Abstract

The VPN Tree Routing Conjecture is a conjecture about the Virtual Private Network Design problem. It states that the symmetric version of the problem always has an optimum solution which has a tree-like structure. In recent work, Hurkens, Keijsper and Stougie (Proc. IPCO XI, 2005; SIAM J. Discrete Math., 2007) have shown that the conjecture holds when the network is a ring. A shorter proof of the VPN Conjecture for rings was found a few months ago by Grandoni, Kaibel, Oriolo and Skutella (to appear in Oper. Res. Lett., 2008).

In their paper, Grandoni et al. introduce another conjecture, called the Pyramidal Routing Conjecture (or simply PR Conjecture), which implies the VPN Conjecture. Here we consider a strengthened version of the PR Conjecture. First we establish several general tools which can be applied in arbitrary networks. Then we use them to prove that outerplanar networks satisfy the PR Conjecture.

1 Introduction

The symmetric Virtual Private Network Design problem (sVPND) takes place in an undirected network. Inside the network there are \( k \) distinguished vertices called terminals. The goal of the problem is to choose a collection of \( \binom{k}{2} \) paths, one between each pair of terminals, and capacity reservations on the edges covered by these paths, in such a way that any admissible traffic demand between the terminals can be routed through the paths and that the cost of the reservation is minimum (a precise definition is given in the next paragraph). So far the question of determining the complexity of sVPND has remained open, see Erlebach and Rüegg [1] and Italiano, Leonardi and Oriolo [6]. The VPN Tree Routing Conjecture (or shortly, VPN Conjecture) states that sVPND always has an optimum solution whose paths determine a tree. As shown by Gupta, Kleinberg, Kumar, Rastogi and Yener [3], if the VPN Conjecture is true then sVPND can be solved in polynomial time by a single all-pairs shortest paths computation. Essentially, the only general class of networks where the VPN Conjecture is known to hold is the class of ring networks (that is, whose underlying graph is a cycle), a result due to Hurkens, Keijsper and Stougie [4, 5]. (Actually, Hurkens et al. prove the VPN Conjecture in other cases too, e.g., when the network is complete graph of size four.) A short proof of the VPN Conjecture for ring networks was very recently found by Grandoni, Kaibel, Oriolo and Skutella [2]. A consequence of our results is that the VPN Conjecture holds for all outerplanar networks.

Let \( G = (V, E) \) be a undirected, finite, simple graph representing the network. Each edge \( e \) of \( G \) has a given cost \( c_e \in \mathbb{R}_+ \). Let \( W \subseteq V \) denote the set of all terminals, thus \( |W| = k \). For each terminal \( u \in W \) we are given an upper bound \( b_u \in \mathbb{Z}_+ \) on the total amount of traffic that \( u \) can send or receive. The traffic demands between terminals are encoded in a traffic matrix, that is, a non-negative \( k \times k \) real matrix \( D = (d_{uv}) \) with lines and columns indexed by the
set of all terminals which is symmetric and has zeroes on the diagonal. We say that a traffic
matrix $D$ is admissible if we have $\sum_{v \in W} d_{uv} \leq b_u$ for all terminals $u \in W$. Now the symmetric
Virtual Private Network Design problem (sVPND) is to choose a simple $u$–$v$ path $P_{uv}$ for each
pair of distinct terminals $u, v \in W$ together with capacity reservations $y_e \in \mathbb{R}_+$ for all edges
$e \in E$ in such a way that every admissible demand matrix $D = (d_{uv})$ can be routed, that is,
$y_e \geq \sum (d_{uv} : u, v \in W$ and $e \in P_{uv})$, and the cost of the capacity reservation $c^T y = \sum_{e \in E} c_e y_e$
is minimum.

Grandoni et al. [2] introduced a new problem related to the symmetric Virtual Private
Network Design problem which they call Pyramidal Routing problem (PR). In this paper, we
consider the following version of their problem. As before, the network is given as a (undirected,
finitive, simple) graph $G = (V, E)$. We will always assume that $G$ is connected. This time $G$ has
a special vertex $r$ called the root. Each vertex $v$ has a certain demand $b_v \in \mathbb{Z}_+$. A vertex $v$ with
$b_v > 0$ is referred to as a terminal. We always assume that the root is a terminal, i.e., we have
$b_r \geq 1$. Let $k$ be the integer defined as

$$k := \sum_{v \in V} b_v.$$ 

So $k$ is simply the total demand. A routing is a collection $P$ of simple paths (repetitions are
allowed) such that (i) all paths in $P$ start at vertex $r$; (ii) for each vertex $v$ exactly $b_v$ paths of
$P$ end in $v$. In particular, any routing $P$ contains $b_r$ trivial paths starting and ending at
the root. The collection of paths $P$ determine two vectors in $\mathbb{Z}^E$: the $n$-vector $n = n(P)$ and the
$y$-vector $y = y(P)$ which are respectively defined as

$$n_e := |\{P : e \in P \in P\}|, \text{ and } y_e := \min\{n_e, k - n_e\}.$$ 

In other words $n_e$ gives the number of paths of $P$ containing edge $e$ and $y_e = p(n_e)$ where
$p : x \mapsto \min\{x, k - x\}$ is the so-called pyramidal function. Let $c_e \in \mathbb{R}_+$ denote the cost of edge
e $e \in E$. The cost of a routing is the total cost of its $y$-vector, that is, $c^T y = \sum_{e \in E} c_e y_e$. The
Pyramidal Routing problem (PR) is to find a routing whose cost is minimum. As mentioned
above, our version of PR is slightly more general than the one of Grandoni et al. [2]. The
original version is obtained by restricting $b_v$ to be 0 or 1 for all vertices $v$. This is not a severe
restriction because a demand $b > 1$ at some vertex $u$ can be simulated, for instance, by adding
$b$ pendant edges $uv_1, \ldots, uv_b$ with cost zero to the graph and letting $b_u = 0$ and $b_{v_i} = 1$ for
$i = 1, \ldots, b$.

**Conjecture (PR Conjecture).** For any instance $(G, r, b, c)$ of the Pyramidal Routing problem
there always exists an optimum routing whose paths form a tree (that is, such that the support
of the corresponding $n$-vector induces a tree).

As shown by Grandoni et al. [2], the PR Conjecture implies the VPN Conjecture. Moreover,
it follows easily from their results that the PR Conjecture restricted to the class of outerplanar
graphs implies the VPN Conjecture restricted to the class of outerplanar graphs.

We conclude this introduction by an outline of this extended abstract. In Section 2 we
consider several results which are at the same time basic and essential. First we note that if
the PR Conjecture holds for all the blocks of a graph $G$ then it also holds for $G$. To a given
graph $G$, root $r$ and demand vector $b$ we can associate in the standard way an upper-monotone
polyhedron which we call the Pyramidal Routing polyhedron (or PR polyhedron). The PR
Conjecture is equivalent to the following statement: all extreme points of the PR polyhedron
correspond to tree routings or, in other words, the $y$-vector of any routing is dominated by (i.e.,
coordinate-wise bigger or equal to) a convex combination of $y$-vectors of tree routings. We thus
obtain a formulation of the PR Conjecture that does not involve edge costs. Finally, we give
a necessary condition for a routing to determine an extreme point of the PR polyhedron. In
particular, our necessary condition implies that in any such extremal routing all paths from the root to a given vertex must coincide.

In Section 3 we show that if the PR Conjecture is satisfied by a graph then it is satisfied by all its minors. The proof uses the block-decomposition result of Section 2 and the PR Conjecture for cycles, which we have to reprove because we consider a strengthened version of the PR Conjecture. The minor-monotonicity of the PR Conjecture is allows to focus on restricted classes of graphs. For instance, we can restrict to graphs with maximum degree at most three.

We prove our main result in Section 4. More specifically, we prove the PR Conjecture for ladders (i.e., 2-connected outerplanar graphs with maximum degree at most three), which implies the PR Conjecture for outerplanar graphs, which in turn implies the VPN Conjecture for outerplanar graphs.

2 Fundamental tools

Our first lemma allows us to reduce the PR Conjecture to 2-connected graphs. We do not include its easy proof here but point out that it relies in an essential way on the fact that, in our version of the Pyramidal Routing problem, the demand $b_r$ at the root can be arbitrary.

Lemma 1. If the PR Conjecture holds for all blocks (maximal connected subgraphs without a cut-vertex) of a graph $G$ then it holds also for $G$.

The following lemma provides a way to state the PR Conjecture without referring to edge costs. Given a graph $G = (V, E)$, root $r \in V$ and demands $b \in \mathbb{Z}^V_+$ we define the Pyramidal Routing polyhedron (or PR polyhedron) $Q = Q(G, r, b)$ as the dominant of the convex hull of the $y$-vectors of routings in $G$. Thus we have $Q := \text{conv}\{y(P) \in \mathbb{R}^E : P$ is a routing in $G\} + \mathbb{R}^E$. Solving a PR instance of the form $(G, r, b, c)$ amounts to minimizing the linear function $y \mapsto c^T y$ over the corresponding PR polyhedron $Q(G, r, b)$.

Lemma 2. The PR Conjecture holds for a certain graph $G$ if and only if all extreme points of the PR polyhedron correspond to tree routings. In other words, the PR Conjecture holds for $G$ if and only if for any routing $P$ in $G$ there exists a collection of tree routings $T_1, \ldots, T_\ell$ and non-negative coefficients $\lambda_1, \ldots, \lambda_\ell$ summing up to 1 such that

$$\sum_{i=1}^\ell \lambda_i y(T_i) \leq y(P).$$

Proof. For the backward implication, let $P$ be an optimum solution to PR with respect to some cost vector $c \in \mathbb{R}^E_+$. Then (1) implies $\sum_{i=1}^\ell \lambda_i c^T y(T_i) \leq c^T y(P)$. So at least one of the tree routings $T_1, \ldots, T_\ell$ has a cost which does not exceed the cost of $P$. That is, at least one of the tree routings is optimum.

Let us now prove the forward implication by contradiction. Suppose that the PR Conjecture holds for $G$ but the PR polyhedron has an extreme point $y(P)$ where $P$ is not a tree routing. Then we can separate $y(P)$ from the other points of the PR polyhedron by a hyperplane. Because dominants are upper-monotone, it follows that there exists a non-negative cost vector $c$ such that $c^T y(P) < c^T y(Q)$ for all routings $Q$ such that $y(Q) \neq y(P)$. In particular, we have $c^T y(P) < c^T y(T)$ for all tree routings $T$, a contradiction. The result follows.

If $y$ and $y'$ are two vectors in $\mathbb{R}^E_+$ such that $y' \leq y$ we say that $y$ is dominated by $y'$. So if a routing $P$ satisfies (1) for some choice of tree routings $T_i$ and non-negative coefficients $\lambda_i$ summing up to 1, then the $y$-vector of $P$ is dominated by the corresponding convex combination of $y$-vectors of tree routings. We call a routing extremal if its $y$-vector is an extreme point of the PR polyhedron. So proving the PR Conjecture amounts to proving that all extremal routings
are tree routings or, equivalently, that the $y$-vector of any routing is dominated by a convex combination of $y$-vectors of tree routings.

The next lemma provides a useful necessary condition for a routing to be extremal. It essentially says that if a routing $P$ is extremal then its $n$-vector has to be an extreme point of the polytope defined as the convex hull of the $n$-vectors of all routings in $G$.

**Lemma 3.** Let $P$ and $P_1, \ldots, P_\ell$ with $n$-vector $n(P)$ and $n(P_1), \ldots, n(P_\ell)$ respectively. Then

$$n(P) = \sum_{i=1}^\ell \lambda_i n(P_i) \quad \text{implies} \quad \sum_{i=1}^\ell \lambda_i y(P_i) \leq y(P).$$

In particular, $P$ is not extremal whenever some routing $P_i$ whose corresponding coefficient $\lambda_i$ is positive has a $y$-vector distinct from that of $P$.

**Proof.** This follows from the concavity of the pyramidal function $p : x \mapsto \min\{x, k-x\}$. Indeed, for each edge $e$ we have

$$\sum_{i=1}^\ell \lambda_i y_e(P_i) = \sum_{i=1}^\ell \lambda_i p(n_e(P_i)) \leq p\left(\sum_{i=1}^\ell \lambda_i n_e(P_i)\right) = p(n_e(P)) = y_e(P).$$

We will use the above lemma in quite an intricate way in Section 4 below. However, it will more often be used in a very simple way in trying to tame the behavior of the paths in an extremal routing. For a path $P$ from the root $r$ to some terminal $u$ and a vertex $v$ on $P$, denote by $P^r v$ the sub-path from $r$ to $v$ and by $P^v u$ the sub-path from $v$ to the terminal $u$. The picture on the right illustrates these definitions in the context of the following “taming” lemma.

**Lemma 4 (Taming).** Suppose $P_1$, $P_2$ are paths in $P$ and $v$ is a vertex contained in both $P$ and $P_2$. Assume that the vertex sets of $P^r v$ and $P^v u$ are disjoint, as well as those of $P^r v$ and $P^v u$. Denote by $P_3$ the concatenation of $P^r v$ and $P^v u$ and by $P_4$ the concatenation of $P^r v$ and $P^v u$, we have

$$\frac{1}{2} y(P \setminus \{P_1 \cup P_4\}) + \frac{1}{2} y(P \setminus \{P_2 \cup P_3\}) \leq y(P).$$

In particular, in an extremal routing, all paths from the root to a fixed terminal coincide. □

### 3 Minor-monotonicity of the conjecture

In this section we prove that the class of graphs for which the PR Conjecture holds is closed under edge deletions and contractions. This is a key ingredient in the proof of our main result because it allows us to focus only on graphs with maximum degree at most 3. Moreover, if the PR Conjecture turns out to be false then Proposition 6 indicates that there could still be a hope to characterize the graphs which do satisfy the PR Conjecture.

The next result states that the our more general PR Conjecture is true for cycles. It is a crucial ingredient in proving the minor-monotonicity of the PR Conjecture, and hence in the result on outerplanar graphs.

**Lemma 5.** The PR Conjecture holds in case $G$ is a cycle.

We will sketch a proof below.
Remark: Since the version of the conjecture we consider here is more general than that of Grandoni et al. [2], Lemma 5 does not follow directly from their results. However, in a class of graphs which is closed under edge-subdivisions, the two versions are equivalent.

This can be shown as follows. First, it is possible to show that an optimal tree routing for 0/1-demands can be chosen to be a shortest path tree (this is proven similarly to the corresponding statement for the VPN, see [3]). Second, one may use the same arguments as Grandoni et al. [2] did for the VPN.

Grandoni et al. show that the y-vector of any routing on a cycle is dominated by the y-vector of a single tree routing. We now sketch a proof of Lemma 5, which is a subtle generalization of their argument: we dominate the y-vector by a convex combination of at least two trees. In fact, there are examples showing that in our version of the conjecture, one tree may not be sufficient. Our approach generalizes to ladder graphs, as we will see in Section 4.

Proof sketch. We prove that any extremal routing in G is a tree routing. Without loss of generality we can assume that all vertices of the cycle are terminals. If there is a non-terminal vertex v, then we can dissolve v, i.e., remove v from the graph and make its two neighbours adjacent. (If the resulting graph is non-simple then instead we use Lemma 4 to conclude.)

Number the vertices and edges of the cycle consecutively as w0, e0, w1, e1, ..., wm, em where w0 =: r is the root. Now let s0 := ∑mj=1 bwj for i = 0, ..., m (i.e., s0 = 0). Given any routing Q, we define its n-function as the continuous function fQ: [0, sm] → R which satisfies fQ(sj) = nej(Q) for all i = 0, ..., m and is affine on every interval [si, si+1]. This is just the interpolation of the n-vector. We define the y-function of Q as

\[ \tilde{f}^Q: [0, sm] \to \mathbb{R}: t \mapsto \min\{f^Q(t), k - f^Q(t)\} = p(f^Q(t)), \]

so \( \tilde{f}^Q(s_j) = y_{s_j}(Q) \) for every index j. Notice that the graph of \( \tilde{f}^Q \) can be obtained from that of \( f^Q \) by mirroring the part of the graph of \( f^Q \) above the \( k/2 \)-line, that is, the horizontal line through the points \( (0, k/2) \) and \( (s_m, k/2) \).

Consider now an extremal routing P in G and denote by f its n-function and by \( \tilde{f} \) its y-function. By Lemma 4 we can show that the affine segments of f all have slopes in \{-1, +1\}.

By arguments similar to those we will use later in the proof of Theorem 7, we can show that the only interesting case is when the n-function f consists of three affine parts: it increases on the interval \( [0, s_j] \) for some j, then it decreases on the interval \( [s_j, f(0) + s_j] \) and then increases again on the interval \( [f(0) + s_j, s_m] \). Moreover, the graph of f crosses the \( k/2 \)-line in the second interval, between \( t = s_j \) and \( t = f(0) + s_j \). Note that \( f(0) + s_j = s_{j'} \) for some \( j' \).

Let \( T_1 \) be the tree routing which omits edge \( e_j \), and let \( T_2 \) be the tree routing which omits edge \( e_{j'} \). Let \( f \) and \( f' \) be the corresponding n-functions, and \( \tilde{f} \) and \( \tilde{f}' \) the corresponding y-functions. We show that \( \tilde{f} \) is dominated by a certain convex combination \( \lambda_1\tilde{f}_1 + \lambda_2\tilde{f}_2 \) of \( \tilde{f} \) and \( \tilde{f}' \). The coefficients \( \lambda_1 \) and \( \lambda_2 \) are defined as

\[ \lambda_1 := \frac{\alpha_1}{\alpha_1 + \beta_1} \quad \text{and} \quad \lambda_2 := \frac{\beta_1}{\alpha_1 + \beta_1}, \]

where \( \alpha_1 = \alpha_2, \beta_1 = \beta_2 \) are as indicated in Fig. 1. For the y-vector, this implies that we have \( \lambda_1y(T_1) + \lambda_2y(T_2) \leq y(P) \). So either P is not extremal, which contradicts our assumption, or \( P \) is a tree routing.

**Proposition 6.** If the PR Conjecture holds for G then it holds for any minor of G.

**Proof.** Graphs verifying the PR Conjecture are clearly closed under edge deletion. So it suffices to consider the case of edge contractions. Let e = st be an edge of G. Consider some instance \((G', r', b')\) of PR with \( G' = G/e \) and some routing \( P' \) in \( G' \). Let \( u_e \) denote the vertex of \( G' \) resulting from the contraction of e. Now define a root r and demands \( b_v \) in G as follows. If
Figure 1: To the left: the graphs of the $n$-functions $f$, $f^1$ and $f^2$ (solid lines) and of the corresponding $y$-functions $\bar{f}$, $\bar{f}^1$ and $\bar{f}^2$ (the part of the graph of the $n$-function which was reflected to obtain the graph of the $y$-function is indicated by dashed lines). To the right: the graphs of $f$, $\bar{f}$ and the convex combination $\lambda_1 \bar{f}^1 + \lambda_2 f^2$ (in bold).

If $r' \neq u_e$ then let $r := r'$, otherwise let $r := s$. Let $b_v := b_v'$ for $v \not\in \{s, t\}$, $b_v := b_{u_v}$ for $v = s$ and $b_v := 0$ for $v = t$. Let $\mathcal{P}$ be the routing in $G$ obtained from $\mathcal{P}'$ by rerouting the paths containing $u_e$ in such a way that no path in $\mathcal{P}$ starts or ends in $t$ or uses an edge of $G$ incident to $t$ and to a common neighbor of $s$ and $t$.

Consider $n := n(\mathcal{P})$, $n' := n(\mathcal{P}')$, $y := y(\mathcal{P})$ and $y' := y(\mathcal{P}')$. In order to relate $n'$ and $n$ (resp. $y'$ and $y$) we associate to each edge $f'$ in $G'$ a unique edge $f$ in $G$ as follows. We let $f := f'$ when $f'$ is not incident to $u_e$ or if $f = vu_e$ and $v$ is not a common neighbor of $s$ and $t$ in $G$; otherwise $f' = vu_e$ for some common neighbor of $s$ and $t$ in $G$ and we let $f := ws$. Then we have $n'_f = n_f$ for all edges $f'$ of $G'$. In particular, we have $y'_f = y_f$ for $f' \in E(G')$.

The PR Conjecture holds for $G$ so, by Lemma 2, the $y$-vector of $\mathcal{P}$ is dominated by a convex combination of $y$-vectors of tree routings, that is, $\sum_{i=1}^{\ell} \lambda_i y(T_i) \leq y(\mathcal{P})$ where the $T_i$'s are some tree routings. Now let $T'_i$ denote the routing obtained from the tree routing $T_i$ by contracting the edge $e$. We remark that $T'_i$ is not necessarily a tree routing. (It is if $T_i$ uses the edge $e$.) Then we have $\sum_{i=1}^{\ell} \lambda_i y(T'_i) \leq y(\mathcal{P}')$. The support of each $y(T'_i)$ is either a tree or a tree plus an edge. In the second case, by Lemmas 1 and 5, $y(T'_i)$ is dominated by a convex combination of $y$-vectors of tree routings. Hence $y(\mathcal{P}')$ is dominated by a convex combination of $y$-vectors of tree routings. The result follows.

From Proposition 6, we can infer the following. The PR-conjecture for graphs with maximum degree three implies the PR-conjecture. The PR-conjecture for (hexagonal) grids implies the PR-conjecture for planar graphs. The PR-conjecture for ladders implies the PR-conjecture for outerplanar graphs.

4 Sketch of the proof for the outerplanar case

We come to the main result of this paper.

**Theorem 7.** The PR-conjecture holds when $G$ is outerplanar.

The remainder of this section is dedicated to sketching the proof for this result.
By the previous remark, it suffices to consider the case where $G$ is a ladder. A **ladder** is any graph obtained from a matching $H$ with $E(H) = \{v_i v'_i : i = 1, \ldots, k\}$ by adding $2k - 2$ $H$-paths with no common internal vertex in such a way that there is precisely one path from $v_i$ to $v_{i+1}$, and one path from $v'_i$ to $v'_{i+1}$ for $i = 1, \ldots, k - 1$; see the picture on the right for an example (the letters refer to definitions below). The matching edges are called **rungs** of the ladder.

Our proof of the PR conjecture for ladders proceeds by induction on the number of rungs. The case when $G$ has two rungs is done in Lemma 5. Suppose now that a ladder $G$ with at least three rungs and demands $b_v, v \in V$, are given, and let $P$ be any routing on $G$.

If there is an edge of $G$ which is not used by $P$, then, possibly by using the block-decomposition Lemma 1, we can invoke the induction hypotheses to write the $y$-vector $y(P)$ of this routing as a convex combination of $y$-vectors of tree routings as in (1), Lemma 2. Otherwise, we will accomplish the induction step by producing a number of routings, $P_1, \ldots, P_n$, each of which omits some edge of $G$, and establish that $y(P)$ is dominated by the convex hull of the $y$-vectors of these routings. The induction hypothesis then yields, for each $P_j$, a convex combination of $y$-vectors of tree routings dominating $y(P_j)$. Thus, a convex combination of tree routings dominating $y(P)$ is found.

The strategy which we use is to focus on the lowest cycle of the ladder. There we are able to “uncross” $P$. This strategy relies crucially on the fact that the degrees of the vertices $u, v$ on the top edge of the lowest cycle are not larger than three. In fact, there are examples which show that the strategy fails in general outerplanar graphs. Thus, the minor-monotonicity is indispensable for our proof.

The strategy is implemented in the following steps.

1. Examine in what ways a path may meddle with the lowest cycle (cf. Section 4.1).
2. “Smooth” the routing to reduce the amount “wobbling” (cf. Section 4.2).
3. For the smoothed routing, identify routings omitting edges and establish a convex combination of their $y$-vectors (cf. Section 4.3).

The last step is the most onerous one. The key ingredient there is $n$-functions and $y$-functions similarly to those we use for the proof of Lemma 5.

### 4.1 Examine in what ways a path may meddle with the lowest cycle

Let $uv$ be the top edge of the lowest cycle, let $U$ be the vertex set of the lower connected component of $G - \{u, v\}$, and let $\bar{U}$ be the subgraph of $G$ induced by $U \cup \{u, v\}$ (see the above picture of a ladder). W.l.o.g., we may assume that the root is not in $U$. In what way may a path intersect $\bar{U}$?

- It may enter $\bar{U}$ and then head for a terminal which is not in $U$. We call these paths **thru paths**, and denote their number by $q$.
- It may enter $\bar{U}$ and end at a terminal in $U$. There are four ways how this can be done. We symbolize them by strings $rXt$, where $X$ is replaced by the intersection of the path with $\{u, v\}$, taking into account the order in which the vertices $u$ and $v$ are visited on the path from the root to the terminal. The following patterns are possible: $rvt, rvut$; $rvt, rvut$. 

---

7
Note that, since $u$ and $v$ have degree three, if a path intersects $\{u, v\}$ in both $u$ and $v$, then it must contain the edge $uv$. An application of the Taming Lemma 4 yields that we can assume that $rut$ and $rvut$ do not both occur in $\mathcal{P}$. Such paths, starting from $r$, enter $U$ via $u$, so their sub-paths connecting $r$ to $u$ can be interchanged. The same holds for $rvut$ and $rut$. We are faced with the matrix of cases depicted on the right. Clearly, the two cases marked $B$ are symmetric. It is easy to see that they can be reduced to the cycle, i.e., to Lemma 5. (We emphasize that we need the full strength of Lemma 5 here, i.e., demands $>1$ on the root of the cycle may occur.)

As for Case $A$, we need to invoke the following easy lemma, which holds independently from our case distinction, and will be used for Case $A'$, too.

**Lemma 8.** The $y$-vector of any routing $\mathcal{P}$ is dominated by a convex combination of tree routings and routings in which no thru path uses the top edge $uv$.

**Proof (sketch).** This follows again by an application of the Taming Lemma 4: we can dominate $y(\mathcal{P})$ by routings in which either all thru paths use the top edge $uv$, or they all walk around $U$. For the first type, we can remove $U$ from $G$, suitably update the demands for the vertices $u$ and $v$, and invoke the induction hypotheses.

Hence, Case $A$ implies that the top edge $uv$ is not used by $\mathcal{P}$. We are left with Case $A'$. The situation on $\hat{U}$ is visualized in the picture on the right. There are, say, $r_u \geq 0$, paths of type $r v u t$ entering $\hat{U}$ through the vertex $u$ and heading for a terminal in $U$ via $uv$, and, say, $r_v \geq 0$, paths of type $r v u t$ entering $\hat{U}$ through the vertex $v$ and heading for terminal in $U$ via $uv$. In addition, there are $q$ thru paths entering $\hat{U}$ at either $u$ or $v$, respectively, walking all the way along $U$, and either ending at $v$ or $u$, respectively, or leaving $\hat{U}$ again. The numbers next to the edges in the picture show known values of the $n$-vector for $\mathcal{P}$: The top edge $uv$ is used by $r_u + r_v$ paths, the topmost vertical edges by $r_v + q$ and $r_u + q$ paths, respectively.

### 4.2 Smoothing the routing

To study how $\mathcal{P}$ behaves on $\hat{U} \setminus \{uv\}$, we use $n$-functions and $y$-functions similarly to those in the proof of Lemma 5. Here, we number the vertices and edges of $\hat{U} \setminus \{uv\}$ consecutively as $u = w_0, e_0, \ldots, w_m, e_m$. Now let $s_i := \sum_{j=1}^i b_{w_j}$ for $i = 0, \ldots, m$. For any routing $\mathcal{Q}$, we then define the $n$-function $f^Q: [0, s_m] \rightarrow \mathbb{R}$ and $y$-function $f^Q: [0, s_m] \rightarrow \mathbb{R}$ precisely as in the proof of Lemma 5. We abbreviate $f := f^P$ and $\hat{f} := \hat{f}^P$.

We say that a crossing of any $n$-function $g: [0, s_m] \rightarrow \mathbb{R}$ is a point $t$ in the open interval $]0, s_m[$ in which the function is smooth and at which the graph of $g$ intersects the horizontal line through $(0, k/2)$ transversely, i.e., $g$ is affine near $t$, $g(t) = k/2$, and the slope of $g$ in $t$ is $\pm 1$. In general $f$ may have have many crossings, peaks (i.e., local maxima in the open interval $]0, s_m[$) and valleys (i.e., local minima in the open interval): the function wobbles. The following lemma describes what can be done in such a situation.

**Lemma 9.** Let $f^P$ be the $n$-function of some routing $\mathcal{P}$. Let $t_1, t_2$ be integers in $[0, s_m]$ with $f^P(t_1) = f^P(t_2)$. By reflecting the graph of $f^P$ in the interval $[t_1, t_2]$ at the horizontal line through $(0, f(t_1))$, we obtain the graph of an $n$-function of some routing.
The figure on the right shows how this lemma is used to reduce the wobbling. A valley where the function is above $k/2$ can be removed by reflecting as shown in the picture. Note that the $y$-function of the old routing is dominated by the $y$-function of the new routing. Similarly, a peak below $k/2$ can be removed. If $t_1$ and $t_2$ are two crossing, and there is no crossing between these two, then by reflection, we can obtain a routing whose $n$-function has fewer crossings, and whose $y$-function dominates the $y$-function of the routing we started with. In this manner, we can smooth the $y$-function to arrive at the following result.

**Lemma 10.** After smoothing, we have an $n$-function which has at most one crossing, at most one peak, and at most one valley. At the peak, the $n$-function must be above $k/2$, at the valley, it must be below $k/2$.

### 4.3 Identify routings omitting edges and establish a convex combination of their $y$-vectors

The following procedure is at the heart of our proof. Similarly to what we did in our proof of Lemma 5, we establish path routings $Q_1$ and $Q_2$ omitting edges $g_1$ and $g_2$ of $\bar{U}$ respectively and identify coefficients $\lambda_1$ and $\lambda_2$ for the $y$-vectors $y(Q_1)$ and $y(Q_2)$ such that $y(P)$ is dominated by $\lambda_1 y(Q_1) + \lambda_2 y(Q_2)$. This completes the proof.

We now describe how these routings, edges, and coefficients are identified in the case when $f$ has a crossing (the case without crossing is easier). The proof that these selections make sense and do the job is grossly beyond the limit of this extended abstract. After smoothing the $y$-function $f = f^P$ looks as drawn in the left picture of Fig. 2. The dashed line is the part were $\bar{f}$ differs from $f$. If $f$ has a peak, then we denote the point by $s_j$ and if it has a valley, we denote it by $s_j'$. 

![Figure 2: Left: Smoothed solution. Right: Routings $Q_i$ and coefficients](image)

A path ending at a terminal vertex might either arrive there in a clockwise or in an anti-clockwise motion. We first construct a routing $Q^1$ by reversing this orientation for every terminal vertex in $U$. $Q^1$ clearly does not use the top edge $uv$, because we are in the Case A', where every path of $P$ uses this edge. The graph of the $n$-function $f^1 := f^{Q_1}$ of $Q^1$ can be obtained by reflecting the graph of $f$ at the horizontal line through $(r_u + r_v)/2 + q$. The right picture in
Fig. 2 shows $f^1$ and $\bar{f}^1 := f^{Q^1}$. Depending on whether $k/2$ is above or below $(r_u + r_v)/2 + q$, $\bar{f}^1$ lies above $\bar{f}$ in $s_j$ and below $\bar{f}$ in $s_j'$ or vice versa. Let us assume the former, as in Fig. 2. For the other point $s_j$, we produce a routing which does not use the edge $e_j$, but coincides with $P$ on every edge not in $P$. Such a routing is uniquely determined except on the top edge $uv$, where for a certain number of paths, we may be able to choose whether they use $uv$ or not (the respective values of $n_{uv}(\cdot)$ are easy to compute). We take the routing $Q^2$ which uses $uv$ as sparingly as possible. Let $f_2 := f^{Q^2}$, and $\bar{f}_2 := \bar{f}^{Q^2}$. Using the values $\alpha_i$ and $\beta_i$ as sketched in the figure, the coefficients for the convex combination of the $y$-vectors are now

$$\lambda_1 := \frac{\alpha_1}{\alpha_1 + \beta_1}, \quad \lambda_2 := \frac{\beta_1}{\alpha_1 + \beta_1}.$$

We then have

$$\bar{f}(s_j) = \lambda_1 f^1(s_j) + \lambda_2 f^2(s_j).$$

Clearly, we also have

$$\bar{f}(s_j') = \frac{\alpha_2}{\alpha_2 + \beta_2} f^1(s_j') + \frac{\beta_2}{\alpha_2 + \beta_2} f^2(s_j'),$$

although this affine combination may not be a convex combination (moreover, there are cases when $\alpha_2 + \beta_2 \leq 0$). But it is possible to show that

$$\alpha_2 = \alpha_1 \quad \text{and} \quad \beta_2 \leq \beta_1,$$

and that the right hand side of equation (2) is a decreasing function in $\beta_2$. So we obtain

$$f(s_j') \geq \lambda_1 f^1(s_j') + \lambda_2 f^2(s_j').$$

The inequality in the other points in $[0, s_m]$ now follows using the fact that $\bar{f}$ is concave on the relevant intervals.

A computation shows that $y_{uv}(P) \geq \lambda_1 y_{uv}(Q^1) + \lambda_2 y_{uv}(Q^2)$ also holds.

This completes our sketch of the proof of the Pyramidal Routing conjecture for ladders.

References


