

This means that $\text{Id}_B$ has a left inverse in $\text{Hom}(B, B)$ and that $\text{Id}_A$ has a right inverse in $\text{Hom}(A, A)$. At this moment, it remains unclear to us whether these one-sided inverses are inverses. We will see in Section 7 that this is true in the case of a smash cross (co)product Hopf algebra.

5. Cross product bialgebras and Hopf data

If $(A, B, \psi, \phi)$ is a cross product algebra-coalgebra datum, then $A$ is a left $B$-module and a left $B$-comodule, and $B$ is a right $A$-module and a right $A$-comodule, see Lemmas 2.2 and 2.4. Now we can ask the following question: suppose that $A$ and $B$ are algebras and coalgebras, that $A$ is a left $B$-module and a left $B$-comodule, and $B$ is a right $A$-module and a right $A$-comodule. Is there a list of necessary and sufficient conditions that these actions and coactions need to satisfy, so that they give rise to bialgebra admissible tuple?

This question was partially answered in [3]. In [3, Def. 2.5], a list of axioms is proposed, we call this list the Bespalov-Drabant list. If these axioms are satisfied, then $(A, B)$ is a Hopf datum. If $(A, B, \psi, \phi)$ is a bialgebra admissible tuple, then $(A, B)$, with actions and coactions given by (2.2-2.4), is a Hopf datum, see [3, Prop. 2.7]. Moreover, the - crucial - conditions (4.4.g,c) show that $\psi$ and $\phi$ can be recovered from the actions and coactions.

Conversely, given a Hopf datum, we can produce a cross product algebra-coalgebra datum $(A, B, \psi, \phi)$, but we don’t know whether it is a bialgebra admissible tuple, see [3, Prop. 2.6]. Otherwise stated, we obtain a cross product algebra and coalgebra, but we don’t know whether it is a bialgebra. We could also say the following: the Bespalov-Drabant list is necessary, but not sufficient. Using the results of Section 4, we are able to present an alternative list of necessary and sufficient conditions. Basically, this is a - technical - restatement of Theorem 4.6. The computations will turned out to be quite lengthy, and this is why we decided to divide them over several Lemmas.

Lemma 5.1. Let $A, B$ be algebras and coalgebras such that $\xi_X \circ \eta_X = \text{Id}_X$, $\xi_X \circ m_X = \xi_X \otimes \xi_X$ and $\Delta_X \circ \eta_X = \eta_X \otimes \eta_X$, for all $X \in \{A, B\}$. Furthermore, assume that $\psi : B \otimes A \rightarrow A \otimes B$ and $\phi : A \otimes B \rightarrow B \otimes A$ are morphisms in $C$ satisfying (2.1.c-d) and (2.3.c-d). Consider the following
equations:

\[ (a) \begin{array}{c}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ A \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ A \ B
\end{array}
\end{array} = \\
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
, \quad (b) \\
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\end{array} = \\
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
, \quad (c) \\
\begin{array}{c}
A \ A \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
B \ A
\end{array}
\end{array} = \\
\begin{array}{c}
A \ A \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ A \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
, \quad (d) \\
\begin{array}{c}
A \ B \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
B \ A
\end{array}
\end{array} = \\
\begin{array}{c}
A \ B \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\begin{array}{c}
B \ A
\end{array}
. \quad (5.1)

We have the following properties:

(i) (5.1.a) holds if and only if (4.4.g) and (4.9.e) hold;
(ii) (5.1.b) holds if and only if (4.4.g) and (4.9.f) hold;
(iii) (5.1.c) holds if and only if (4.4.c) and (4.9.c) hold;
(iv) (5.1.d) holds if and only if (4.4.c) and (4.9.d) hold.

Proof. We only prove (i). The proof of (ii), (iii) and (iv) is similar. Actually (iii) and (iv) follow from (i) and (ii) by duality arguments.

The direct implication in (i) follows easily by composing the given equality to the left with $\varepsilon_A \otimes \text{Id}_{A \otimes B}$, to obtain (4.4.g), and with $\text{Id}_{A \otimes A} \otimes \varepsilon_B$, to obtain (4.9.e). To prove the converse, we compute

\[ \begin{array}{c}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ A \ B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ A
\end{array}
\begin{array}{c}
A \ B \ B
\end{array}
\end{array}, \quad (5.2)
\]

as required. We used the coassociativity of $\Delta_B$ and $\Delta_A$ resp. in the first equality and in (*). \qed

Our next aim is to show that (2.1.a-b) are satisfied if (4.4.g), (4.9.a,b,e,f), and

\[ \begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (a) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}, \quad (b) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (c) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}, \quad (d) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (e) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}, \quad (f) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (g) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}, \quad (h) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (i) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}, \quad (j) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (k) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}, \quad (l) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
A
\end{array}
\end{array}, \quad (m) \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
B \ B \ A
\end{array}
\begin{array}{c}
B
\end{array}
\end{array}. \quad (5.2)\]
are satisfied. More precisely, we have the following result; we omit the proof as it is similar to the proof of Lemma 5.1.

**Lemma 5.2.** Under the same hypotheses as in Lemma 5.1, we have

- (i) (4.4.g), (4.9.b,c) and (5.2.a-c) imply (2.1.a);
- (ii) (4.4.g), (4.9.a,f) and (5.2.a,c,d) imply (2.1.b).

Without proof we present the dual version of Lemma 5.2. To this end we need the dual versions of the equations in (5.2), namely,

\[
\begin{align*}
(a) & \quad B \triangleright A = B \triangleright B A, \\
(b) & \quad A \triangleright B = A \triangleright B B A, \\
(c) & \quad B \triangleright A A = B \triangleright B A, \\
(d) & \quad A \triangleright B A = A \triangleright B B A.
\end{align*}
\]

**Lemma 5.3.** Under the same assumptions as in Lemma 5.1, we have that

- (i) (4.4.c), (4.9.b,c), and (5.3.a-c) imply (2.3.a);
- (ii) (4.4.c), (4.9.a,d), and (5.3.b-d) imply (2.3.b).

**Theorem 5.4.** Let \(A\) and \(B\) be algebras and coalgebras. There exist \(\psi : B \otimes A \rightarrow A \otimes B\) and \(\phi : A \otimes B \rightarrow B \otimes A\) such that \((A, B, \psi, \phi)\) is a bialgebra admissible tuple, that is, \(A \# \phi B\) is a cross product bialgebra if and only if the following assertions hold.

- (i) \(\varepsilon_X \circ \eta_X = 1_{\text{Id}_A}\), \(\varepsilon_X \circ \eta_X = 1_{\text{Id}_A} \otimes \varepsilon_X\) and \(\Delta_X \circ \eta_X = \eta_X \otimes \varepsilon_X\), for all \(X \in \{A, B\}\);
- (ii) \(A \in \mathcal{C}_B\) via \(\mu = \begin{array}{c}
B \\
A
\end{array}\) such that \(A \in \mathcal{C}_B\) for all \(X \in \{A, B\}\);
- (iii) \(A \in \mathcal{C}_B\) via \(\nu_1 = \begin{array}{c}
\frac{1}{A}
\end{array}\) such that \(1_B \in \mathcal{C}_B\) and \(A \in \mathcal{C}_B\);
- (iv) \(B \in \mathcal{C}_A\) via \(\mu_1 = \begin{array}{c}
\frac{1}{B}
\end{array}\) such that \(A \in \mathcal{C}_B\) and \(B \in \mathcal{C}_B\);
- (v) \(B \in \mathcal{C}_A\) via \(\nu_1 = \begin{array}{c}
\frac{1}{B}
\end{array}\) such that \(A \in \mathcal{C}_B\) and \(B \in \mathcal{C}_B\).
(vi) these actions and coactions are compatible in the sense that

\[
\begin{align*}
B A A & \rightarrow B A A \\
A & \rightarrow A \\
\end{align*}
\]

(vii) one of the four following sets of three equations is satisfied:

(vii.1) = \{(5.4), (5.5.b)\}, (vii.2) = \{(5.5.a), (5.6.b), (5.7.b)\},
(vii.3) = \{(5.6.a), (5.5.b), (5.6.b)\}, (vii.4) = \{(5.7.a), (5.4.b), (5.7.b)\}.

These four sets are equivalent if (i-vi) are satisfied.
Proof. Suppose that there exist \( \psi \) and \( \phi \) such that \( A \#_\psi B \) is a cross product bialgebra. Then \( A \) is a left \( B \)-module and \( B \)-comodule, and \( B \) is a right \( A \)-module and \( A \)-comodule, see (2.2,2.4). (ii)-(v) follow from (4.2), (4.3), (2.1.c,d), (2.3.c,d), and the fact that \( \varepsilon_X \circ \eta_X = \text{Id}_X \), for all \( X = A,B \). Now observe that it follows from (4.4.g,c) that \( \psi \) and \( \phi \) can be recovered from the actions and coactions

\[
\psi = \begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\) and \( \phi = \begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\end{array}\). Then the six formulas in (vi) are reformulations of (5.2.d), (5.3.d), (4.9.a), (5.2.b), (5.3.a) and (4.9.b). In a similar fashion, we have that

- (vii.1) is a reformulation of (4.9.c,d) and (4.4.d);
- (vii.2) is the reformulation of (4.9.e,f) and (4.4.h) in terms of actions and coactions
- (vii.3) follows from (4.9.d), (4.9.e) and (4.10.a);
- (vii.4) is a reformulation of (4.9.c), (4.9.f) and (4.10.b).

It follows from Theorem 4.6 that these sets of conditions are equivalent. Conversely, assume that \( A \) is a left \( B \)-module and \( B \)-comodule, and that \( B \) is a right \( A \)-module and \( A \)-comodule, satisfying all the conditions of the Theorem. Then we define \( \psi \) and \( \phi \) using (5.8). The actions and coactions are then given by (2.2,2.4) because of the unit-counit conditions in (ii-v). A simple verification tells us that \( \psi \) and \( \phi \) satisfy (4.4.g,c). As in the proof of the direct implication, we show that (i-vi) imply that \( \varepsilon_X \circ \eta_X = \text{Id}_X \) for \( X = A,B \), (4.2), (4.3), (2.1.c,d), (2.3.c,d), (5.2) and (5.3). In addition, the last equalities in (vii.1-vii.4) turn out to be (4.4.d), (4.4.h), (4.10.a) and (4.10.b). (4.9) follows immediately from (vi), using (5.8). It then follows from Lemmas 5.2 and 5.3 that \((A,B,\psi,\phi)\) is a cross product algebra-coalgebra datum. The result now follows from the equivalence of the conditions (iv-vii) in Theorem 4.6, verification of the details is left to the reader. \( \square \)
Let us compare the conditions in Theorem 5.4 to the Bespalov-Drabant list. Conditions (i-v) appear in the Bespalov-Drabant list. Conditions (vi) are also in the Bespalov-Drabant list, namely they are the module-algebra, the comodule-coalgebra, and the algebra-coalgebra compatibility. The remaining conditions in the Bespalov-Drabant list are the module-comodule, module-coalgebra and comodule-algebra compatibility. In order to obtain sufficient conditions, these three conditions have to be replaced by our condition (vii), which appears in four equivalent sets of three equations. Each of the four equations (5.4.a-5.7.a) can be regarded as the appropriate substitute of the module-comodule compatibility, while (5.4.b, 5.5.b) are the comodule-algebra compatibility conditions and (5.6.b, 5.7.b) are the module-coalgebra compatibility conditions in the Bespalov-Drabant list. Thus, a simple inspection shows that the set of Hopf data in our sense is contained in the set of Hopf data in the sense of Bespalov and Drabant. In the next section we will see that in some particular cases this inclusion is actually an equality. Whether this holds in general is still an open problem. We end this Section with a reformulation of Proposition 4.7 in terms of actions and coactions. The proof is left to the reader.

Proposition 5.5. Let $A \times^\phi \psi B$ be a cross product bialgebra. If $\text{Id}_A$ and $\text{Id}_B$ have convolution inverses $S$ and $s$, then $A \times^\phi \psi B$ is a Hopf algebra with antipode

$$(\mu_l \otimes \mu_r)(\text{Id}_B \otimes c_{A,B} \otimes \text{Id}_A)(\Delta_B \otimes \Delta_A)(s \otimes S)(\mu_A \otimes \mu_B)(\text{Id}_B \otimes c_{A,B} \otimes \text{Id}_A)(\nu_l \otimes \nu_r),$$

where the notations are as in the statement of Theorem 5.4.

6. Smash cross (co)product bialgebras

As a general conclusion so far, we can conclude that there are essentially three ways to describe cross product bialgebras:

(1) by bialgebra admissible tuples, these are characterized in Theorem 4.6;
(2) by actions and coactions, this is discussed in Theorem 5.4;
(3) by injections and projections, this result will be recalled in Proposition 7.1.

The second and third description are not entirely satisfactory in the following sense. As we have remarked above, the substitute of the module-comodule compatibility in Theorem 5.4 appears in four different forms, which are equivalent if some other conditions are satisfied. What is missing is a kind of unified module-comodule compatibility. The objection to the injection/projection description is that we need two algebras/coalgebras and two projections. For some classical results, see a brief survey in the introduction, one projection is sufficient.

Let $A \times^\phi \psi B$ be a cross product bialgebra. If $A \#^\psi B$ is a (left or right) smash product algebra in the sense of Proposition 3.1 (or its right handed version), then we call $A \times^\phi \psi B$ a (left or right) smash cross product bialgebra. In a similar way, if $A \#^\phi B$ is a (left or right) smash product coalgebra in the sense of Proposition 3.2, then we say that $A \times^\phi \psi B$ is a (left or right) smash cross coproduct bialgebra. In this Section, we will characterize smash cross product bialgebras and smash cross coproduct bialgebras, and we will see that the four module-comodule compatibility relations unify in this case. Moreover, they coincide with the Bespalov-Drabant module-comodule compatibility relation, so our list coincide with the Bespalov-Drabant list in the case when $\psi$ or $\phi$ is left (or right) conormal, respectively normal. Furthermore, we will also see that in one of these particular situations the bialgebra admissible tuples are in one-to-one correspondence with a set of certain trivalent Hopf data, and so all the trivalent cross product bialgebras are either a smash cross product or coproduct bialgebra. This improves [3, Theorem 2.16].

As an application, we will see that if a cross product bialgebra comes with a tensor product (co)algebra structure then it is necessarily a double cross (co)product bialgebra in the sense of Majid [9]. When we apply this result to the category of sets, then we obtain that the only cross product Hopf algebra structure is the bicross product of groups introduced by Takeuchi in [18]. We will also describe the cross product bialgebras that are a biproduct in the sense of Radford [13]. The second objection can be overcome if we restrict attention to smash cross (co)product Hopf algebras; then it turns out that one projection suffices, the other one can be recovered from it. This will be the topic of Section 7.
First we will establish that smash cross product bialgebras and smash cross coproduct bialgebras are completely determined by normality properties of the morphisms ψ and φ. This is mainly due to the crucial relations (4.9.c,g).

**Definition 6.1.** Let $A, B$ be algebras and coalgebras and $ψ : B ⊗ A → A ⊗ B$, $φ : A ⊗ B → B ⊗ A$ morphisms in $C$.

(i) $ψ$ is called left (right) conormal if

\[
\begin{array}{c}
\text{B} & \text{A} \\
\text{B} & \text{A} \end{array} = \begin{array}{c}
\text{B} & \text{A} \\
\text{B} & \text{A} \end{array}
\]

(ii) $φ$ is called left (right) normal if

\[
\begin{array}{c}
\text{B} & \text{A} \\
\text{B} & \text{A} \end{array} = \begin{array}{c}
\text{B} & \text{A} \\
\text{B} & \text{A} \end{array}
\]

**Lemma 6.2.** Let $A ×^φ B$ be a cross product bialgebra. Then:

(i) $ψ$ is left (right) conormal if and only if $A#ψ B$ is a left (right) smash product algebra, if and only if $\varepsilon_A ⊗ \text{Id}_B : A ×^φ B → B$ (Id$_A ⊗ \varepsilon_B : A ×^φ B → A$) is a bialgebra morphism.

(ii) $φ$ is left (right) normal if and only if $A#φ B$ is a left (right) smash product coalgebra, if and only if $\eta_A ⊗ \text{Id}_B : B → A ×^ψ B$ (Id$_A ⊗ \eta_B : A → A ×^ψ B$) is a bialgebra morphism.

**Proof.** Since $A ×^φ B$ is a cross product bialgebra the equalities (4.4,g,c) and (4.9.b) hold. Thus if $ψ$ is left conormal then $B$ is a bialgebra in $C$ and $ψ$ satisfies (3.1). It then follows from Proposition 3.1 that $A#ψ B$ is a smash product. Conversely, if $A#ψ B$ is a left smash product algebra, then $B$ is a bialgebra in $C$ and $ψ$ satisfies (3.1), see Proposition 3.1. Compose (3.1) to the left with $\varepsilon_A ⊗ \text{Id}_B$; using (4.3.c), it follows that $ψ$ is left conormal. The second equivalence in (i) is immediate, and the proof of the right handed version is similar. The second assertion is the dual of the first one. □

A cross product bialgebra $A ×^φ B$ is called a left (right) Radford biproduct if $A#ψ B$ is a left (right) smash product algebra and $A#φ B$ is a left (right) smash product coalgebra.

**Corollary 6.3.** A cross product bialgebra $A ×^φ B$ is a left (right) Radford biproduct if and only if $ψ$ is left (right) conormal and $φ$ is left (right) normal. If, moreover, $B$ is a Hopf algebra and Id$_A$ is convolution invertible, then $A ×^φ B$ is a Hopf algebra.

Our next aim is to describe smash cross product bialgebras. Obviously Radford biproducts are special cases, and this is why we did not provide an explicit construction of the Radford biproduct. Theorem 6.4 is a generalization of [6, Theorem 4.5], where the special case where $A$ and $B$ are bialgebras is discussed. Recall from [3, Def. and Prop. 2.15] that a Bespalov-Drabant Hopf datum is called trivalent if one of the morphisms $\varepsilon_A ⊗ \text{Id}_B$, $\eta_A ⊗ \varepsilon_B$, $\eta_A ⊗ \text{Id}_B$ or $\varepsilon_A ⊗ \eta_B$ is both an algebra and a coalgebra morphism or, equivalently, if one of the (co)actions $\mu_{l/r}$, $\nu_{l/r}$, defined in Theorem 5.4 is trivial. If this is the case then $A ×^φ B$ is called a trivalent cross product bialgebra, see [3, Def. 2.4].

**Theorem 6.4.** For a cross product algebra-coalgebra datum $(A, B, ψ, φ)$, the following assertions are equivalent:

(i) $A#ψ B$ is a cross product bialgebra and $ψ$ is left conormal.

(ii) $A#φ B$ is a left smash cross product bialgebra.

(iii) $B$ is a bialgebra in $C$, $A$ is a left $B$-module algebra and a left $B$-comodule algebra, $B$ is a right $A$-comodule and the following relations hold:
A \times \phi \psi B \text{ is a Hopf algebra in } C \text{ with antipode } \\
(\mu_l \otimes \Id_B)(\Id_B \otimes c_{B,A})(\Delta_B \otimes \Id_A)(s \otimes S)(m_B \otimes m_A)(\Id_B \otimes c_{A,B} \otimes \Id_A)(\nu_l \otimes \nu_r).

Proof. 
(i) \Leftrightarrow (ii). Follows from Lemma 6.2.

(iii) \Leftrightarrow (iv). Follows easily by comparing the conditions in (iii) with the the ones in [3, Definition 2.5], specialized for \mu_r trivial (or, equivalently, for \psi left conormal).

(ii) \Leftrightarrow (iii). A #^\psi B \text{ is a cross product bialgebra if and only if conditions (i-vi) and (vii.2) from Theorem 5.4 are fulfilled. Using the left normality of } \psi, \text{ it follows easily that these conditions reduce to condition (iii) in Theorem 6.4, with one exception: we will show that the third equality in (vii.2) is equivalent to the seventh and eighth compatibility condition in Theorem 6.4 and the fact that } A \text{ is a left } B\text{-comodule algebra. Indeed, using the left normality of } \psi, \text{ the third equality in (vii.2) takes the form}

Composing this equality to the right with \eta_A \otimes \Id_B \otimes \eta_A \otimes \Id_B, \text{ we obtain the seventh compatibility condition. Composing it to the right with } \Id_A \otimes 2_B \otimes \Id_A \otimes \eta_B, \text{ we find that the left } B\text{-coaction
on $A$ is a morphism in $\mathcal{C}$. Together with $\frac{1}{B} \otimes \frac{1}{A} = \frac{1}{A} \otimes \frac{1}{B}$, this tells us that $A$ is a left $B$-comodule algebra. Finally, composition to the right with $\eta_A \otimes \text{Id}_{B \otimes A} \otimes \eta_B$ gives the eighth compatibility condition.

The proof of the converse implication is based on a direct calculus. In the computation below we use the following properties. At $(\ast_1)$: $A$ is a left $B$-comodule algebra, and the seventh compatibility condition; at $(\ast_2)$: $\Delta_B$ is coassociative and $\mu_A$ is associative; at $(\ast_3)$ and $(\ast_8)$: $\mu_A$ and $\mu_B$ are associative; at $(\ast_4)$ and $(\ast_6)$: $\mu_B$ is associative; at $(\ast_5)$: $\Delta_B$ coassociative, and the eighth compatibility condition; at $(\ast_7)$: naturality of the braiding, $A$ is a left $B$-comodule algebra, and the fact that $\frac{1}{B} \otimes \frac{1}{A}$ is a morphism in $\mathcal{C}$. With all this in mind we have
as needed. The assertion concerning the antipode of $A \times_{\phi} B$ follows from Proposition 4.7. \hfill \Box

Obviously we also have a right handed version of Theorem 6.4.

Let $A$ and $B$ be bialgebras such that $A$ is a left $B$-comodule algebra and $B$ is a right $A$-comodule algebra. Assume that the equalities

\[ A \]
are satisfied. Recall from [11, Ch. 7] that the double cross coproduct bialgebra $A \triangleright \triangleleft B$ is the tensor product algebra $A \otimes B$ with coalgebra structure given by the formulas

$$
\Delta_{A \triangleright \triangleleft B} = A B A B \quad \text{and} \quad \varepsilon_{A \triangleright \triangleleft B} = \varepsilon_A \otimes \varepsilon_B.
$$

**Corollary 6.5.** If $A \times \triangleright \triangleleft B$ is a cross product bialgebra then $\psi$ is left and right conormal if and only if $A \times \triangleright \triangleleft B = A \triangleright \triangleleft B$ is a double cross coproduct bialgebra. If $A$ and $B$ are Hopf algebras, then $A \triangleright \triangleleft B$ is also a Hopf algebra.

**Proof.** First observe that $\psi$ is left and right conormal if and only if $\psi$ is equal to the braiding of $B$ and $A$ in $C$: one implication follows from (4.4.g), and the other one is immediate. Thus a double cross coproduct bialgebra is a cross product bialgebra with $\psi$ left and right conormal.

Conversely, assume that $A \times \triangleright \triangleleft B$ is a cross product bialgebra with $\psi$ left and right conormal, that is, $\psi$ is the braiding of $B$ and $A$ in $C$. Then (iii) in Theorem 6.4 tells us that $A$ and $B$ carry the structure that is needed to define the double cross coproduct bialgebra $A \triangleright \triangleleft B$, and that it is equal to $A \times \triangleright \triangleleft B$.

Finally, if $A$ and $B$ are Hopf algebras with antipodes $S$ and $\tilde{s}$, then $A \triangleright \triangleleft B$ is a Hopf algebra with antipode $c_{B,A}(s \otimes \tilde{S})(m_B \otimes m_A)(\text{Id}_B \otimes c_{A,B} \otimes \text{Id}_A)(\nu_l \otimes \nu_r)$. □

Now we investigate the dual situation.

**Theorem 6.6.** For a cross product algebra-coalgebra datum $(A, B, \psi, \phi)$, the following assertions are equivalent.

(i) $A \# \phi B$ is a cross product bialgebra with $\phi$ left normal.

(ii) $A \# \psi B$ is a smash cross coproduct bialgebra.

(iii) $B$ is a bialgebra, $A$ is a left $B$-comodule coalgebra and a left $B$-module coalgebra, $B$ is a right $A$-module and the following compatibility relations hold:
(iv) \((A, B, \mu, \nu, \mu_r, \nu_r)\) is a trivalent Bespalov-Drabant Hopf datum (with \(\nu_r\) trivial). If \(B\) is a Hopf algebra with antipode \(S\) and \(\text{Id}_A\) has a convolution inverse \(S\), then

\[
(\mu \otimes \mu_r)(\text{Id}_B \otimes c_{B,A} \otimes \text{Id}_A)(\Delta_B \otimes \Delta_A)(S \otimes S)(m_B \otimes \text{Id}_A)(\text{Id}_B \otimes c_{A,B})(\nu_l \otimes \text{Id}_B)
\]

makes \(A \times_{\phi}^\circ B\) a Hopf algebra in \(\mathcal{C}\).

**Proof.** We omit the proof, as it is merely a dual version of the proof of Theorem 6.4. Let us just mention that the left normality of \(\phi\) implies that the conditions (i-vi) and (vii.1) in Theorem 5.4 are equivalent to the eight compatibility conditions in the present Theorem.

We invite the reader to state the right handed version of Theorem 6.6. Combining the left and right handed versions of Theorem 6.6, we can characterize cross product bialgebras having the property \(\phi\) is left and right normal. For the definition of a matched pair of bialgebras and the double cross product bialgebra associated to it we invite the reader to consult \([7, 11]\).

**Corollary 6.7.** If \(A \times_{\phi}^\circ B\) is a cross product bialgebra then \(\phi\) is left and right normal if and only if \((A, B)\) is a right-left matched pair and \(A \times_{\phi}^\circ B = A \times B\), the double cross product bialgebra associated to \((A, B)\). If \(A\) and \(B\) are Hopf algebras, then \(A \times B\) is also a Hopf algebra, with antipode

\[
(\mu_l \otimes \mu_r)(\text{Id}_B \otimes c_{B,A} \otimes \text{Id}_A)(\Delta_B \otimes \Delta_A)(S \otimes S)(m_B \otimes \text{Id}_A)(\text{Id}_B \otimes c_{A,B})(\nu_l \otimes \text{Id}_B).
\]

**Proof.** It can be easily seen from (4.4.c) that \(\phi\) is left and right normal if and only if it is equal to the braiding of \(A\) and \(B\). The rest of the proof is then similar to the proof of Corollary 6.5. We obtain relations that tell us that \((A, B)\) is a right-left matched pair. Moreover, \(A \times_{\phi}^\circ B\) is the tensor product coalgebra, and \(A \times_{\phi}^\circ B\) is a double cross product bialgebra.

We refer to \([7, 18]\) for detail on the bicross product of two groups.

**Corollary 6.8.** A cross product Hopf algebra in the category of sets is a bicross product of two groups.

**Proof.** It is well-known that an algebra in \(\text{Sets}\) is a monoid, and that any set \(X\) has a unique coalgebra structure given by the comultiplication \(\Delta_X(x) = (x, x)\), for all \(x \in X\), and the counit \(S_X = *\), where the singleton \(*\) is the unit object of the monoidal category \(\text{Sets}\). In this way any monoid \(M\) is a bialgebra in \(\text{Sets}\) and it is, moreover, a Hopf algebra if and only if \(M\) is a group. Consequently, the only cross coproduct in \(\text{Sets}\) is the tensor product coalgebra, and the statement then follows from Corollary 6.7.

7. THE STRUCTURE OF A HOPF ALGEBRA WITH AN APPROPRIATE PROJECTION

As we have already mentioned several times, cross product bialgebras can be characterized using injections and projections. We now recall this classical result, see \([3, \text{Prop. 2.2}]\) or \([6, \text{Theorem 4.3}]\).

**Proposition 7.1.** For a bialgebra \(H\), the following statements are equivalent:

(i) \(H\) is isomorphic to a cross product bialgebra;

(ii) There exist algebras and coalgebras \(A, B\) and morphisms \(i : B \to H, \pi : H \to B, j : A \to H, p : H \to A\) such that

- \(i, j\) are algebra morphisms, \(p, \pi\) are coalgebra morphisms and \(pj = \text{Id}_A\) and \(\pi i = \text{Id}_B\);
- \(\zeta = m_H(j \otimes i) : A \otimes B \to H\) is an isomorphism in \(\mathcal{C}\) with inverse \(\zeta^{-1} = (p \otimes \pi)\Delta_H : H \to A \otimes B\).
Proof. For the complete proof, we refer to [3]. For later reference, we give a brief sketch of the proof of (ii) ⇒ (i). ψ and φ are defined by the formulas

\[
\psi = \begin{array}{c}
B \\
A
\end{array} \begin{array}{c}
A \\
B
\end{array}
\text{ and } \phi = \begin{array}{c}
A \\
B
\end{array} \begin{array}{c}
B \\
A
\end{array}.
\]

Then we show that \( A\#_\psi B \) is a cross product bialgebra, and that \( \zeta \) is an isomorphism of bialgebras. \( \square \)

In Proposition 7.1, we need two data, namely \((A, p, j)\) and \((B, \pi, i)\). We will see that one of the two data can be recovered from the other one if some additional conditions are satisfied.

**Lemma 7.2.** Let \( H = A\#_\psi B \) be a (left) smash cross product bialgebra and \( \pi = \varepsilon_A \otimes \text{Id}_B \) and \( i = \eta_A \otimes \text{Id}_B \) the canonical morphisms. Then the following assertions hold.

(i) \( \pi \) is a bialgebra morphism, \( i \) is an algebra morphism, \( \pi i = \text{Id}_B \) and

\[
\begin{array}{c}
B \\
H
\end{array} \begin{array}{c}
B \\
H
\end{array} = \begin{array}{c}
B \\
H
\end{array} \begin{array}{c}
B \\
H
\end{array} \quad \text{and} \quad \begin{array}{c}
B \\
H
\end{array} \begin{array}{c}
B \\
H
\end{array} = \begin{array}{c}
B \\
H
\end{array} \begin{array}{c}
B \\
H
\end{array}.
\]

For (7.2.b), we need the additional assumption that \( B \) is a Hopf algebra, with antipode \( s \).

(ii) If \( H \) is a Hopf algebra with antipode \( S \) then \( B \) is a Hopf algebra with antipode \( \overline{S} \) defined in (7.3.a), satisfying (7.3.b).

\[
\begin{array}{c}
A \\
B
\end{array} = \begin{array}{c}
A \\
B
\end{array} \quad \text{and} \quad \begin{array}{c}
B \\
H
\end{array} = \begin{array}{c}
B \\
H
\end{array}.
\]

Proof. The proof of (i) is straightforward, and is left to the reader. Observe that the conormality of \( \psi \) is needed in order to show that \( \pi \) is a bialgebra morphism, but is not needed in the proof of (7.2). (7.2.a) tells us that \( i : B \to H \) is right \( B \)-colinear. Here \( H \in C_B \) via \( \pi \circ \Delta_H \) and \( B \in C_B \) via \( \Delta_B \).

We will only prove that the morphism \( s \) as defined in (7.3) is antipode for \( B \). We have seen in Remark 4.8 that \( \text{Id}_B \) has always a left convolution inverse. We prove that it also has a right inverse. Compose (4.11.a) to the left with \( \varepsilon_A \otimes \text{Id}_B \) and to the right with \( \text{Id}_A \otimes \eta_B \). Using the left conormality of \( \psi \) we obtain that \( \begin{array}{c}
A \\
B
\end{array} = \begin{array}{c}
A \\
B
\end{array} \). Now compose (4.11.b) to the left with \( \varepsilon_A \otimes \text{Id}_B \) and to the right

\[
\begin{array}{c}
B \\
H
\end{array} = \begin{array}{c}
B \\
H
\end{array}.
\]
with \( \eta_A \otimes \text{Id}_B \). Again using the left conormality of \( \psi \), we now find that

\[
\begin{array}{c}
\text{(1.6)} \\
\end{array}
\]

This shows that \( \xi \), as defined in (7.3.a), is a right inverse for \( \text{Id}_B \) in \( \text{Hom}(B, B) \).

In Theorem 7.5 we will show that Lemma 7.2 has a converse, at least if some additional technical assumptions are satisfied. In the sequel, we assume that \( B \) is a Hopf algebra, \( H \) is a bialgebra and \( B \xrightarrow{\pi} H \) are morphisms in \( C \) such that \( \pi \) is a bialgebra morphism, \( i \) is an algebra morphism, \( \pi i = \text{Id}_B \) and (7.2.a,b) hold. At some places, we will consider the situation where \( H \) is also a Hopf algebra, and then we will assume that (7.3.b) holds as well. In addition, we assume that \((\text{Id}_H \otimes \eta_H)\Delta_H, \text{Id}_H \otimes \eta_B : H \to H \otimes B \) have a equalizer in \( C \). This means that there exists \( A \in C \) and \( j : A \to H \) such that \( \pi \) is an algebra morphism, \( \text{Id} \) and \( (\text{Id}_H \otimes \pi)\Delta_H, \text{Id}_H \otimes \eta_B : H \to H \otimes B \) have an algebra structure such that \( j \) is a right inverse for \( \text{Id}_B \) and \( j \) is a coalgebra morphism.

\[\begin{array}{c}
\text{(7.4)} \\
\end{array}\]

\[\begin{array}{c}
\text{(7.5)} \\
\end{array}\]

Lemma 7.3. Let \( B, H, \pi, i \) be as above. Then \( A \) has an algebra structure such that \( j : A \to H \) is an algebra morphism.

Proof. Applying the universal property of the equalizer \((A, j)\), we find unique morphisms \( m_A : A \otimes A \to A \) and \( \eta : 1 \to A \) morphisms in \( C \) making the diagrams

\[
\begin{array}{c}
\end{array}
\]

commutative, which means that \( jm_A = m_H(j \otimes j) \) and \( j\eta_A = \eta_H \). Furthermore, a simple inspection shows that \( jm_A \) is associative and \( j\eta_A \) has the unit property. This shows that \( A \) is an algebra and \( j \) is an algebra morphism.

The next step is more complicated, and consists in proving that \( A \) also has a coalgebra structure. We will need an extra assumption, namely that \( A \in C \) is flat: \(- \otimes A \) and \( A \otimes - \) preserve equalizers. Actually, we need that \( \text{Id}_A \otimes j \) and \( j \otimes \text{Id}_A \) are monomorphisms, in order to obtain that \( j \otimes j \) is a monomorphism.

Lemma 7.4. Let \( A, B, H, \pi, i, j \) be as above, and consider \( \bar{p} = \eta_H((\text{Id}_H \otimes \text{Id}_A)\Delta_H) : H \to H \). Then there exists a morphism \( p : H \to A \) such that \( jp = \bar{p} \) and \( pj = \text{Id}_A \). Furthermore, \( A \) is a coalgebra, and \( p \) is a coalgebra morphism.

If \( H \) is a Hopf algebra with antipode \( \bar{S} \), such that (7.3.b) is fulfilled, then \( \text{Id}_A \) is convolution invertible.
Proof. We compute that

\[
\begin{array}{cccc}
H & \xrightarrow{(1.4)} & H & \xleftarrow{(1.5)} H \\
\xrightarrow{(7.2.a)} & \xrightarrow{(1.6)} & \xrightarrow{(1.2)} & \xrightarrow{(7.6)} \\
H & \xleftarrow{H} & H & \xrightarrow{H} \\
B & \xrightarrow{B} & B & \xleftarrow{B}
\end{array}
\]

It follows from the universal property of the equalizer \((A, j)\) that there exists a unique morphism \(p : H \to A\) such that \(jp = \tilde{p}\). Then \(jp\tilde{p} = \tilde{pj}\). From (7.4), we deduce that \(\tilde{pj} = j\), so \(jp\tilde{p} = j\), hence \(p\tilde{p} = \text{Id}_A\).

Now we construct the coalgebra structure on \(A\). We claim that \((A, \tilde{p})\) is the coequalizer of \(m_H(\text{Id}_H \otimes i)\), \(\text{Id}_H \otimes \varepsilon_B : H \otimes B \to B\). First of all, we have that

\[
jp(m_H(\text{Id}_H \otimes i)) = \tilde{p}(m_H(\text{Id}_H \otimes i)) = \tilde{p}(\text{Id}_H \otimes \varepsilon_B) = jp(\text{Id}_H \otimes \varepsilon_B),
\]

since

\[
\begin{array}{cccc}
H & \xrightarrow{(1.4)} & H & \xleftarrow{(1.5)} H \\
\xrightarrow{(7.2.a)} & \xrightarrow{(1.6)} & \xrightarrow{(1.3)} & \xrightarrow{(7.6)} \\
H & \xleftarrow{H} & H & \xrightarrow{H} \\
B & \xrightarrow{B} & B & \xleftarrow{B}
\end{array}
\]

Secondly, we need to prove the universal property. Assume that \(\tilde{f} : H \to X\) is such that \(\tilde{f}m_H(\text{Id}_H \otimes i) = f(\text{Id}_H \otimes \varepsilon_B)\). We have to show that there is a unique \(f : A \to X\) such that \(fp = \tilde{f}\). If \(f\) exists, then \(f = f\text{Id}_A = fpj = \tilde{f}j\), hence \(f\) is unique. To prove the existence, let \(f = \tilde{f}j\), then

\[
f\tilde{p} = \tilde{fp} = \tilde{f}\tilde{p} = \tilde{f}m_H(\text{Id}_H \otimes i)(\text{Id}_H \otimes \varepsilon_B)\Delta_H = \tilde{f}(\text{Id}_H \otimes \varepsilon_B\varepsilon)\Delta_H = \tilde{f},
\]
as required. Now we use the universal property of the coequalizer to construct the comultiplication on \(A\). First observe that

\[
\begin{array}{cccc}
\tilde{p}i & = & \tilde{p}i & = \\
\xrightarrow{(7.2.a)} & \xrightarrow{(1.5)} & \xrightarrow{B} & \xrightarrow{B} \\
H & \xrightarrow{H} & H & \xrightarrow{H} \\
B & \xrightarrow{B} & B & \xrightarrow{B}
\end{array}
\]
$\tilde{f} = (p \otimes p)\Delta_H : H \rightarrow A \otimes A$ satisfies the equality

$$(j \otimes j)\tilde{f}_m^H(Id_H \otimes i) = (\tilde{p} \otimes \tilde{p})\Delta_Hm^H(Id_H \otimes i) = \tilde{f}(Id_H \otimes \varepsilon_B).$$

where we freely used associativity and coassociativity of the multiplications and comultiplications that are involved, and the fact that $\pi$ is a bialgebra morphism and $i$ is an algebra morphism. We have shown that

$$(j \otimes j)\tilde{f}_m^H(Id_H \otimes i) = (\tilde{p} \otimes \tilde{p})\Delta_Hm^H(Id_H \otimes \varepsilon_B) = \tilde{f}(Id_H \otimes \varepsilon_B).$$

Now $j \otimes j$ is a monomorphism in $C$, see the notes preceding the Lemma, and it follows that $\tilde{f}_m^H(Id_H \otimes i) = \tilde{f}(Id_H \otimes \varepsilon_B)$. Applying the universal property of the coequalizer $(A, p)$, we find a unique morphism $\Delta_A : A \rightarrow A \otimes A$ such that $\Delta_Ap = (p \otimes p)\Delta_H$. Arguments dual to those presented in the proof of Lemma 7.3 show that $\Delta_A$ is coassociative.

Finally, $\tilde{f} = \varepsilon_A : H \rightarrow 1$ satisfies $\tilde{f}_m^H(Id_H \otimes i) = \tilde{f}(Id_H \otimes \varepsilon_B)$. Applying the universal property again, we find a unique morphism $\varepsilon_A : A \rightarrow 1$ such that $\varepsilon_Ap = \varepsilon_A$. It is immediate that $\varepsilon_A$ is a counit for $\Delta_A$, and hence $(A, \Delta_A, \varepsilon_A)$ is a coalgebra in $C$. The construction of $\Delta_A$ and $\varepsilon_A$ is such that $p$ is a coalgebra morphism, and this finishes the proof of the first statement.

Applying the formulas that we obtained above, we easily see that $\Delta_A = \Delta_Ap = (p \otimes p)\Delta_Hj$. 
Furthermore

\[ A H = \]

Now assume that \( H \) is a Hopf algebra with antipode \( S \), and that (7.3.b) is fulfilled. \( \tilde{f} = m_H(i\pi \otimes S)\Delta_H : H \to H \) satisfies the equation

\[ (\text{Id}_H \otimes \pi)\Delta_H \tilde{f} = \]

Applying the universal property of the equalizer \((A, j)\), we obtain a unique morphism \( \tilde{S}_A : H \to A \) such that \( j\tilde{S}_A = \tilde{f} \). We will show that \( S = \tilde{S}_A j \) is the convolution inverse of \( \text{Id}_A \).
The first equality in (1.5) now follows from the fact that \( j \) is a monomorphism. The second one follows in a similar way by using (7.8) and (1.5) twice, the details are left to the reader.

**Theorem 7.5.** Let \( \mathcal{C} \) be a braided monoidal category with equalizers, such that every object is flat. For a Hopf algebra \( B \) in \( \mathcal{C} \), there is a one-to-one correspondence modulo isomorphism (in fact, a category equivalence) between

1. the triples \((A, \psi, \phi)\) such that \((A, B, \psi, \phi)\) is a bialgebra admissible tuple, \( A \times_{\psi} B \) has an antipode, and \( \psi \) is left conormal, and
2. the triples \((H, i, \pi)\) consisting of a Hopf algebra \( H \), an algebra morphism \( i : B \rightarrow H \) and a Hopf algebra morphism \( \pi : H \rightarrow B \) satisfying \( \pi i = \text{Id}_B \) and conditions (7.2) and (7.3.b).

In particular, if \((H, i, \pi)\) is a triple of type (b), then \( H \) is isomorphic to a (left) smash cross product Hopf algebra.

**Proof.** Given a triple \((A, \psi, \phi)\) of type (a), we construct a triple \((H, i, \pi)\) of type (b) using Lemma 7.2. Conversely, let \((H, i, \pi)\) be a triple of type (b). Let \((A, j)\) be the equalizer of \((\text{Id}_H \otimes \pi) \Delta_H \) and \( \text{Id}_H \otimes \eta_B : H \rightarrow H \otimes B \). It follows from Lemmas 7.3 and 7.4 that \( A \) is an algebra and a coalgebra, and that there is an algebra morphism \( j : A \rightarrow H \) and a coalgebra morphism \( p : H \rightarrow A \) such that \( pj = \text{Id}_A \). We claim that \( \zeta = \rho_H(j \otimes i) : A \otimes B \rightarrow H \) and \( \zeta^{-1} = (p \otimes \pi) \Delta_H : H \rightarrow A \otimes B \) are inverses. Indeed, \( \zeta^{-1} \zeta = \text{Id}_{A \otimes B} \). We also have that

\[
\zeta \zeta^{-1} = \begin{array}{c}
H \\
(p \otimes i) \Delta_H
\end{array} = \text{Id}_H.
\]

Let \( \psi \) and \( \phi \) be defined by (7.1). We apply the implication \((ii) \Rightarrow (i)\) in Proposition 7.1, and obtain

that \( \zeta^{-1} : H \rightarrow A \times_{\psi} B \) is a bialgebra isomorphism. Now

\[
\begin{array}{c}
B \\
A \\
B
\end{array} = \begin{array}{c}
B \\
A \\
B
\end{array} = \begin{array}{c}
B \\
A \\
B
\end{array} \quad \text{(7.4)} = \begin{array}{c}
B \\
A \\
B
\end{array} \quad \text{(1.5)} = j \otimes \text{Id}_B,
\]

and this implies that \( \zeta^{-1} \zeta = \text{Id}_{A \otimes B} \). We also have that

\[
\zeta \zeta^{-1} = \begin{array}{c}
H \\
(p \otimes i) \Delta_H
\end{array} = \text{Id}_H.
\]

Finally, according to Lemma 7.4 \( \text{Id}_A \) is convolution invertible. Together with the fact that \( B \) is a
Hopf algebra, this implies that $A \times_{\psi} B$ is a Hopf algebra, see Proposition 4.7, and that $\zeta^{-1}$ is a Hopf algebra isomorphism. We conclude that $(A, \psi, \phi)$ is a triple of type (a).

Clearly these two constructions are inverse to each other, up to isomorphism. □

We leave it to the reader to formulate the right handed version of Theorem 7.5.

Recall that a biproduct Hopf algebra $A \times_{\psi} B$ is a left smash cross product Hopf algebra for which $\phi$ is left normal.

Corollary 7.6. [13] Let $C$ be a braided monoidal category with equalizers, such that every object is flat. For a Hopf algebra $B$ in $C$, there is a one-to-one correspondence modulo isomorphism (in fact, a category equivalence) between

(a1) the triples $(A, \psi, \phi)$ such that $(A, B, \psi, \phi)$ is a bialgebra admissible tuple, $A \times_{\psi} B$ has an antipode, $\psi$ is left conormal, $\phi$ is left normal and

(b1) the triples $(H, i, \pi)$ consisting of a Hopf algebra $H$, and Hopf algebra morphisms $i : B \to H$ and $\pi : H \to B$ satisfying $\pi i = \text{Id}_B$.

In particular, if $(H, i, \pi)$ is a triple of type (b1), then $H$ is isomorphic to a biproduct Hopf algebra.

Proof. If $(A, \psi, \phi)$ is a triple of type (a1), then it is of type (a) as in Theorem 7.5, and we have $(H, i, \pi)$ of type (b). From the left normality of $\phi$, it is easy to see that $i$ is a coalgebra morphism, and therefore a Hopf algebra morphism, so that $(H, i, \pi)$ is of type (b1).

Conversely, if $(H, i, \pi)$ is of type (b1), then it can be checked easily that (7.2) and (7.3.b) are satisfied. Thus $(H, i, \pi)$ is of type (b), and Theorem 7.5 gives us $(A, \psi, \phi)$ such that $H \cong A \times_{\psi} B$.

It follows from (7.1) that

We conclude that $\phi$ is left normal, and $(A, \psi, \phi)$ is of type (a1). Then $H$ is isomorphic to the biproduct Hopf algebra $A \times_{\psi} B$. □

Now we focus attention to double cross coproduct Hopf algebras. Recall that $X \in C$ is called right (left) coflat if $X \otimes -$ (resp. $- \otimes X$) preserves coequalizers. $X$ is coflat if it is left and right coflat.

Also recall that a double cross coproduct Hopf algebra is a cross product Hopf algebra for which $\psi$ is left and right conormal.

Corollary 7.7. [9] Let $C$ be a braided monoidal category with equalizers, such that every object in $C$ is flat and right coflat. For a Hopf algebra $B$ in $C$, there is a one-to-one correspondence modulo isomorphism (in fact, a category equivalence) between

(a2) the triples $(A, \psi, \phi)$ such that $(A, B, \psi, \phi)$ is a bialgebra admissible tuple, $A \times_{\psi} B$ has an antipode, $\psi$ is left and right conormal, and
(b2) the triples \((H, i, \pi)\) of type (b) such that (7.9) holds, that is, the left adjoint action of \(B\) on \(H\) induced by ~\(i\) is trivial on the image of \(\tilde{p}\).

In particular, if \((H, i, \pi)\) is a triple of type (b2), then \(H\) is isomorphic to a double cross coproduct Hopf algebra.

Proof. A triple \((A, \psi, \phi)\) of type (a2) is automatically of type (a); Theorem 7.5 produces a triple \((H, i, \pi)\) of type (b). Since \(\psi\) is right conormal as well an immediate calculation shows that it satisfies (7.9), so that it is of type (b2).

Conversely, consider a triple \((H, i, \pi)\) of type (b2). Applying Theorem 7.5, we obtain a triple \((A, \psi, \phi)\) of type (a). We now show that \(\psi\) is right conormal. Indeed, we have

The right coflatness of \(B\) implies that \(\text{Id}_{B} \otimes \hat{p}\) is an epimorphism, and then the right conormality of \(\psi\) follows from the fact that \(j\) is a monomorphism. Then it follows that \((A, \psi, \phi)\) is a triple of type (a2).

To make our story complete, we present the dual version of Theorem 7.5. We need some Lemmas first. Most of the proofs are omitted, as they are dual versions of proofs that we presented above.

**Lemma 7.8.** Let \(H = A \times_{\phi}^{\psi} B\) be a (left) smash cross coproduct bialgebra and let \(\pi : H \rightarrow B\) and \(i : B \rightarrow H\) be the canonical morphisms.
(i) If $B$ is a Hopf algebra with antipode $\varepsilon$, then $\pi$ is a coalgebra morphism, $i$ is a bialgebra morphism, $\pi i = \text{Id}_B$ and

\[
\begin{array}{cccc}
H & B & H & B \\
\downarrow & \uparrow & \downarrow & \uparrow \\
\varepsilon & i & \varepsilon & i
\end{array}
\]

and

\[
\begin{array}{cccc}
H & H & H & H \\
\downarrow & \uparrow & \downarrow & \uparrow \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{array}
\]

(ii) If $H$ is a Hopf algebra with antipode $S$ then $B$ is also a Hopf algebra and

\[
\begin{array}{cccc}
H & H & H & H \\
\downarrow & \uparrow & \downarrow & \uparrow \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{array}
\]

For the converse of Lemma 7.8, we need additional assumptions: $C$ has coequalizers, and every object of $C$ is coflat, that is, it is left and right coflat.

Lemma 7.9. Let $H$ be a bialgebra and let $B$ be a Hopf algebra with antipode $\varepsilon$ and suppose that we have a bialgebra morphism $i : B \to H$ and a coalgebra morphism $\pi : H \to B$ such that $\pi i = \text{Id}_B$ and (7.10) holds. Let $(A, p)$ be the coequalizer of $m_H(\text{Id}_H \otimes i)$, $\text{Id}_H \otimes \varepsilon_B : H \otimes B \to H$. Then we have the following results.

(i) $A$ has a coalgebra structure such that $p : H \to A$ is a coalgebra morphism in $C$.

(ii) $(A, j)$ is the equalizer of $\text{Id}_H \otimes \pi) \Delta_H$, $\text{Id}_H \otimes \eta_B : H \to H \otimes B$, where $\tilde{j} = m_H(\text{Id}_H \otimes i \varepsilon \pi) \Delta_H$ and $j$ is defined by commutativity of the diagram

\[
\begin{array}{cccc}
H \otimes B & m_H(\text{Id}_H \otimes i) & H & A \\
m_H(\text{Id}_H \otimes \varepsilon) & \downarrow & \downarrow & \downarrow \\
\text{Id}_H \otimes \varepsilon_B & \downarrow & \downarrow & p \\
\text{Id}_H \otimes \varepsilon_B & \downarrow & \downarrow & \downarrow \\
H & H & H & \tilde{S}_A
\end{array}
\]

Consequently $A$ is an algebra and $j : A \to H$ is an algebra morphism.

(iii) If $H$ is a Hopf algebra with antipode $S$ satisfying (7.11), then $\text{Id}_A$ is convolution invertible.

Proof. (i) follows from Lemma 7.3 by duality arguments. (ii) We just mention that the algebra structure on $A$ is obtained using (7.5).

(iii) Applying the universal property of the coequalizer $(A, p)$, we find $\tilde{S}_A : A \to H$ such that the diagram

\[
\begin{array}{cccc}
H \otimes B & m_H(\text{Id}_H \otimes i) & H & A \\
m_H(\text{Id}_H \otimes \varepsilon) & \downarrow & \downarrow & \downarrow \\
\text{Id}_H \otimes \varepsilon_B & \downarrow & \downarrow & \tilde{S}_A \\
\text{Id}_H \otimes \varepsilon_B & \downarrow & \downarrow & \downarrow \\
H & H & H & \text{Id}_A
\end{array}
\]

commutes. It is straightforward to show that $\tilde{S} = p \tilde{S}_A$ is the convolution inverse of $\text{Id}_A$. \qed

Theorem 7.10 is the dual version of Theorem 7.5. The proof is based on Lemma 7.9 and is omitted.
**Theorem 7.10.** Let $C$ be a braided monoidal category with coequalizers, such that every object in $C$ is coflat. For a Hopf algebra $B$ in $C$, there is a one-to-one correspondence modulo isomorphism (in fact, a category equivalence) between

- (c) the triples $(A,\psi,\phi)$ such that $(A,B,\psi,\phi)$ is a bialgebra admissible tuple, $A \times^\phi B$ has an antipode, and $\phi$ is left normal, and
- (d) the triples $(H,i,\pi)$ consisting of a Hopf algebra $H$, a Hopf algebra morphism $i : B \to H$ and a coalgebra morphism $\pi : H \to B$ satisfying $\pi i = \text{Id}_B$ and conditions (7.10-7.11).

In particular, if $(H,i,\pi)$ is a triple of type (b), then $H$ is isomorphic to a (left) smash cross coproduct Hopf algebra.

A triple $(A,\psi,\phi)$ of type (c) with the additional property that $\psi$ is left conormal is precisely a triple of type (b1), as in Corollary 7.11. In fact, using the (dual versions of) arguments in the proof of Corollary 7.11, we recover Corollary 7.6. This is due to the fact that $A$ can be defined as an equalizer or as a coequalizer. In a similar way, we can look at triples $(A,\psi,\phi)$ of type (c) with the additional property that $\phi$ is right normal, too. Proceeding as in the proof of Corollary 7.7, we obtain Corollary 7.11, giving a characterization of double cross product Hopf algebras. Recall that a double cross product Hopf algebra is a cross product Hopf algebra for which $\phi$ is left and right normal. We omit the proof, let us just mention that (7.12) tells us that the left coadjoint coaction of $B$ on $H$ induced by $\pi$ is trivial on the image of $\tilde{j} = jp$.

**Corollary 7.11.** Let $C$ be a braided monoidal category with coequalizers, such that every object in $C$ is coflat. For a Hopf algebra $B$ in $C$, there is a one-to-one correspondence modulo isomorphism (in fact, a category equivalence) between

- (c2) the triples $(A,\psi,\phi)$ such that $(A,B,\psi,\phi)$ is a bialgebra admissible tuple, $A \times^\phi B$ has an antipode, and $\phi$ is left and right normal, and
- (d2) the triples $(H,i,\pi)$ of type (d) satisfying

$$\require{amssymb}
\begin{align*}
H & \quad \xrightarrow{\pi} \quad B \\
B & \quad \xrightarrow{i} \quad H
\end{align*}
$$

(7.12)

In particular, if $(H,i,\pi)$ is a triple of type (d2), then $H$ is isomorphic to a double cross product Hopf algebra.

8. **Examples of smash cross (co)product Hopf algebras**

In this Section, we present some examples of smash product and coproduct Hopf algebras. In general this is not an easy task, mostly because the triple $(H,i,\pi)$ should satisfy the restrictive conditions in Theorem 7.5, respectively the ones in Theorem 7.10. Nevertheless a class of triples $(H,i,\pi)$ with $H$ a Hopf algebra, $i$ a Hopf algebra morphism and $\pi$ a coalgebra morphism satisfying the first condition in (7.10) and such that $\pi i = \text{Id}_B$, for a certain Hopf algebra $B$, is given by Hopf algebras having a Hopf algebra coradical filtration. More precisely, if the coradical $B$ of $H$ is a Hopf algebra then by [8, Theorem 3.1] or [1, Corollary 2.18 a)] we know that there exists a right $H$-linear coalgebra morphism $\pi : B \to H$ such that $\pi i = \text{Id}_B$, where $i : B \to H$ is the Hopf algebra inclusion morphism of $B$ in $H$. Now the fact that $\pi$ is right $H$-linear means that the first condition in (7.10) is fulfilled.
This suggests that concrete examples of smash coproduct Hopf algebras can be found among Hopf algebras with a Hopf algebra coradical filtration. If such an example is finite dimensional, then its dual offers an example of a smash product Hopf algebra. We have investigated some examples in low dimension, see for example [17] for classification results. We arrived at two examples for which the computations can be carried out explicitly, namely the Hopf algebras $H_4$ and $A''_{C_{4,q}}$. These Hopf algebras can also be written as biproducts; this is due to the fact that there are several possibilities for choosing the morphisms $i$ and $\pi$. We will now present more detail.

**Example 8.1.** Sweedler’s Hopf algebra $H_4$ can be described either as a smash cross product Hopf algebra or as a biproduct Hopf algebra. Recall that $H_4$ is the unital algebra generated by $g$ and $x$ with relations $g^2 = 1$, $x^2 = 0$ and $gx = -xg$. The coalgebra structure of $H_4$ is given by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1; \quad \Delta(x) = x \otimes g + 1 \otimes x, \quad \epsilon(x) = 0.$$ 

The antipode is defined by $S(g) = g$ and $S(x) = gx$, extended by linearity and as an anti-algebra automorphism of $H$.

We first describe $H_4$ as a smash cross product Hopf algebra. For this let $B = k[[g]]$ be the group algebra associated to the cyclic group generated by $g$ and let $\pi : H_4 \to B$ be the canonical projection:

$$\pi(g^m) = \delta_{m,0}g^i, \quad \text{for } l, m \in \{0, 1\}. \quad$$

It is easy to check that $i : B \to H_4$ defined by $i(1) = 1$ and $i(g) = g + x$ is an algebra map and a section of $\pi$. Furthermore, the pair $(\pi, i)$ satisfies all the conditions in Theorem 7.5, and so $H_4$ is isomorphic to a smash cross product Hopf algebra.

The smash cross product Hopf algebra structure on $H_4$ can be described explicitly. First we compute the algebra $A$ following the proof of Lemma 7.3. It turns out that $A$ is the $k$-subspace of $H_4$ generated by $1$ and $y = gx$, and that $j : A \to H_4$ is the canonical embedding. Moreover, $A$ is a unital algebra with unit 1 and is generated by $y$ with relation $y^2 = 0$.

Then we apply Lemma 7.4 to compute the coalgebra structure on $A$. A simple inspection shows that the map $p : H_4 \to A$ is given by the formulas

$$p(1) = 1, \quad p(g) = 1 + y, \quad p(x) = -y \quad \text{and} \quad p(gx) = y.$$

This leads to the following coalgebra structure on $A$,

$$\Delta(1) = 1 \otimes 1, \quad \Delta(1) = 1; \quad \Delta(y) = y \otimes 1 + 1 \otimes y + y \otimes y, \quad \Delta(y) = 0.$$

In addition, by Lemma 7.4 we know that $\text{Id}_A$ admits a convolution inverse $\delta$ that is given by $\delta(1) = 1$ and $\delta(y) = -y$.

Finally, using Proposition 7.1 we obtain the maps $\psi$ and $\phi$, namely,

$$\psi(1 \otimes 1) = 1 \otimes 1, \quad \psi(1 \otimes y) = y \otimes 1, \quad \psi(g \otimes 1) = (1 - y) \otimes g, \quad \psi(g \otimes y) = -y \otimes g;$$

$$\phi(1 \otimes 1) = 1 \otimes 1, \quad \phi(y \otimes 1) = g \otimes y, \quad \phi(1 \otimes g) = g \otimes 1 - 1 \otimes y + g \otimes y, \quad \phi(y \otimes y) = 1 \otimes y.$$

We conclude that $H_4$ is isomorphic to $A \times^\phi B$ as a Hopf algebra, and that the connecting isomorphism $\zeta : A \times^\phi B \to H_4$ is given by the formulas

$$\zeta(1 \times^\phi 1) = 1, \quad \zeta(1 \times^\phi y) = g + x, \quad \zeta(y \times^\phi 1) = gx \quad \text{and} \quad \zeta(y \times^\phi y) = -x.$$ 

Secondly, it is well-known that $H_4$ is the Radford biproduct of $k[x]$ by $k[[g]]$. Indeed, if $\pi, B$ are as above and take $i' : B \to H_4$ to be the canonical Hopf algebra embedding of $B$ in $H_4$ it then follows that the triple $(H, i', \pi)$ satisfies (b1) in Corollary 7.6.

If $\text{char}(k) \neq 2$, then $H_4$ is a selfdual Hopf algebra, see for instance the end of Sec. 2 in [14]. Thus in this case $H_4$ can also be described as a smash cross coproduct Hopf algebra. Another example of such a Hopf algebra is presented in Example 8.2.

**Example 8.2.** Let $k$ be a field containing a primitive fourth root of unit $\lambda$ and let $H$ be the unital Hopf algebra generated by $a, b$ with relations $a^3 = 1, b^2 = 0$ and $ab = \lambda ba$. Then $H$ is a Hopf algebra with structure maps

$$\Delta(a) = a \otimes a, \quad \Delta(b) = a \otimes b + b \otimes a^{-1}, \quad \epsilon(b) = 0, \quad S(b) = -\lambda^{-1} b.$$ 

Note that $H$ is isomorphic to the 8-dimensional Hopf algebra $A''_{C_{4,q}}$ considered in [17], $\{a^lb^m \mid 0 \leq \ l \leq 3, \ 0 \leq m \leq 1\}$ is a basis for $H$. 


We claim that \( H \) is isomorphic to a smash cross coproduct Hopf algebra. Let \( B = k[\langle a \rangle] \) be the group algebra of the cyclic group \( \langle a \rangle \), and let \( i : B \to H \) be the inclusion map. Consider
\[
\pi : H \to B, \quad \pi(a^l) = a^l \quad \text{and} \quad \pi(a^l b) = \lambda^l (a - a^{-1}) a^l \quad (0 \leq l \leq 3).
\]
A straightforward computation shows that \((\pi, i)\) satisfies all the conditions in Theorem 7.10, and we conclude that \( H \) is isomorphic to a smash cross coproduct Hopf algebra.

Explicitly, \( H \) is isomorphic to \( A \times_\psi^b B \), with \( A, \psi \) and \( \phi \) defined as follows. As a vector space, \( A = H/V \), where \( V \) is the subspace of \( V \) spanned by \( \{a^l b^m(a-1) \mid 0 \leq l \leq 3, 0 \leq m \leq 1 \} \). Writing \( \overline{a} \) for the class in \( A \) containing \( h \in H \), we find that \( \overline{a^l} = \overline{a} \) and \( \overline{a^l b} = \lambda^l \overline{b} \), for \( 0 \leq l \leq 3 \). Let \( p : H \to A \) be the canonical projection and let \( j : A \to H \) be the \( k \)-linear map determined by the formulas \( j(\overline{a}) = 1 \) and \( j(\overline{b}) = 1 - a^2 + \lambda^{-1} ab \).

The algebra structure on \( A \) is the following: \( \overline{T} \) is the unit, and \( \overline{b}^2 = 2\overline{b} \). Furthermore, \( A \) is a coalgebra via the structure
\[
\Delta(\overline{T}) = \overline{T} \otimes \overline{T}, \quad \varepsilon(\overline{T}) = 1; \quad \Delta(\overline{b}) = \overline{b} \otimes \overline{T} + \overline{T} \otimes \overline{b}, \quad \varepsilon(\overline{b}) = 0.
\]
\( \text{Id}_A \) is convolution invertible with inverse \( \overline{S} \) defined by \( \overline{S}(\overline{T}) = \overline{T} \) and \( \overline{S}(\overline{b}) = -\overline{b} \).

The morphisms \( \psi \) and \( \phi \) are determined by the formulas
\[
\psi(\overline{a^l} \otimes \overline{T}) = \overline{a^l} \otimes \overline{T}, \quad \psi(\overline{a^l} \otimes \overline{b}) = (\lambda^l - 1) \overline{T} \otimes (a^2 - 1)a^l + \lambda^l \overline{b} \otimes a^l, \quad (0 \leq l \leq 3),
\]
and
\[
\phi(\overline{T} \otimes \overline{a^l}) = a^l \otimes \overline{T} \quad \text{and} \quad \phi(\overline{b} \otimes \overline{a^l}) = a^{l+2} \otimes \overline{b}.
\]

Finally, we have a Hopf algebra isomorphism \( \zeta : A \times_\psi^b B \to H \), given by
\[
\zeta(\overline{T} \otimes \overline{a^l}) = a^l \quad \text{and} \quad \zeta(\overline{b} \otimes \overline{a^l}) = (1 - a^2) a^l + \lambda^{-l-1} a^{l+1} b, \quad (0 \leq l \leq 3).
\]

We leave the verification of the details to the reader.

We also claim that \( H \) is isomorphic to a biproduct Hopf algebra. Toward this end, consider \( B \) and \( i \) as above and take \( \pi' : B \to H \) defined by \( \pi'(a^l b^m) = \delta_{m,0} a^l \), for all \( 1 \leq l \leq 3 \) and \( 0 \leq m \leq 1 \). A simple inspection shows that the triple \((H, i, \pi')\) obeys (b1) in Corollary 7.6, and thus \( \overline{H} \) is isomorphic to a biproduct Hopf algebra. Further detail is left to the reader.

From [17] we know that \( H \cong A'' \) is not a selfdual Hopf algebra. Thus \( H^* \) provides an example of a smash cross product Hopf algebra.

As we have remarked, our two examples of a smash cross (co)product Hopf algebra are isomorphic to a biproduct Hopf algebra. It would be interesting to have examples that cannot be written as biproduct Hopf algebras. The classification program for pointed Hopf algebras has supplied a large variety of examples of Hopf algebras, and these could be possible candidates. But this leads us to another problem: suppose we have a pointed Hopf algebra that can be written as a smash cross (co)product. How can we be sure that it is not isomorphic to a biproduct Hopf algebra?

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