STABLE EQUIVALENCE OF MORITA TYPE AND FROBENIUS EXTENSIONS

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Abstract. A.S. Dugas and R. Martínez-Villa proved in [4, Corollary 5.1] that if there exists a stable equivalence of Morita type between the \( k \)-algebras \( \Lambda \) and \( \Gamma \), then it is possible to replace \( \Lambda \) by a Morita equivalent \( k \)-algebra \( \Delta \) such that \( \Gamma \) is a subring of \( \Delta \) and the induction and restriction functors induce inverse stable equivalences. In this note we give an affirmative answer to a question of Alex Dugas about the existence of a \( \Gamma \)-coring structure on \( \Delta \). We do this by showing that \( \Delta \) is a Frobenius extension of \( \Gamma \).

As in [4], we will assume throughout that the algebras \( \Lambda \) and \( \Gamma \) are finite dimensional over a field \( k \) and have no semisimple blocks.

The algebras \( \Lambda \) and \( \Gamma \) are said to be stably equivalent if the categories of finitely generated modules modulo projectives for \( \Lambda \) and \( \Gamma \) are equivalent (see [1]).

A pair of left-right projective bimodules \( \Lambda M \Gamma \) and \( \Gamma N \Lambda \) is said to induce a stable equivalence of Morita type between \( \Lambda \) and \( \Gamma \) if we have the following isomorphisms of bimodules:

\[
\Lambda M \otimes \Gamma N \Lambda \cong \Lambda \Lambda \Lambda \oplus \Lambda P \Lambda \text{ and } \Gamma N \otimes \Lambda M \Gamma \cong \Gamma \Gamma \Gamma \oplus \Gamma Q \Gamma
\]

where \( \Lambda P \Lambda \) and \( \Gamma Q \Gamma \) are projective bimodules (see [2]).

We begin by stating the result of Dugas and Martínez-Villa mentioned in the abstract:

**Theorem 1.** (see [4, Corollary 5.1]) Let \( \Lambda \) and \( \Gamma \) be finite dimensional \( k \)-algebras whose semisimple quotients are separable. If at least one of them is indecomposable, then the following are equivalent:

1. There exists a stable equivalence of Morita type between \( \Lambda \) and \( \Gamma \).
2. There exists a \( k \)-algebra \( \Delta \), Morita equivalent to \( \Lambda \), and an injective ring homomorphism \( \Gamma \hookrightarrow \Delta \) such that the restriction and induction functors are exact and induce inverse stable equivalences.
3. There exists a \( k \)-algebra \( \Delta \), Morita equivalent to \( \Lambda \), and an injective ring homomorphism \( \Gamma \hookrightarrow \Delta \) such that

\[
\Gamma \Delta \Gamma = \Gamma \Gamma \Gamma \oplus \Gamma P \Gamma \text{ and } \Delta \Delta \otimes \Gamma \Delta \Delta \cong \Delta \Delta \Delta \oplus \Delta Q \Delta
\]

for projective bimodules \( \Gamma P \Gamma \) and \( \Delta Q \Delta \).

We recall now the definition of Frobenius extension, and its dual notion, Frobenius coring.

**Definition 2.** (see [5]) Let \( i : R \rightarrow S \) be a ring homomorphism. Then \( S/R \) is called a Frobenius extension if one of the following equivalent conditions is satisfied:

The work on this note was started while the last two authors were visiting the Mathematics Department at Mount Allison University. They thank the department for its warm hospitality.
(1) $S$ is finitely generated and projective as a right $R$-module and $\text{Hom}_R(S, R)$ and $S$ are isomorphic as $(R, S)$-bimodules;

(2) there exists a Frobenius system $(e, \varepsilon)$, consisting of

\[ e = e_1 \otimes e^2 \in (S \otimes_R S)^S = \{e_1 \otimes e^2 \in S \otimes_R S \mid se_1 \otimes e^2 = e_1 \otimes e^2 s, \forall s \in S\} \]

and $\varepsilon : S \to R$ an $R$-bimodule map such that $\varepsilon(e_1)e^2 = e_1\varepsilon(e^2) = 1$.

For the proof of the equivalence of the two conditions, see for example [3, Theorem 28].

**Definition 3.** (see [7]) If $R$ is a ring, a coring is a comonoid in the monoidal category of $R$-bimodules. So a coring consists of an $R$-bimodule $C$, together with a coassociative comultiplication $C \to C \otimes_R C$ and counit $C \to R$ which are both $R$-bimodule maps. $C$ is called a Frobenius $R$-coring if there exists a Frobenius system $(\theta, 1)$, consisting of an element $1 \in C$ and an $R$-bimodule map $\theta : C \otimes_R C \to R$ satisfying the conditions

\[ c_{(1)} \theta(c_{(2)} \otimes d) = \theta(c \otimes d_{(1)})d_{(2)} \text{ and } \theta(c \otimes 1) = \theta(1 \otimes c) = \varepsilon(c). \]

Let $(S, m, 1, e, \varepsilon)$ be a Frobenius extension of $R$, and consider $\Delta : S \to S \otimes_R S$, $\Delta(s) = se = es$. An easy verification shows that $(S, \Delta, \varepsilon, \theta = \varepsilon \circ m, 1)$ is a Frobenius coring.

Conversely, if $(C, \Delta, \varepsilon, \theta, 1)$ is a Frobenius $R$-coring, then $(C, m, 1, \Delta(1), \varepsilon)$ is a Frobenius extension. Here $m : C \otimes_R C \to C$, $m(c \otimes d) = c_{(1)} \theta(c_{(2)} \otimes d) = \theta(c \otimes d_{(1)})d_{(2)}$.

These two assertions basically tell us that Frobenius extension structures on an $R$-bimodule $M$ correspond bijectively to Frobenius $R$-coring structures on $M$.

Let $S$ be a Frobenius extension. Then the categories $\mathcal{M}_S$ and $\mathcal{M}^S$ are isomorphic: on a right $S$-module, we define a right $S$-coaction by $\rho(m) = me_1 \otimes e^2$. On a right $S$-comodule, we define a right $S$-action $ms = m_{[0]} \varepsilon(m_{[1]} s)$.

The restriction functor $G : \mathcal{M}_S \to \mathcal{M}_R$ has a left adjoint, the induction functor $F$; the forgetful functor $\mathcal{M}_S \to \mathcal{M}_R$ has a right adjoint. These functors are compatible with the above isomorphism. This implies that $G$ is at the same time a left and a right adjoint of $F$.

**Definition 4.** (see [6] or [3, p.91]) Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. If there exists a functor $G : \mathcal{D} \to \mathcal{C}$ which is at the same time a right and a left adjoint of $F$, then we call $F$ a Frobenius functor, and we say that $(F, G)$ is a Frobenius pair for $\mathcal{C}$ and $\mathcal{D}$.

**Remark 5.** (see [5] or [3, Theorem 28, p.103]) Let $i : S \to R$ be a ring homomorphism, $F$ the induction functor and $G$ the restriction functor. If $S/R$ is a Frobenius extension, then we have seen above that $(F, G)$ is a Frobenius pair; in fact, it can be shown that the converse also holds: $(F, G)$ is a Frobenius pair if and only if $S/R$ is a Frobenius extension.

We can now state and prove our result. Assertion (3) gives an affirmative answer to a question asked by Alex Dugas.

**Theorem 6.** Let $\Lambda$ and $\Gamma$ be finite dimensional $k$-algebras whose semisimple quotients are separable. Assume that at least one of them is indecomposable, and that there exists a stable equivalence of Morita type between $\Lambda$ and $\Gamma$. Then the following assertions hold:

(1) There exists a $k$-algebra $\Delta$, Morita equivalent to $\Lambda$, and an injective ring homomorphism $\Gamma \to \Delta$ such that the restriction and induction functors are a Frobenius pair.
(2) There exists a \( k \)-algebra \( \Delta \), Morita equivalent to \( \Lambda \), and an injective ring homomorphism \( \Gamma \hookrightarrow \Delta \) such that \( \Delta/\Gamma \) is a Frobenius extension.

(3) There exists a \( k \)-algebra \( \Delta \), Morita equivalent to \( \Lambda \), and an injective ring homomorphism \( \Gamma \hookrightarrow \Delta \) such that

\[
\Gamma \Delta = \Gamma \Gamma \oplus \Gamma P \Gamma \quad \text{and} \quad \Delta \Delta \otimes \Gamma \Delta \simeq \Delta \Delta \oplus \Delta Q \Delta
\]

for projective bimodules \( \Gamma P \Gamma \) and \( \Delta Q \Delta \), and \( \Delta \) is a Frobenius \( \Gamma \)-coring with comultiplication given by the injection of \( \Delta \Delta \) into \( \Delta \otimes \Gamma \Delta \), and counit given by the projection of \( \Gamma \Delta \) onto \( \Gamma \Gamma \).

**Proof.** (1) Suppose \( \Lambda M \Gamma \) and \( \Gamma N \Delta \) are indecomposable bimodules that induce a stable equivalence of Morita type. Let \( \Delta = \text{End}_\Lambda (M) \). By the proof of (1) \( \Rightarrow \) (2) of [4, Corollary 5.1], we have that

\[
\text{Res}^\Delta_{\Gamma} \simeq (- \otimes_{\Lambda} M \Gamma) \circ \text{Hom}_\Delta (M, -)
\]

and

\[
\text{Ind}^\Delta_{\Gamma} \simeq (- \otimes_{\Lambda} M \Delta) \circ (- \otimes_{\Gamma} N \Lambda).
\]

Now \( - \otimes_{\Lambda} M \Gamma \) is a right and left adjoint of \( - \otimes_{\Gamma} N \Lambda \) by [4, Corollary 3.1,(2)], and \( \text{Hom}_\Delta (M, -) \) is a right and left adjoint of \( - \otimes_{\Lambda} M \Delta \) because they are inverse equivalences, so \( \text{Res}^\Delta_{\Gamma} \) is a right and left adjoint of \( \text{Ind}^\Delta_{\Gamma} \).

(2) follows from (1) and Remark 5.

(3) follows immediately from the above observation that a Frobenius extension is also a Frobenius coring. \( \square \)

**References**


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