STATIONARY DISTRIBUTION OF A PERTURBED QUASI-BIRTH-AND-DEATH PROCESS

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Introduction

Quasi-Birth-and-Death (QBD) processes have applications in many areas. They are used in the modeling of telecommunication networks, queueing theory, computer systems, etc. As in most of mathematical models, the input parameters have to be estimated from the real world. The parameters in the modeled systems represent quantities, which are sometimes hard to measure accurately. Furthermore, the real world is not constant and the parameters can evolve and become slightly different. The results obtained through the models should be interpreted cautiously. Analysis of the modified model behavior will show how important it is to take appropriate values and will point to the needed precision level. If the model is complex and computationally feasible there may be no other solution than to change the initial parameters and have a look at the new results. But it is more satisfying if we can establish equations for the modified models.

Our question is: how is the stationary probability vector of the QBD process modified if we change slightly its initial parameters. More precisely, let $Q$ be the infinitesimal generator of an infinite dimensional QBD process with stationary distribution $\pi$, assume that it is perturbed by a matrix $\tilde{Q}$ such that

$$Q(\varepsilon) = Q + \varepsilon \tilde{Q},$$

with $\varepsilon$ a small number, is the infinitesimal generator of another QBD process. Our purpose is to describe the effect on $\pi(\varepsilon)$, the stationary distribution of $Q(\varepsilon)$. It is primordial to assess the impact of small variations of the initial parameters on the stationary distribution, compared with the initial stationary probability vector because it is a fundamental quantity in the study of QBD process and moreover a lot of performance measures are calculated by using this distribution.

Recall that an infinite dimensional QBD process $\{X_t : t \in \mathbb{R}^+\}$ is a two-dimensional Markov process defined on the state space $S = \{(n, i) : n \in \mathbb{N}, i \in E\}$ where $E$ denotes the
set \{1, \ldots, m\} \text{ with } m \in \mathbb{N} \text{. Its generator has the following tridiagonal block-structure:}

\[
Q = \begin{bmatrix}
B & A_1 \\
A_{-1} & A_0 & A_1 \\
& A_{-1} & A_0 & \ddots \\
& & \ddots & \ddots
\end{bmatrix},
\]

where the block entries \( B, A_{-1}, A_0 \) and \( A_1 \) are square matrices of order \( m \). We assume that the Quasi-Birth-and-Death process is positive recurrent and we denote by \( \pi^\top = (\pi_0^\top, \pi_1^\top, \pi_2^\top, \pi_3^\top, \ldots) \) its stationary probability vector, where \( \pi_n \) denotes the stationary probability vector of level \( n \in \mathbb{N} \), i.e. \( \pi \) is the unique non-negative solution of the equations \( \pi^\top Q = 0, \pi^\top 1 = 1 \). An important result in matrix-analytic methods is that the entire stationary probability vector can be computed by the knowledge of \( \pi_0 \) and \( R \), a matrix called the rate-matrix, which is the minimal non-negative solution of the matrix-quadratic equation

\[
R^2 A_{-1} + RA_0 + A_1 = 0.
\]

The link between the stationary distribution and the rate-matrix is given by the so-called matrix-geometric property:

\[
\pi_n^\top = \pi_0^\top R^n,
\]

for \( n \geq 1 \). In this work, \( \pi_0 \) the stationary probability vector of level 0 and \( R \) the rate-matrix will be key quantities. We will study them in the first chapter. Notice that, although in general \( R \) cannot be computed explicitly, nevertheless some rate matrices of QBDs with a particular structure are known explicitly. We will discuss them at the end of the first chapter.

For finite dimensional infinitesimal generators, Schweitzer [?] provided the first perturbation analysis in terms of Kemeny and Snell’s [?] fundamental matrix \( Z = (Q + 1 \pi^\top)^{-1} \). Various authors have used Schweitzer’s results to explore the effects of perturbing simple chains, such as birth-death chains. It has also been used by Haviv and van der Heyden [?] to find bounds on the effect of perturbation on stationary distributions.

Another generalized inverse used abundantly by Meyer ([?], [?]) and Rising [?] is \( Q^\# \), the group inverse of \( Q \), defined by the three equations \( QQ^# = Q^# Q, QQ^# Q = Q \) and \( Q^# QQ^# = Q^# \) when it exists. This matrix and the fundamental matrix are related by the relation

\[
Z = Q^# + 1 \pi^\top,
\]

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where $\mathbf{1}$ denotes the column vector of ones. Generally, the term $\mathbf{1}\pi^\top$ is superfluous in applications involving $Z$: all relevant information is essentially contained in $Q^\#$. For instance, Cao and Chen [?] established that

$$\frac{\partial}{\partial \varepsilon} \left( \pi^\top(\varepsilon) \right) \big|_{\varepsilon=0} = -\pi^\top \tilde{Q} Q^\#.$$  (1)

The notion of "group inverse" has to be used with caution for infinite dimensional matrices, because it is not well-defined in this case. We will introduce $D$ the deviation matrix defined in Coolen-Schrijner and Van Doorn [?] by

$$D = \int_0^\infty (e^{Qt} - \mathbf{1}\pi^\top) \, dt,$$

which in fact is related to the group inverse of $Q$, when the QBD process is finite, by the obvious relation $D = -Q^\#$. The topic of generalized inverses will be the subject of the second chapter of this work.

The last chapter will be dedicated to the study of the sensitivity of the stationary distribution of the perturbed QBD. For this purpose, we will use both matrix analytic methods and the theory of generalized inverses developed in the previous chapters. Two approaches will be discussed. The first approach is based on (??). Here we analyze the structure of the deviation matrix. The second approach is based on the repetitive structure of the QBD and the matrix geometric formula. Here we censor the process to the first levels and deal with a finite QBD.

In this work, we reserve the particular symbol $I$ for the unit matrix and 0 for the vector of zeros. The dimension of the matrices can often be deduced from the context. If there is ambiguity, we will mention it explicitly.