Surfaces That Define Facets of the Linear Ordering Polytope

Samuel Fiorini

Department of Mathematics
Université Libre de Bruxelles (Belgium)

DIMACS Workshop: Polyhedral Combinatorics of Random Utility
The linear ordering polytope for $n = 3$

Ground set: $X = \{1, 2, 3\}$

Vertices $\in \mathbb{R}^{3 \times 3}$:

1 $\preceq$ 2 $\preceq$ 3
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 \\ 1 \end{pmatrix}$$

1 $\preceq$ 3 $\preceq$ 2
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 \\ 1 \end{pmatrix}$$

2 $\preceq$ 1 $\preceq$ 3
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 \end{pmatrix}$$

2 $\preceq$ 3 $\preceq$ 1
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 \end{pmatrix}$$

3 $\preceq$ 1 $\preceq$ 2
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 & 1 \\ 1 \end{pmatrix}$$

3 $\preceq$ 2 $\preceq$ 1
$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$
The linear ordering polytope for $n = 3$

Ground set: $X = \{1, 2, 3\}$

Vertices $\in \mathbb{R}^{3 \times 3}$:

1 $\preceq$ 2 $\preceq$ 3
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

1 $\preceq$ 3 $\preceq$ 2
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

2 $\preceq$ 1 $\preceq$ 3
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

2 $\preceq$ 3 $\preceq$ 1
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

3 $\preceq$ 1 $\preceq$ 2
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

3 $\preceq$ 2 $\preceq$ 1
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]
The linear ordering polytope for $n = 3$

Ground set: $X = \{1, 2, 3\}$

Vertices $\in \mathbb{R}^{3 \times 3}$:

- $1 \preceq 2 \preceq 3$
  \[
  \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
  \end{pmatrix}
  \]

- $1 \preceq 3 \preceq 2$
  \[
  \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
  \end{pmatrix}
  \]

- $2 \preceq 1 \preceq 3$
  \[
  \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
  \end{pmatrix}
  \]

- $2 \preceq 3 \preceq 1$
  \[
  \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
  \end{pmatrix}
  \]

- $3 \preceq 1 \preceq 2$
  \[
  \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
  \end{pmatrix}
  \]

- $3 \preceq 2 \preceq 1$
  \[
  \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
  \end{pmatrix}
  \]
The linear ordering polytope for \( n = 3 \) (continued)

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

All the vertices satisfy:

\[
\begin{align*}
x_{ii} &= 1 & \forall i \in X \\
x_{ij} + x_{ji} &= 1 & \forall i, j \in X, i \neq j
\end{align*}
\]
The linear ordering polytope for $n = 3$ (continued)

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

All the vertices satisfy:

\[
\begin{cases}
   x_{ii} = 1 & \forall i \in X \\
   x_{ij} + x_{ji} = 1 & \forall i, j \in X, i \neq j
\end{cases}
\]
The linear ordering polytope for $n = 3$ (continued)

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

All the vertices satisfy:

\[
\begin{cases}
    x_{ii} = 1 & \forall i \in X \\
    x_{ij} + x_{ji} = 1 & \forall i, j \in X, i \neq j
\end{cases}
\]
A definition of the polytope for general $n$

$X$ ground set of size $n$ (e.g., $X = \{1, 2, \ldots, n\}$)

$$P^n_{LO} := \text{conv}\{\chi^L \in \{0, 1\}^{n \times n} : L \text{ linear order on } X\}$$

WANTED:

An explicit system of linear inequalities $Ax \geq b$ such that

$$P^n_{LO} = \{x \in \mathbb{R}^{n \times n} : Ax \geq b\}$$

or (better) the facet-defining inequalities (FDIs) of $P^n_{LO}$
A definition of the polytope for general $n$

$X$ ground set of size $n$ \quad (e.g., $X = \{1, 2, \ldots, n\}$)

$$P^n_{LO} := \text{conv}\{\chi^L \in \{0, 1\}^{n \times n} : L \text{ linear order on } X\}$$

**WANTED:**

An **explicit** system of linear inequalities $Ax \geq b$ such that

$$P^n_{LO} = \{x \in \mathbb{R}^{n \times n} : Ax \geq b\}$$

or (better) the facet-defining inequalities (FDIs) of $P^n_{LO}$
A definition of the polytope for general $n$

$X$ ground set of size $n$ (e.g., $X = \{1, 2, \ldots, n\}$)

$$P_{LO}^n := \text{conv}\{\chi^L \in \{0, 1\}^{n \times n} : L \text{ linear order on } X\}$$

WANTED:

An explicit system of linear inequalities $Ax \geq b$ such that

$$P_{LO}^n = \{x \in \mathbb{R}^{n \times n} : Ax \geq b\}$$

or (better) the facet-defining inequalities (FDIs) of $P_{LO}^n$
Motivations for studying the linear ordering polytope

1. probabilistic preference theory:
   - characterize binary choice probabilities

2. combinatorial optimization:
   - rank aggregation (median linear orders)
   - minimum feedback arc set problem
Motivations for studying the linear ordering polytope

1. probabilistic preference theory:
   - characterize binary choice probabilities

2. combinatorial optimization:
   - rank aggregation (median linear orders)
   - minimum feedback arc set problem
Motivations for studying the linear ordering polytope

1. probabilistic preference theory:
   - characterize binary choice probabilities

2. combinatorial optimization:
   - rank aggregation (median linear orders)
   - minimum feedback arc set problem
Motivations for studying the linear ordering polytope

1 probabilistic preference theory:
   ▶ characterize binary choice probabilities

2 combinatorial optimization:
   ▶ rank aggregation (median linear orders)
   ▶ minimum feedback arc set problem
Motivations for studying the linear ordering polytope

1. probabilistic preference theory:
   ▶ characterize binary choice probabilities

2. combinatorial optimization:
   ▶ rank aggregation (median linear orders)
   ▶ minimum feedback arc set problem
Basic facet-defining inequalities

- **Trivial inequality:**
  \[ x_{ij} \leq 1 \iff x_{ji} \geq 0 \]

- **Transitivity (or 3-dicycle) inequality:**
  \[ x_{ij} + x_{jk} + x_{ki} \leq 2 \iff x_{ij} + x_{jk} - x_{ik} \leq 1 \iff x_{ik} + x_{kj} + x_{ji} \geq 1 \]

(Naive) Question
Is that all?
Basic facet-defining inequalities

- **Trivial** inequality:
  \[ x_{ij} \leq 1 \iff x_{ji} \geq 0 \]

- **Transitivity** (or 3-dicycle) inequality:
  \[ x_{ij} + x_{jk} + x_{ki} \leq 2 \iff x_{ij} + x_{jk} - x_{ik} \leq 1 \iff x_{ik} + x_{kj} + x_{ji} \geq 1 \]

(Naive) Question

Is that all? **OF COURSE NOT!**
Basic facet-defining inequalities

- **Trivial inequality:**
  \[ x_{ij} \leq 1 \iff x_{ji} \geq 0 \]

- **Transitivity** (or 3-dicycle) inequality:
  \[ x_{ij} + x_{jk} + x_{ki} \leq 2 \iff x_{ij} + x_{jk} - x_{ik} \leq 1 \iff x_{ik} + x_{kj} + x_{ji} \geq 1 \]

(Naive) Question
Is that all?  **OF COURSE NOT!**
Basic facet-defining inequalities

- **Trivial inequality:**
  \[ x_{ij} \leq 1 \iff x_{ji} \geq 0 \]

- **Transitivity (or 3-dicycle) inequality:**
  \[ x_{ij} + x_{jk} + x_{ki} \leq 2 \iff x_{ij} + x_{jk} - x_{ik} \leq 1 \iff x_{ik} + x_{kj} + x_{ji} \geq 1 \]

(Naive) Question

Is that all? **OF COURSE NOT!**
Basic facet-defining inequalities

- **Trivial** inequality:
  \[ x_{ij} \leq 1 \iff x_{ji} \geq 0 \]

- **Transitivity** (or 3-dicycle) inequality:
  \[ x_{ij} + x_{jk} + x_{ki} \leq 2 \iff x_{ij} + x_{jk} - x_{ik} \leq 1 \iff x_{ik} + x_{kj} + x_{ji} \geq 1 \]

(Naive) Question
Is that all? **OF COURSE NOT!**
Möbius ladder inequalities
Nice inequalities from the 80’s

For $k$ odd, $k \geq 3$:

$$\sum_{ij \in M} x_{ij} \geq \frac{k + 1}{2}$$

Theorem (Grötschel, Jünger and Reinelt 1985)

If $M$ is a Möbius ladder generated by $k$ directed cycles then it defines a facet of $P^n_{LO}$. 
Möbius ladder inequalities
Nice inequalities from the 80’s

For $k$ odd, $k \geq 3$:

$$\sum_{ij \in M} x_{ij} \geq \frac{k + 1}{2}$$

Theorem (Grötschel, Jünger and Reinelt 1985)

If $M$ is a Möbius ladder generated by $k$ directed cycles then it defines a facet of $P_{LO}^n$. 
Why are Möbius ladder inequalities important?
A ‘small $n$’ perspective

$P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>vertices of $P^n_C$</th>
<th>facets of $P^n_{LO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?
A ‘small $n$’ perspective

$$P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>vertices of $P^n_C$</th>
<th>facets of $P^n_{LO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?
A ‘small n’ perspective

\[ P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \} \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>vertices of ( P^n_C )</th>
<th>facets of ( P^n_{LO} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic nor Möbius ladders</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?
A ‘small $n$’ perspective

$P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>vertices of $P^n_C$</th>
<th>facets of $P^n_{LO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?

A ‘small $n$’ perspective

$$P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>vertices of $P^n_C$</th>
<th>facets of $P^n_{LO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?
A ‘small $n$’ perspective

$$P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>vertices of $P^n_C$</th>
<th>facets of $P^n_{LO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?
A ‘small n’ perspective

\[ P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \} \]

<table>
<thead>
<tr>
<th>n</th>
<th>vertices of ( P^n_C )</th>
<th>facets of ( P^n_{LO} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Why are Möbius ladder inequalities important?
A ‘small \( n \)’ perspective

\[ P^n_C := \{ x \in \mathbb{R}^{n \times n} : x \text{ satisfies all basic constraints} \} \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>vertices of ( P^n_C )</th>
<th>facets of ( P^n_{LO} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>3</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>4</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>5</td>
<td>all integral</td>
<td>only basic ones</td>
</tr>
<tr>
<td>6</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>7</td>
<td>half-integral</td>
<td>basic ones + Möbius ladders</td>
</tr>
<tr>
<td>8+</td>
<td>most are non-half-integral</td>
<td>most are neither basic neither ML</td>
</tr>
</tbody>
</table>
Möbius ladder inequalities . . .

- are essentially the only inequalities we can separate in polynomial time (Müller & Schulz, Caprara & Fischetti, F, Letchford)

- have low (Chvátal-Gomory, lift-and-project, . . . ) rank

- cut all \( \{0, \frac{1}{2}, 1\} \)-points which are not in the polytope (F, unpublished)
Why are Möbius ladder inequalities important?
A combinatorial optimization perspective

Möbius ladder inequalities . . .

- are essentially the only inequalities we can separate in polynomial
time (Müller & Schulz, Caprara & Fischetti, F, Letchford)

- have low (Chvátal-Gomory, lift-and-project, . . . ) rank

- cut all \( \{0, \frac{1}{2}, 1\} \)-points which are not in the polytope
  (F, unpublished)
Why are Möbius ladder inequalities important?
A combinatorial optimization perspective

Möbius ladder inequalities . . .

- are essentially the only inequalities we can separate in polynomial time (Müller & Schulz, Caprara & Fischetti, F, Letchford)

- have low (Chvátal-Gomory, lift-and-project, . . .) rank

- cut all \{0, \frac{1}{2}, 1\}-points which are not in the polytope (F, unpublished)
Why are Möbius ladder inequalities important?
A combinatorial optimization perspective

Möbius ladder inequalities . . .

- are essentially the only inequalities we can separate in polynomial time (Müller & Schulz, Caprara & Fischetti, F, Letchford)

- have low (Chvátal-Gomory, lift-and-project, . . . ) rank

- cut all \( \{0, \frac{1}{2}, 1\} \)-points which are not in the polytope (F, unpublished)
Why are Möbius ladder inequalities valid?
An example with $k = 3$ and $n = 6$

Goal:

1. Sum the following valid inequalities:

$$x_{ij} + x_{jk} + x_{ki} \geq 1 \quad \text{for } ijk \in \{123, 341, 634, 456, 561, 125\}$$

Result:
Why are Möbius ladder inequalities valid?

An example with $k = 3$ and $n = 6$

Goal:

1. Sum the following valid inequalities:

$$x_{ij} + x_{jk} + x_{ki} \geq 1 \quad \text{for } ijk \in \{123, 341, 634, 456, 561, 125\}$$

Result:
2. Add the inequalities:

\[ x_{ij} \geq 0 \quad \text{for } ij \in \{23, 41, 45, 63, 61, 25\} \]

Result:

3. Finally add the inequalities:

\[ -x_{ij} - x_{ji} \geq -1 \quad \text{for } ij \in \{13, 46, 15\} \]

Final result:

\[ \sum_{ij \in M} 2x_{ij} \geq k \iff \]
2. Add the inequalities:
\[ x_{ij} \geq 0 \quad \text{for } ij \in \{23, 41, 45, 63, 61, 25\} \]

Result:

3. Finally add the inequalities:
\[ -x_{ij} - x_{ji} \geq -1 \quad \text{for } ij \in \{13, 46, 15\} \]

Final result:
\[ \sum_{ij \in M} 2x_{ij} \geq k \iff \]
\[ \sum_{ij \in M} 2x_{ij} \geq k \iff \sum_{ij \in M} x_{ij} \geq \frac{k}{2} \]

\( k \) is odd, and the vertices of \( P_{LO}^n \) are integral \( \} \implies \sum_{ij \in M} x_{ij} \geq \frac{k + 1}{2} \) is valid
An alternative representation of the derivation

\[ x_{ij} + x_{jk} + x_{ki} \geq 1 \rightarrow \]

\[ x_{ij} \geq 0 \rightarrow \]

An oriented triangulation of the projective plane!
An alternative representation of the derivation

\[ x_{ij} + x_{jk} + x_{ki} \geq 1 \rightarrow \]

\[ x_{ij} \geq 0 \rightarrow \]

\[ \text{An oriented triangulation of the projective plane!} \]
An alternative representation of the derivation

\[ x_{ij} + x_{jk} + x_{ki} \geq 1 \rightarrow \]

An oriented triangulation

of the projective plane!

\[ x_{ij} \geq 0 \rightarrow \]
Another example of a Möbius ladder inequality
And of the corresponding triangulation
The classification of surfaces

Orientable surfaces:

Nonorientable surfaces:

Definition

A surface is a connected, compact, Hausdorff topological space locally isomorphic to $\mathbb{R}^2$. 
The classification of surfaces

Orientable surfaces:

Nonorientable surfaces:

Definition

A surface is a connected, compact, Hausdorff topological space locally isomorphic to $\mathbb{R}^2$. 
The classification of surfaces

Orientable surfaces:

Nonorientable surfaces:

Definition

A surface is a connected, compact, Hausdorff topological space locally isomorphic to $\mathbb{R}^2$. 
Theorem (Classification of surfaces)

Every surface is homeomorphic to:

- $S_h$ for a certain $h \geq 0$ (orientable case), or
- $N_b$ for a certain $b > 0$ (nonorientable case).

The surfaces $S_0, S_1, N_1, S_2, N_2, \ldots$ are pairwise non-homeomorphic.
**Question A**
What are the ‘facet-defining’ surfaces?

**Theorem (F 2003)**
*No orientable surface... and all nonorientable surfaces!*

**Question B**
Consider a nonorientable surface $S$ and a triangulation of $S$. What are the facet-defining orientations of this triangulation?

**Status**: still an *open* problem...
Question A
What are the ‘facet-defining’ surfaces?

Theorem (F 2003)
No orientable surface... and all nonorientable surfaces!

Question B
Consider a nonorientable surface $S$ and a triangulation of $S$. What are the facet-defining orientations of this triangulation?

Status: still an open problem...
Question A
What are the ‘facet-defining’ surfaces?

Theorem (F 2003)
No orientable surface... and all nonorientable surfaces!

Question B
Consider a nonorientable surface $S$ and a triangulation of $S$. What are the facet-defining orientations of this triangulation?

Status: still an open problem...
Question A
What are the ‘facet-defining’ surfaces?

Theorem (F 2003)
No orientable surface... and all nonorientable surfaces!

Question B
Consider a nonorientable surface $S$ and a triangulation of $S$. What are the facet-defining orientations of this triangulation?

Status: still an open problem...
Question A
What are the ‘facet-defining’ surfaces?

Theorem (F 2003)
*No orientable surface... and all nonorientable surfaces!*

Question B
Consider a nonorientable surface $S$ and a triangulation of $S$. What are the facet-defining orientations of this triangulation?

**Status:** still an open problem...
Example: a Klein bottle inequality
Example: a Klein bottle inequality
Example: a Klein bottle inequality
Zones of an oriented triangulation

\[ \vec{K} \text{ oriented triangulation} \]

\[ G \text{ ‘dual’ graph} \]

‘Dual’ graph \( G \):

- \( V(G) = \text{triangles of } \vec{K} \)
- \( E(G) = \text{pairs of compatibly oriented triangles} \)
Definition

A zone is a connected component of $G$

Zone graph $\mathcal{Z}$:
- $V(\mathcal{Z}) = \text{zones of } \tilde{K}$
- $E(\mathcal{Z}) = \text{pairs of adjacent zones}$

Lemma (F 2003)

If $\tilde{K}$ is facet-defining then:

(i) each zone is a triangulated polygon

(ii) there is an odd number of zones

(iii) $\mathcal{Z}$ is factor-critical
Definition

A zone is a connected component of $G$.

Zone graph $\mathcal{Z}$:
- $V(\mathcal{Z}) = \text{zones of } \vec{K}$
- $E(\mathcal{Z}) = \text{pairs of adjacent zones}$

Lemma (F 2003)

If $\vec{K}$ is facet-defining then:

(i) each zone is a triangulated polygon
(ii) there is an odd number of zones
(iii) $\mathcal{Z}$ is factor-critical
Definition

A zone is a connected component of $\mathcal{G}$

Zone graph $\mathcal{Z}$:

- $V(\mathcal{Z})$ = zones of $\mathcal{K}$
- $E(\mathcal{Z})$ = pairs of adjacent zones

Lemma (F 2003)

If $\mathcal{K}$ is facet-defining then:

(i) each zone is a triangulated polygon

(ii) there is an odd number of zones

(iii) $\mathcal{Z}$ is factor-critical
Factor-critical graphs

$G$ is **factor-critical** if $G - v$ has a perfect matching for all $v \in V(G)$

**Proposition (Lovász 1972)**

$G$ is factor-critical iff $G$ has an odd ear decomposition.
Factor-critical graphs

\( G \) is **factor-critical** if \( G - v \) has a perfect matching for all \( v \in V(G) \)

**Proposition (Lovász 1972)**

\( G \) is factor-critical iff \( G \) has an odd ear decomposition.
Factor-critical graphs

$G$ is factor-critical if $G - v$ has a perfect matching for all $v \in V(G)$

Proposition (Lovász 1972)

$G$ is factor-critical iff $G$ has an odd ear decomposition.
The orientable case

**Proposition (F 2003)**

If $S$ is orientable and $\vec{K}$ is an oriented triangulation of $S$ then $\vec{K}$ is not facet-defining.

**Proof.** (By contradiction)

\[ \begin{align*}
S \text{ orientable and } & \quad \vec{K} \text{ defines a facet } \\
& \quad \Rightarrow \quad \mathcal{Z} \text{ bipartite and factor-critical}
\end{align*} \]

Hence $\mathcal{Z} = \emptyset$, a contradiction.
The orientable case

Proposition (F 2003)

If \( S \) is orientable and \( \vec{K} \) is an oriented triangulation of \( S \) then \( \vec{K} \) is not facet-defining.

**Proof.** (By contradiction)

\[
\begin{align*}
S \text{ orientable and} & \quad \vec{K} \text{ defines a facet} \\
\Rightarrow & \quad \mathcal{Z} \text{ bipartite and factor-critical}
\end{align*}
\]

Hence \( \mathcal{Z} = \bullet \), a contradiction.
Representing a graph by a digraph

G

C1

C2

C3

C4

C5

D

C1

C2

C3

C4

C5

I

II

III
Proposition (F 2003)

Let $H$ be a graph without isolated vertex, let $G$ be the graph obtained from $H$ by replacing every edge by a vertex, and let $D$ be a representation of $G$. Then $\tau(D) = \rho(G)$ and

$$\sum_{ij \in D} x_{ij} \geq \tau(D) \text{ is a FDI} \iff G \begin{cases} \text{is factor-critical} \\ \text{has no ‘bad’ vertex} \end{cases}$$

A vertex $v$ is bad if . . .
The nonorientable case

Corollary (F 2003)

*If S is a nonorientable surface then S has a facet-defining oriented triangulation $\vec{K}$.*

**Proof.** Choose $b$ such that $S \cong \mathbb{N}_b$ ($\chi = 2 - b$ is the Euler characteristic of $S$), and $H$ as follows:

$$H \Rightarrow G \text{ subdivision} \quad G \Rightarrow D \text{ representation} \quad D \Rightarrow \vec{K} \text{ triangulation.}$$

The inequality defined by $\vec{K}$ is:

$$\sum_{ij \in D} x_{ij} \geq \tau(D)$$
The nonorientable case

Corollary (F 2003)

If $S$ is a nonorientable surface then $S$ has a facet-defining oriented triangulation $\vec{K}$.

Proof. Choose $b$ such that $S \cong \mathbb{N}_b$ ($\chi = 2 - b$ is the Euler characteristic of $S$), and $H$ as follows:

![Diagram](image)

$H \Rightarrow G$ subdivision $\quad G \Rightarrow D$ representation $\quad D \Rightarrow \vec{K}$ triangulation.

The inequality defined by $\vec{K}$ is:

$$\sum_{ij \in D} x_{ij} \geq \tau(D)$$
An illustration of the proof
Outlook

How should we study the linear ordering polytope?

The ‘Rosetta stone’ approach

The ‘mirror of Erised’ approach
Outlook

How should we study the linear ordering polytope?

The ‘Rosetta stone’ approach

The ‘mirror of Erised’ approach
Outlook

How should we study the linear ordering polytope?

The ‘Rosetta stone’ approach

The ‘mirror of Erised’ approach
In this talk:

**edge cover problem** $\leadsto$ **dicycle cover problem** (FAS problem)

Natural extension:

**set cover problem** $\leadsto$ **dicycle cover problem** (FAS problem)
In this talk:

- **edge cover problem** $\leadsto$ **dicycle cover problem** (FAS problem)

Natural extension:

- **set cover problem** $\leadsto$ **dicycle cover problem** (FAS problem)